Chapter 2

KAZHDAN-LUSZTIG CELLS AND THEIR COMBINATORICS

In this chapter we begin with some background material on Coxeter groups and Hecke algebras associated to them. We soon specialize to $\mathcal{S}_n$ and after introducing the basic combinatorial objects there, we describe the RSK-correspondence in §2.3.2. In §2.3.3 we recall certain combinatorial results which play a significant role in the rest of the chapters and finally in §2.3.4 we introduce some notations and discuss a few preliminary results that will be used repeatedly later.

All the results mentioned in §§2.1.2-2.2 can be found in [Lus93] or [Hum90]. In §2.3 we gather together various preliminaries which deal specifically with the combinatorial aspects of $\mathcal{S}_n$ that play a key role in the thesis. The concepts discussed in §§2.3.1, 2.3.2 are covered in greater detail in [Sag01]. The last sub-section is fairly self-contained, while the results mentioned in §2.3.3 forms the main theme of [Gec06] and we do not undertake the task of going through their proofs here.

2.1 Coxeter System $(W, S)$

Let $W$ denote a group (written multiplicatively), with identity element denoted as 1, and $S$ be a set of generators for $W$. Then the pair $(W, S)$ is a Coxeter system if there exists a matrix $(m(s, s'))_{(s, s') \in S \times S}$ with entries in $\mathbb{N} \cup \{\infty\}$ satisfying the conditions:

i) $m(s, s) = 1$ for all $s \in S$ and

ii) $m(s, s') = m(s', s) \geq 2$ for all $s \neq s'$

and such that the natural map from the free group generated by the set $S$ to $W$ has kernel the normal subgroup generated precisely by the elements

$$(ss')^{m(s, s')}$$

where $s, s' \in S$ and $m(s, s') < \infty$. 

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We assume familiarity with the basic notions regarding Coxeter systems. However, we briefly recall here the notions that we will use. (See [Bou94, Chapter 4] for more details).

Each $w \in W$ can be written as a product of a finite sequence of elements of $S$. For a given $w$, the smallest possible integer $r \geq 0$ such that $w$ is a product of a sequence of $r$ elements from $S$ is called the length of $w$, denoted as $l(w)$. Thus, $l(1) = 0$ and $l(s) = 1$ for all $s \in S$ (it is easily seen that $1 \notin S$). Let $w = s_1 \cdots s_r$ for some $s_i$ (not necessarily distinct) in $S$. Then $s_1 \cdots s_r$ is said to be a reduced expression for $w$ if $r = l(w)$. A subexpression of a given reduced expression $s_1 \cdots s_r$ is a product of the form $s_{i_1} \cdots s_{i_t}$ where $1 \leq i_1 < \cdots < i_t \leq r$. An element $u \in W$ is a prefix of $w \in W$ if there is a reduced expression $s_1 \cdots s_r$ for $w$ such that the subexpression $s_1 \cdots s_j$ for some $j \leq r$ gives a reduced expression for $u$.

**Bruhat order:** For $w$, $w' \in W$, we write $w \leq w'$ if $w$ can be obtained as a subexpression of some reduced expression for $w'$. This defines a partial ordering on $W$ called the Bruhat order. We sometimes also write $w' \geq w$ to mean $w \leq w'$. By $w < w'$ or $w' > w$, we mean $w \leq w'$ and $w \neq w'$.

**Deletion condition:** ([Hum90, §5.8]) Suppose $w = s_1 \cdots s_r$, ($s_i \in S$), with $l(w) < r$. Then there exist indices $i < j$ for which $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$, where $\hat{s}_k$ means that $s_k$ is omitted.

**Remark 2.1.1**

(a) For a given Coxeter system $(W, S)$ where $W$ is finite, there is a unique element in $W$ which is of maximal length. (See [Hum90, §9.8], for example).

(b) For $J \subset S$, let $W_J$ denote the subgroup of $W$ generated by $J$. Then $(W_J, J)$ is a Coxeter system in its own right. Further, if we denote the length function on $W_J$ as $l_J$ then $l_J(w) = l(w)$ for all $w \in W_J$. The subgroup $W_J$ is called a parabolic subgroup of $W$ (with respect to $S$). (See [Hum90, §5.5], for example).

