CHAPTER 3

THERMAL ANALYSIS OF VISCOUS DISSIPATION EFFECTS IN FREE AND FORCED CONVECTION OVER A NON ISOTHERMAL VERTICAL FLAT PLATE WITH SUCTION AND INJECTION

3.1 INTRODUCTION

The problem of laminar mixed convection flow over a heated vertical flat plate provides one of the most fundamental problems in heat transfer and is of considerable theoretical and practical interest. Mixed convection from a vertical flat plate has been used in transport process devices, electronic devices and chemical processing devices etc. The effect of buoyancy forces on flow and heat transfer is usually ignored when a forced convection flow over a cooled or heated surface is studied. However, neglecting buoyancy effects cannot be justified, because under certain circumstances the buoyancy forces influences the flow and temperature functions despite the presence of forced convection flow.

1The contents of this chapter was accepted for publication in International Review of Pure and Applied Mathematics.
In the free convection boundary layer flow, Clarke (1973) has presented solutions for the outer region of the flow field for blowing under which similar solutions to the boundary layer equations exist. The steady mixed convection from a vertical flat plate has been studied by Merkin (1969). Lloyd and Sparrow (1970) have obtained a complete solution to the aiding mixed convection from a vertical isothermal flat plate by using the local similarity method for different values of the Prandtl number $Pr = 0.003, 0.01, 0.03, 0.72, 10, 100$ and for Richardson number in the range 0 to 4. Merkin and Pop (2002) have applied a similarity transformation to analyze mixed convection boundary layer flow over a vertical semi-infinite plate in which the free stream velocity is uniform and the wall temperature is inversely proportional to the distance along the plate. In this paper the solutions are analyzed for both small and large Prandtl numbers, for different values of mixed convection parameter including free, forced and mixed convection flows.

Watanabe and Kawakami (1989) have solved a non-similar free convection boundary layer flow with uniform suction or injection over a vertical flat plate by employing the difference-differential method. Watanabe (1990) has solved the non-similar problem of forced flow past a wedge with uniform suction or injection. The effect of free and forced mixed convection boundary layer with uniform fluid injection or suction was studied by Wang and Kleinstreuer (1990). Sacid (2005) has employed implicit finite difference scheme for the isothermal flat plate, the periodic oscillation and for the temperature on the plate. The result reveals that the steady periodic variation of the Nusselt number and friction co-efficient for both aiding and
opposing flows with different amplitudes and frequencies of the oscillating surface temperature. Recently Orhan Aydin and Ahemet Kaya (2007) have solved the governing equations using the finite difference method for laminar, mixed convection heat transfer from a heated bar, cooled vertical flat plate in both aiding and opposing buoyancy situation. The Richardson number and Eckert number have been analyzed on the velocity and temperature profiles as well as the friction and heat transfer co-efficient.

The objective of this chapter is to consider free and forced convection over a non-isothermal vertical flat plate with suction and injection, in the presence of viscous dissipation effect. Numerical results are presented for some representative values of governing parameters, mainly, the buoyancy parameter($K$) representing the free and forced convection flow and the Eckert number($e$) representing the effect of the viscous dissipation.

3.2 GOVERNING EQUATIONS

The conservation equations of mass, momentum and energy for steady, laminar, incompressible, two dimensional, free and forced convection boundary layer flow, with constant physical properties with viscous dissipation effects are

\[ u_x + v_y = 0 \quad (\text{mass}), \quad (3.1) \]
\[ uu_x + vu_y = \nu u_{yy} + gB(T - T_\infty) \quad (\text{momentum}), \quad (3.2) \]
\[ uT_x + vT_y = \alpha T_{yy} + \left(\frac{\nu}{C_p}\right) (u_y)^2 \quad (\text{energy}) \quad (3.3) \]
The compatibility conditions are

\[ y = 0; \quad u = 0, \quad v = v_0, \quad T = T_0, \]
\[ y = \infty; \quad u = U_\infty, \quad T = T_\infty (\text{uniform}), \quad (3.4) \]

### 3.3 Solution Procedure

It is considered that the system of two dimensional laminar, free and forced convection flow over a non isothermal vertical flat plate with viscous dissipation effects together with uniform suction and injection. The lower end of the plate is taken as the origin and the \( x \)-dimension is taken along the plate direction. The wall temperature are assumed to vary as power functions of distance from the start of the boundary layer respectively as

\[ T_0 - T_\infty = ax^n, \quad (3.5) \]

where \( "a" \) and \( "n" \) are constants.

The velocity and temperature fields in their respective boundary layer regions are suitably described by the following functions

\[ u/U_\infty = 2y/d_1 - 2y^3/d_1^3 + y^4/d_1^4, \quad (y < d_1) \]
\[ u = U_\infty, \quad (y \geq d_1) \]
\[ (T - T_\infty)/(T_0 - T_\infty) = 1 - 2y/d_2 + 2y^3/d_2^3 - y^4/d_2^4, \quad (y < d_2) \]
\[ T = T_\infty, \quad (y \geq d_2) \quad (3.6) \]

where \( d_1 \) and \( d_2 \) are momentum and thermal boundary layer thicknesses respectively.
The velocity and thermal profiles \((3.6)\) satisfy the following compatibility conditions:

\[
y = 0; \quad u = 0, \quad v = v_0, \quad T = T_0,
\]

\[
y = d_1; \quad u = U_\infty, \quad \partial u/\partial y = 0 \text{ (smoothfit)}, \quad \partial^2 u/\partial y^2 = 0, \quad (3.7)
\]

\[
y = d_2; \quad T = T_\infty, \quad \partial T/\partial y = 0 \text{ (smoothfit)}, \quad \partial^2 T/\partial y^2 = 0.
\]

The smooth fit conditions \(u_y = 0\) and \(T_y = 0\) correspond to \(P_{12} = 0\) and \(J_q = 0\) at the respective edges of the boundary layers. In view of the boundary conditions \((3.7)\) the trial functions \((3.6)\) are appropriate ones for the case when viscous dissipation effects are neglected. However, the analysis is carried out on the principle of perturbation that the result hold good for small viscous dissipation effects. Here \(d_1\) and \(d_2\) are unknown parameters and they are to be determined by the present thermodynamic analysis.

In order to formulate the governing principle of dissipative processes for the present system the balance equations are required for which can be rewritten as,

\[
\nabla \cdot \vec{V} = 0, \quad (\vec{V} = iu + jv) \quad (3.8)
\]

\[
\rho(\vec{V} \cdot \nabla)\vec{V} + \nabla \cdot \vec{P} = -g B\rho(T - T_\infty)\vec{I}, \quad (3.9)
\]

\[
\rho C_p(\vec{V} \cdot \nabla)T + \nabla \cdot \vec{J}_q = \mu (u_y)^2. \quad (3.10)
\]

These equations represent the mass, momentum and energy balances respectively. In the energy picture, the energy dissipation for the present system is given by

\[
T\sigma = -J_q(\partial nT/\partial y) - P_{12}(\partial u/\partial y), \quad (3.11)
\]
the heat flux $J_q$ and $P_{12}$ the only component of momentum flux $\hat{\mathbf{P}}_{us}$, satisfy the constitutive relations connecting the independent forces and fluxes as

$$\begin{align*}
P_{12} &= -L_s(\partial u/\partial y) \\
J_q &= -L_\lambda(\partial lnT/\partial y).
\end{align*} \quad (3.12)$$

The dissipation potentials $\Psi$ and $\Phi$ in the energy picture are given by

$$\begin{align*}
T\Psi &= (1/2) \left[ L_\lambda (\partial lnT/\partial y)^2 + L_s (\partial u/\partial y)^2 \right]. \\
T\Phi &= (1/2) \left[ R_\lambda J_q^2 + R_s P_{12}^2 \right].
\end{align*} \quad (3.13)$$

The Governing Principle of Dissipative Processes in the energy picture

$$\delta \int_V \left[ T\sigma - T\Psi - T\Phi \right] dV = 0, \quad (3.15)$$

is formulated and it has the following form,

$$\delta \int_0^l \int_0^\infty \left[ -J_q(\partial lnT/\partial y) - P_{12}(\partial u/\partial y) - (L_\lambda/2)(\partial lnT/\partial y)^2 \\
- (L_s/2)(\partial u/\partial y)^2 - (R_\lambda/2)J_q^2 - (R_s/2)P_{12}^2 \right] dy dx = 0. \quad (3.16)$$