### 2.2 Hecke Algebra corresponding to $(W, S)$

Let $R$ be a commutative ring with unity. We begin with the definition of a generic Iwahori-Hecke algebra associated to a Coxeter system. By the Hecke algebra, we shall mean a particular case of a generic Iwahori-Hecke algebra.

**Definition 2.2.1** Let $(W, S)$ be a Coxeter system. Let $a_s, b_s \in R$ ($s \in S$) be such that $a_s = a_t$ and $b_s = b_t$ whenever $s$, $t$ are conjugate under $W$. Then the generic Iwahori-Hecke algebra associated with $(W, S)$ over $R$ with parameters $\{a_s, b_s \mid s \in S\}$ is the free $R$-module $E$ with basis $\{T_x \mid x \in W\}$ and multiplication given by

$$T_s T_w = \begin{cases} 
T_{sw} & \text{if } l(sw) = l(w) + 1 \\
 a_s T_w + b_s T_{sw} & \text{if } l(sw) = l(w) - 1
\end{cases} \quad (2.1)$$
for \( s \in S, w \in W \), making it an associative algebra with \( T_1 \) as identity.

That an algebra structure on \( \mathcal{E} \) as in the above definition, exists and that it is unique is guaranteed by the following result:

**Theorem 2.2.2** ([Hum90, p.146]) Let \((W, S)\) be a Coxeter system. Given \( a_s, b_s \in R \) \((s \in S)\) satisfying the conditions as in the above definition, there exists a unique structure of an associative algebra on the free \( R \)-module \( \mathcal{E} \) with basis \( \{T_x \mid x \in W\} \) such that \( T_1 \) acts as identity and the conditions as in (2.1) are satisfied.

The group algebra of \( W \over R \) is an example of a generic Iwahori-Hecke algebra, where the parameters are chosen to be \( a_s = 0, b_s = 1 \) for all \( s \in S \).

To obtain the Hecke algebra associated to \((W, S)\), we take in Definition 2.2.1 the ring \( R \) to be \( \mathbb{Z}[v, v^{-1}] \), the ring of Laurent polynomials with coefficients in \( \mathbb{Z} \), and the parameters to be \( a_s = v - v^{-1}, b_s = 1 \) for all \( s \in S \).

Written explicitly, we have

**Definition 2.2.3** The Hecke algebra associated to \((W, S)\), denoted as \( \mathcal{H} \), is a free \( \mathbb{Z}[v, v^{-1}] \)-module with basis \( T_w, w \in W \), and multiplication being given by

\[
T_s T_w = \begin{cases} 
T_{sw} & \text{if } l(sw) = l(w) + 1 \\
(v - v^{-1})T_w + T_{sw} & \text{if } l(sw) = l(w) - 1
\end{cases}
\tag{2.2}
\]

for \( s \in S, w \in W \).

The Hecke algebra associated with the Coxeter system \((W, S)\) is a “deformation” of the group algebra of \( W \) over \( \mathbb{Z} \); taking \( v = 1 \) in Definition 2.2.3 we recover the group algebra of \( W \) over \( \mathbb{Z} \). (See also Specializations of the Hecke algebra below).

In the next lemma we summarize a few basic facts about \( \mathcal{H} \).

**Lemma 2.2.4**

1. Let \( s_1 \cdots s_r \) be a reduced expression for \( w \in W \). Then \( T_w = T_{s_1} \cdots T_{s_r} \).

2. Let \( s \in S \) and \( w \in W \). Then,

\[
T_w T_s = \begin{cases} 
T_{ws} & \text{if } l(ws) = l(w) + 1 \\
(v - v^{-1})T_w + T_{ws} & \text{if } l(ws) = l(w) - 1
\end{cases}
\tag{2.3}
\]

3. For \( w \in W \), the element \( T_w \) is invertible in \( \mathcal{H} \) with inverse \( T_w^{-1} = T_{s_1}^{-1} \cdots T_{s_k}^{-1} \) where \( s_1 \cdots s_k \) is a reduced expression for \( w \). For \( s \in S \), the element \( T_s^{-1} = T_s - (v - v^{-1})T_1 \).

4. In the case \(^1\) when \( W = S_n \) and \( S \) is the set of simple transpositions \((2.3)\) we have for \( x, y \in W \), let \( T_x T_y = \sum_{w \in W} r_w T_w, r_w \in \mathbb{Z}[v, v^{-1}] \). Then \( r_{xy} = 1 \); and \( r_w \neq 0 \) only if \( xy \leq w \). In particular, \( r_1 \neq 0 \) iff \( x = y^{-1} \).