The transverse velocity component $v$ is obtained from the mass balance Equation (3.1) as

$$v = U_\infty[y^2/d_1^2 - 3y^4/2d_1^4 + 4y^5/5d_1^5]d'_1 + v_0. \quad (3.17)$$

To formulate Gyarmati's variational principle the velocity and temperature functions (3.6) are substituted in the momentum and energy balance Equations (3.2) and (3.3), and on direct integration with respect to $y$ with the help of smooth fit boundary conditions ($u_y = 0$ and $T_y = 0$) the expressions
of $P_{12}$ and $J_q$ are obtained respectively as given below.

$$-P_{12}/L_s = \left( U_\infty/d_1 \right) + \left( U_\infty^2 d'_1/v \right) \left[ 0.056111111 - 2y^3/3d_1^3 + 7y^5/5d_1^5 \right. $$

$$ -11y^6/15d_1^6 - 3y^7/7d_1^7 + 2y^8/5d_1^8 - 4y^9/45d_1^9 \right] $$

$$ + \left[ v_0 U_\infty/v \right] \left[ -0.7 + 2y/d_1 - 2y^3/d_1^3 + y^4/d_1^4 \right] $$

$$ + \left[ gB(T_0 - T_\infty)d_1/v \right] \left[ 1/2 - d_1/3d_2 + d_1^3/10d_2 - d_1^4/30d_2^4 \right] $$

$$ - y/d_1 + y^2/d_1 d_2 - y^4/2d_1 d_2^2 + y^5/5d_1^3 d_2^2 \right] $$

(3.18)

$$ -J_q/L_\chi = \left[ nU_\infty Pr(T_0 - T_\infty)/\nu \right] \left[ -2d_2^2/15d_1 + 3d_2^4/140d_1^3 \right. $$

$$ -d_2^3/180d_1^4 + y^2/d_1 - 4y^3/3d_1 d_2 - y^4/2d_1^2 + y^5/5d_1^4 $$

$$ + 4y^5/5d_1^3 d_2 + 4y^5/5d_1^3 d_2 - y^6/3d_1^4 d_2 - y^6/3d_1^4 d_2 $$

$$ - 4y^6/7d_1^5 d_2 - 4y^7/4d_1^6 d_2 + 4y^7/4d_1^6 d_2 - y^8/9d_1^7 d_2 $$

$$ + \left[ U_\infty Pr(T_0 - T_\infty)/\nu \right] d_1^3 \left[ -2d_2^2/15d_1 + 3d_2^4/35d_1^3 \right. $$

$$ - d_2^3/36d_1^4 + 4y^3/3d_1 d_2 - 12y^5/5d_1 d_2 - 4y^5/5d_1^3 d_2 $$

$$ + 4y^6/3d_1^2 d_2 + y^6/3d_1^2 d_2 + 12y^7/7d_1^3 d_2 - y^8/d_1^4 d_2 $$

$$ - 3y^8/4d_1^4 d_2 + 4y^9/9d_1^4 d_2 \right]$$

$$ + \left[ U_\infty Pr(T_0 - T_\infty)/\nu \right] d_1^3 \left[ 2d_2^2/15d_1^2 - 9d_2^4/140d_1^3 + d_2^5/45d_1^5 - 2y^3/3d_1^3 d_2 \right. $$

$$ + 3y^5/5d_1^3 d_2 + 6y^5/5d_1^3 d_2 - 4y^6/15d_1^3 d_2 - 2y^6/3d_1^3 d_2 $$

$$ - 9y^7/7d_1^4 d_2 + 3y^8/5d_1^4 d_2 + 3y^8/4d_1^4 d_2 - 16y^9/45d_1^5 d_2 \right]$$

$$ + \left[ v_0 Pr(T_0 - T_\infty)/\nu \right] \left[ 1 - 2y/d_2 + 2y^3/d_2^3 - y^4/d_2^4 \right. $$

$$ - \left[ U_\infty^2 Pr/C_p \right] \left[ -4d_2^2/d_1^2 + 8d_2^2/d_1^4 - 4d_2^3/3d_1^3 - 36d_2^5/5d_1^5 \right. $$

$$ + 7d_2^5/d_1^4 - 16d_2^7/7d_1^7 + 4y/d_2 - 8y^3/d_2^4 + 4y^4/d_2^5 $$

$$ + 36y^5/5d_1^6 - 7y^6/d_1^7 + 16y^7/7d_1^8 \right]$$

(3.19)
The prime indicates the differentiation with respect to \( x \). The above expressions for \( P_{12} \) and \( J_q \) are obtained with the assumption that the Prandtl number \( Pr \geq 1 \).