\(^1\) It can be verified, in the light of Remark 2.2.3(b), that if \( W \) is a parabolic subgroup of \( S_n \) then the statement still holds. In fact the statement can be proved more generally for any Coxeter system (see for example, [Hum90, Exercise 1.13]).
PROOF: ([DJ86] Lemma 2.1; note the Remark on notation below)

1. This is elementary using the relations in (2.2).

2. This is proved easily by inducting on $l(w)$. When $l(w) \leq 1$ the relations in (2.3) are just those in (2.2). Also, by (1) we know that if $w$ has a reduced expression of the form $s_1 \cdots s_r$, then $T_{s_1} \cdots T_{s_r} = T_{s_1} \cdots T_{s_r}$. So, if $ws > w$ for $s \in S$, we have the required relation by (1). In the case when $ws < w$, since $l(ws) < l(w)$, we apply induction hypothesis to $ws$ to get $T_{ws}T_s = T_w$. Now, multiplying both sides of this equation by $T_s$ and using the expression for $T_s^2$, we obtain the required relation even in this case.

3. By (2.2), we have $T_s^2 = (v - v^{-1})T_s + T_1$ which immediately gives the second part of (3). The first part now follows immediately from (1).

4. A routine inductive argument on $l(x)$ along with relations in (2.2) and (2) above proves (4). (The description of the set $S$ as the simple transpositions in $S_n$ is used; See also [Mat99] §1.16))

Henceforth, by $A$ we mean the ring $\mathbb{Z}[v, v^{-1}]$ and by $\mathcal{H}$ the Hecke algebra associated to $(W, S)$ as in Definition 2.2.3.

Notation 2.2.5 We repeatedly use the following short-hand notation:
- $\epsilon_w := (-1)^{l(w)}$
- $v_w := v^{l(w)}$

Remark on notation: The notation that we use is as in [Lus83] (also in [Gec03]), while in [DJ86], [KL79], [MP03] (also [Hum90], [Sha86]) the notation used is slightly different. To pass from our notation to that of [DJ86], [KL79] or [MP03], we need to replace $v$ by $q^{1/2}$ and $T_w$ by $q^{-l(w)/2}T_w$.

Specializations of the Hecke algebra

Let $k$ be a commutative ring with unity and $a$ an invertible element in $k$. There is a unique ring homomorphism $A \to k$ defined by $v \mapsto a$. We denote by $\mathcal{H}_k$ the $k$-algebra $\mathcal{H} \otimes_A k$ obtained by extending the scalars to $k$ via this homomorphism. We have a natural $A$-algebra homomorphism $\mathcal{H} \to \mathcal{H}_k$ given by $h \mapsto h \otimes 1$. By abuse of notation, we continue to use the same symbols for the images in $\mathcal{H}_k$ of the elements of $\mathcal{H}$. If $M$ is a (right) $\mathcal{H}$-module, $M \otimes_A k$ is naturally a (right) $\mathcal{H}_k$-module.

An important special case is when we take $a$ to be the unit element $1$ of $k$. We then have a natural identification of $\mathcal{H}_k$ with the group ring $kW$, under which $T_w$ maps to the element $w$ in $kW$.

Convention: When the value of $a$ is not specified, by the “specialization of $\mathcal{H}$ to $k$”, we mean the algebra $\mathcal{H}_k$ defined via the map $v \mapsto 1$. 

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An involution on $\mathcal{H}$

We introduce an involution (order 2 ring automorphism) on $\mathcal{H}$ defined as follows:

$$\sum a_w T_w := \sum a_w T_{w^{-1}}$$

(2.4)

where $a \mapsto \overline{a}$ on $A$ is defined by $v \mapsto \overline{v} := v^{-1}$ extended $\mathbb{Z}$-linearly to give an involution on $A$. This is called the bar involution.