With the help of Equations (3.6), (3.18) and (3.19) the GPDP given by (3.16) is formulated and the integration of the Lagrangian with respect to \( y \) is carried out. The variational principle, after simplification, is written in a simple form as

\[
\delta \int_{0}^{l} L(d_1, d_2, d_1', d_2') \, dx = 0. \tag{3.20}
\]

The explicit expression for the above function is as follows,

\[
\delta \int_{0}^{l} \lambda \left[ n(T_0 - T_\infty)^2 U_\infty Pr / \nu \right] [ -0.016955266 d_2^4 / d_1^4 + 0.004678654 \\
+ 0.07936507936 d_2^2 / d_1^2 ] + [(T_0 - T_\infty)^2 U_\infty Pr / \nu ] d_2^2 [ 0.255855255 \\
d_2^2 / d_1^2 - 0.0767676767 d_2^3 / d_1^3 + 0.212698412698 d_2 / d_1 ] + [(T_0 - T_\infty)^2 \\
U_\infty Pr / \nu ] d_1^2 [ -0.020468420 d_2^5 / d_1^5 - 0.1063920635 d_2^4 / d_1^4 + 0.057575757 \\
d_2^2 / d_1^2 ] + [(T_0 - T_\infty)^2 v_0 Pr / \nu ] [ -0.5 - (T_0 - T_\infty)^2 [ 0.742857142857 / d_2 ] \\
- [ n^2 (T_0 - T_\infty)^2 U_\infty Pr^2 / \nu^2 x^2 ] [ 0.000007453 d_2^{11} / d_1^8 - 0.000053679 d_2^{10} / d_1^7 \\
+ 0.000097337 d_2^9 / d_1^6 + 0.000244847 d_2^8 / d_1^5 - 0.000905919 d_2^7 / d_1^4 \\
+ 0.002253302 d_2^6 / d_1^3 ] - [(T_0 - T_\infty)^2 U_\infty^2 Pr^2 / \nu^2 ] (d_2')^2 [ 0.000233174 \\
d_2^2 / d_1^2 - 0.001382515 d_2^6 / d_1^7 + 0.002055787 d_2^5 / d_1^6 + 0.0036874236 d_2^4 / d_1^5 \\
- 0.011061953 d_2^3 / d_1^4 + 0.015273615 d_2^2 / d_1^3 ] - [(T_0 - T_\infty)^2 U_\infty^2 Pr^2 / \nu^2 ] \\
(d_1')^2 [ 0.000149231 d_1^{11} / d_1^{10} - 0.000829509 d_1^{10} / d_1^9 + 0.001156380 d_1^9 / d_1^8 \\
+ 0.001474969 d_1^8 / d_1^7 + 0.003818403 d_1^7 / d_1^6 ] - [(T_0 - T_\infty)^2 v_0^2 Pr^2 / \nu^2 ] \\
[ 0.091269841 d_2 ] - [ n(T_0 - T_\infty)^2 U_\infty^2 Pr^2 / \nu^2 x ] d_2 [ 0.000819849 d_2^{10} / d_1^8 
\]
\[-0.000536798d_2^6/d_1^6 + 0.000876035d_2^4/d_1^6 + 0.001958781d_2^2/d_1^6\]
\[-0.006341436d_2^4/d_1^4 + 0.011266511d_2^3/d_1^4 \right] - \left[n(T_0 - T_\infty)^2U_2^2 Pr^2/v^2 x]\right] d_1^3 \right]
\[-0.000655879d_2^{11}/d_1^3 + 0.00041759d_2^{10}/d_1^3 - 0.000657026d_2^2/d_1^3\]
\[-0.00136489d_2^6/d_1^6 + 0.004147122d_2^5/d_1^6 - 0.005633255d_2^5/d_1^6\]
\[-\left[n(T_0 - T_\infty)^2U_\infty v_0 Pr^2/v^2 x]\right] - \left[0.001534021d_2^6/d_1^4 + 0.005682650d_2^5/d_1^4\right]\]
\[-0.028585858d_2^4/d_1^4 - \left[(T_0 - T_\infty)^2U_\infty v_0 Pr^2/v^2 x\right]d_1^3 d_2^2 \right]\]
\[-0.00373079d_2^{10}/d_1^3 + 0.001248999d_2^8/d_1^4 - 0.003083681d_2^7/d_1^4 - 0.004793650d_2^7/d_1^4\]
\[+0.013827442d_2^6/d_1^4 + 0.005273615d_2^5/d_1^4 - \left[(T_0 - T_\infty)^2U_\infty v_0 Pr^2/v^2 x\right]d_2^2\]
\[-0.00568608d_2^5/d_1^4 + 0.024244404d_2^4/d_1^4 - 0.070735930d_2^4/d_1^4\]
\[-\left[(T_0 - T_\infty)^2U_\infty v_0 Pr^2/v^2 x\right]d_1^3 d_2^2 \right]\]
\[-0.00373079d_2^{10}/d_1^4 + 0.001248999d_2^8/d_1^4 - 0.003083681d_2^7/d_1^4 - 0.004793650d_2^7/d_1^4\]
\[+0.013827442d_2^6/d_1^4 + 0.005273615d_2^5/d_1^4 - \left[(T_0 - T_\infty)^2U_\infty v_0 Pr^2/v^2 x\right]d_2^2\]
\[-0.00568608d_2^5/d_1^4 + 0.024244404d_2^4/d_1^4 - 0.070735930d_2^4/d_1^4\]
\[+\left[v_0 U_\infty^2/v\right][0.2] + \left[gB(T_0 - T_\infty)U_\infty/v\right]d_1[0.2 - 0.2d_1/d_2 + 0.078571428\]
\[d_3^2/d_2^2 - 0.027777777d_3^4/d_2^4 - \left[U_\infty^2/d_1^2[0.742857142] - \left[U_\infty^2/d_1^2[0.5\right]\]
\[-\left[g^2 B^2(T_0 - T_\infty)^2/v^2\right]d_1^3[0.0416666666d_1 - 0.0833333333d_1^3/d_2\]
\[+0.044444444d_1^3/d_2^3 + 0.0333333333d_1^4/d_2^3 - 0.05d_1^3/d_2^3 + 0.013888888\]
\[d_3^2/d_2^2 + 0.008888888d_1^3/d_2^3 - 0.006666666d_1^3/d_2^3 + 0.0012626262d_1^3/d_2^3\]
\[-\left[U_\infty^3d_1^4/v\right][0.0000000011] - \left[U_\infty^3v_0 d_1^4/v^2\right][0.012453239d_1\]
\[-\left[U_\infty^2d_1^4gB(T_0 - T_\infty)/v^2\right]d_1[0.012644300d_1 - 0.012789801d_1^3/d_2\]
\[+0.005036019d_1^4/d_2^3 - 0.001776427d_1^4/d_2^3 - \left[v_0 U_\infty gB\right]\]
\[(T_0 - T_\infty)/v^2\]d_1[0.083333333d_1 + 0.071690476d_2^2/d_2 - 0.027222222\]
\[d_1^4/d_2^3 + 0.009365079d_1^4/d_2^3]\right]dx = 0.\] (3.21)
In the above equations $d_1$ and $d_2$ are the independent parameters which are to be calculated and the Euler-Lagrange equations corresponding to this variational principle are

$$\frac{\partial L}{\partial d_1} - \frac{d}{dx}\left(\frac{\partial L}{\partial d_1'}\right) = 0, \quad (3.22)$$
and $$\frac{\partial L}{\partial d_2} - \frac{d}{dx}\left(\frac{\partial L}{\partial d_2'}\right) = 0. \quad (Pr \geq 1) \quad (3.23)$$

Equations (3.22) and (3.23) are non-linear second order ordinary differential equations interms of $d_1$ and $d_2$ whose coefficients are functions of $Re$, $Pr$, $Gr$, $K$, $\epsilon$, $n$ and $H$.

Where $Re = \frac{U_\infty x}{\nu}$ (Reynolds number),

$Pr = \frac{\mu C_p}{K}$ (Prandt number),

$Gr = \frac{g B (T_0 - T_\infty) x^3}{\nu^2}$ (Grashof number),

$K = \frac{Gr}{(Re^2)}$ (Buoyancy parameter),

$\epsilon = \frac{g B x}{C_p}$ (Local dissipation number of the fluid),

$n = \text{wall temperature exponent},$

and $H = \text{suction/injection parameter}.$

Although the Equations (3.22) and (3.23) can be solved directly by using a numerical method, the solution is obtained for the considered problem by employing the following transformations

$$d_{1,2} = d_{1,2}' \sqrt{\nu x / U_\infty} \quad (3.24)$$

in the variational principle (3.20). Thus the Euler-Lagrange equations of the transformed principle assume the simple forms as

$$\frac{\partial L}{\partial d_{1,2}'} = 0. \quad (3.25)$$
Equations (3.25) are obtained as coupled polynomial equations in terms of non-dimensional boundary layer thicknesses \( d_1^* \) and \( d_2^* \) and the coefficients of these equations depend on the independent parameter \( Pr \), buoyancy parameter \( K \), viscous dissipation effect \( \epsilon \), wall temperature exponent \( n \) and \( H \), where \( H \) is the non-dimensional suction/injection speed and is given by

\[
H = \frac{v_0}{v} \sqrt{\nu_x/U_\infty}. \tag{3.26}
\]

Since suction and injection are represented by \( H < 0 \) and \( H > 0 \) respectively, Equations (3.25) can be solved for any value of \( Pr \), \( H \), \( K \), \( \epsilon \) and \( n \).

### 3.4 ANALYSIS OF RESULTS

After determining simultaneous solutions of \( d_1^* \) and \( d_2^* \) for the given combinations of \( Pr \), \( n \), \( H \) and \( K \) for small values of \( \epsilon \), the skin friction (shear stress) and local heat transfer (Nusselt number) are computed with the help of the following expressions respectively.

\[
\tau_{\omega} = \sqrt{\nu_x/U_\infty^3} \left(-P_{12}/L_s\right)_{y=0}, \tag{3.27}
\]

\[
Nu_t = \sqrt{\nu_x/U_\infty} (T_0 - T_\infty)^2 \left(-J_{q}/L_\lambda\right)_{y=0}. \tag{3.28}
\]

To study the effects of viscous dissipation on these characteristics of the problem, the most important cases of the problem are selected as (a) isothermal surface \((n = 0)\), (b) constant surface heat flux \((n = 1)\).

The system of governing Equations (3.1)-(3.3) was numerically calculated for various values of Prandtl number \((Pr)\), wall temperature exponent \((n)\), non-dimensional suction/injection speed \((H)\), buoyancy parameter \((K)\).
and for small viscous dissipation values of $\epsilon$.

In order to establish the accuracy and validity of our present technique, the results are compared with known solutions, which yield remarkable accuracy. Table 3.1 displays the heat transfer values for various $K$ when $Pr=7$, $H=0$, $\epsilon=0$ and $n=0$, and the obtained results are compared with the solutions of Saied (2005) and with those of Aydin (2007), and the comparison shows that the present results are well comparable with known solution. From this table it is evident that, the increase of $Pr$ results in the increase of heat transfer due to the decreasing thermal boundary layer thickness.

**Table 3.1 Comparison of heat transfer values with Saied (2005) and Aydin (2007) for various $K$ when $Pr = 7$, $H = 0$, $\epsilon = 0$ and $n = 0$**

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<tr>
<td>0</td>
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<td>0.625</td>
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<td>0.2</td>
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<td>0.851</td>
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</table>
The heat transfer values for various $K$ when $Pr=10, 100$, $H=0$, $\epsilon=0$ and $n=0$ are exhibited in Table 3.2 and the obtained results are compared with numerical solutions of Lloyd and Sparrow (1970). This table states that, the heat transfer values increase with increasing values of $K$. The comparison evidently reveals that the present results with the existing results is in good agreement.

Table 3.2 Comparison of heat transfer values with Lloyd and Sparrow (1970) for various $K$

when $Pr = 10, 100$, $H = 0$, $\epsilon = 0$ and $n = 0$

<table>
<thead>
<tr>
<th>$K$</th>
<th>Present result of Lloyd and Sparrow (1970)</th>
<th>Present result</th>
<th>Result of Lloyd and Sparrow (1970)</th>
</tr>
</thead>
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<td>$Pr = 10$</td>
<td>$Pr = 100$</td>
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</tr>
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<td>0.4</td>
<td>0.85780642</td>
<td>0.8259</td>
<td>1.72457351</td>
</tr>
<tr>
<td>1</td>
<td>0.91682678</td>
<td>0.9212</td>
<td>1.98095485</td>
</tr>
<tr>
<td>2</td>
<td>1.05815214</td>
<td>1.029</td>
<td>2.2003943</td>
</tr>
<tr>
<td>4</td>
<td>1.21045717</td>
<td>1.173</td>
<td>2.4669225</td>
</tr>
</tbody>
</table>

Table 3.3 represents the skin friction values which are obtained by the present thermodynamic technique for various values of $K$ when $Pr=7$, $\epsilon=0$, $n=0$ and $H=0$. From this table, one can also note that, the skin friction values are increasing with the values of $K$. 
Table 3.3  Comparison of skin friction values

with Saeid (2005) for various $K$

when $Pr = 7$, $\epsilon = 0$ and $n = 0$

<table>
<thead>
<tr>
<th>$K$</th>
<th>Present result $H = 0$</th>
<th>Result of Saeid (2005) $H = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3356</td>
<td>0.3320</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4756</td>
<td>0.4686</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6023</td>
<td>0.5938</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6905</td>
<td>0.6875</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8225</td>
<td>0.8125</td>
</tr>
<tr>
<td>1</td>
<td>0.9435</td>
<td>0.9376</td>
</tr>
</tbody>
</table>

Figures 3.1 and 3.2 exhibit the values of heat transfer with $K$ for various Prandtl numbers when $H=0$, $\epsilon = -0.1$, 0, 0.1 and $n=0$. From these two figures it is observed that the heat transfer values increase with the increasing values of buoyancy parameter $K$ irrespective of Prandtl numbers. Figures 3.3-3.12 represent the heat transfer values with $K$ for various Prandtl number and for different values of $H$, $\epsilon$ and $n$. From these figures, it is also noted that the same trend remains prevail for all cases.
Figure 3.1 Variation of local heat transfer with $K$ for various values of $Pr$ when $H = 0$, $\varepsilon = 0$ and $n = 0$
Figure 3.2 Variation of local heat transfer with $K$ for various values of $Pr$
when $H = 0$, $\varepsilon = 0.1$, -0.1 and $n = 0$
Figure 3.3  Variation of local heat transfer with $K$ for various values of $Pr$
when $H = 0$, $\varepsilon = 0$ and $n = 1
Figure 3.4  Variation of local heat transfer
with $K$ for various values of $Pr$
when $H = 0$, $\varepsilon = 0.1$, -0.1 and $n = 1$
Figure 3.5  Variation of local heat transfer
with $K$ for various values of $Pr$
when $H = 1$, $\varepsilon = 0$ and $n = 0$
Figure 3.6  Variation of local heat transfer with $K$ for various values of $Pr$ when $H = 1$, $\varepsilon = 0.1$, -0.1 and $n = 0$
Figure 3.7  Variation of local heat transfer with $K$ for various values of $Pr$
when $H = -1$, $\varepsilon = 0$ and $n = 0$
Figure 3.8 Variation of local heat transfer with $K$ for various values of $Pr$ when $H = -1$, $\varepsilon = 0.1$, -0.1 and $n = 0$
Figure 3.9  Variation of local heat transfer with $K$ for various values of $Pr$
when $H = 1$, $\varepsilon = 0$ and $n = 1$
Figure 3.10  Variation of local heat transfer with $K$ for various values of $Pr$
when $H = 1$, $\varepsilon = 0.1$, -0.1 and $n = 1$
Figure 3.11  Variation of local heat transfer with $K$ for various values of $Pr$ when $H = -1$, $\varepsilon = 0$ and $n = 1$
Figure 3.12 Variation of local heat transfer with $K$ for various values of $Pr$ when $H = -1$, $\varepsilon = 0.1$, $-0.1$ and $n = 1$
3.5 CONCLUSION

An analytical solution to flow and heat transfer in free and forced convection with uniform suction and injection over a non-isothermal vertical plate is obtained. The governing partial differential equations are reduced to simple polynomial equations, the coefficients of which are functions of independent parameters $Pr$, $H$, $n$, $K$ and $\epsilon$. These equations are used as, a rapid way of obtaining heat transfer at the isothermal wall surface ($n = 0$) and constant surface heat flux ($n = 1$) for any values of $Pr$, $H$, $K$ and $\epsilon$. The great advantage involved in the present technique is that the results are obtained with remarkable accuracy and the amount of calculation is considerably less when compared with more conventional exact methods.