### 2.2.1 Kazhdan - Lusztig bases of $\mathcal{H}$

Apart from the basis $T_w, w \in W$, there is another $A$-basis of $\mathcal{H}$ which is of interest to us. It is determined uniquely by the conditions:

$$\overline{C_w} = C_w \quad \text{and} \quad C_w \equiv T_w \mod \mathcal{H}_{>0} \quad (\dagger)$$

where $\mathcal{H}_{>0} := \sum_{w \in W} A_{>0} T_w, A_{>0} := v\mathbb{Z}[v]$. This is called the $C$-basis of $\mathcal{H}$.

The existence and uniqueness of a basis as above follows from:

**Theorem 2.2.6** [KL79, Theorem 1.1] For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}$ having the following properties:

1. $\overline{C_w} = C_w$,

2. $C_w = \sum_{y \leq w} \epsilon_y \epsilon_w p_{y,w} T_y$ where $p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $y < w$ and $p_{w,w} = 1$.

The existence of these elements is proved by constructing $C_w$ by induction on the length of $w$. Set $C_1 := T_1$. For $w = su$ such that $l(w) = l(u) + 1$ define

$$C_w := C_s C_u - \sum_{z:sz < z < u} \nu(z, u) C_z$$

(2.5)

where $\nu(z, u)$ is the coefficient of $v^{-1}$ in $p_{z,u}$ and $C_s := T_s - vT_1$. One then verifies that the properties (1) and (2) hold. That these elements form a basis follows then from property (2) above.

**Remark 2.2.7** As is verified easily, the involutive anti-automorphism of the algebra $\mathcal{H}$ given by $h \mapsto h^*$, where $(\sum a_w T_w)^* := \sum a_w T_{w^{-1}}$, commutes with the bar involution on $\mathcal{H}$. Therefore,

i) it follows from the defining conditions $(\dagger)$ that $C_w^* = C_{w^{-1}}$.

ii) applying $*$ to the relation $C_w = T_w + \sum_{y \leq w} \epsilon_y \epsilon_w p_{y,w} T_y$ (Theorem 2.2.6(2) above), we get $C_{w^{-1}} = T_{w^{-1}} + \sum_{y \leq w} \epsilon_y \epsilon_w p_{y,w} T_{y^{-1}}$ (using (i)). Thus by the uniqueness condition in Theorem 2.2.6 we note that $p_{y,w} = p_{y^{-1},w^{-1}}$. 

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Using the above remark it can be deduced, by applying * to \textbf{(2.5)}, that for \( w = us \) such that \( l(w) = l(u) + 1 \) we have
\[
C_w = C_u C_s - \sum_{z : z < u} \mu(z, u) C_z
\] (2.6)
where \( \mu(z, u) := \nu(z^{-1}, u^{-1}) \).

From the above equation we obtain the following basic result.

**Lemma 2.2.8** [KL79] § 2.3\] *Let \( s \in S \) and \( ws < w \). Then \( C_w T_s = -v^{-1} C_w \).*

**Proof:** We induct on \( l(w) \). If \( w = s \) then inserting the expression \( C_s = T_s - vT_1 \) in \( C_s T_s \) and simplifying it using \textbf{(2.3)}, we get \( C_s T_s = -v^{-1} C_s \). Under induction hypothesis assume the statement for \( z \) such that \( l(z) < l(u) \) and \( zs < z \). Replacing \( u \) by \( ws \) in \textbf{(2.6)} and using the expression thus obtained we get
\[
C_w T_s = C_w (C_s T_s) - \sum_{zs < z < ws} \mu(z, ws)(C_z T_s)
\]
Applying the induction hypothesis, we then have
\[
C_w T_s = -v^{-1} C_w + \sum_{zs < z < ws} v^{-1} \mu(z, ws) C_z
\]
as required. \( \square \)

The above observation leads us to some useful properties of the polynomials \( p_{x,w} \) which appear in statement \( (2) \) of Theorem 2.2.6.

**Corollary 2.2.9** [KL79] §2.3\] *Let \( s \in S \) and \( x, w \in W \).

1. If \( x < w \), \( ws < w \) and \( xs > x \) then \( p_{x,w} = v^{-1} p_{xs,w} \).

2. Let \( w_0 \) be the longest element of \( W \). Then we have \( p_{x,w_0} = v^{l(x) - l(w_0)} \) for all \( x \in W \).

**Proof:** By Lemma 2.2.8 we have the relation \( C_w T_s = -v^{-1} C_w \) whenever \( ws < w \). Inserting into it the expression for \( C_w \) given by Theorem 2.2.6 \textbf{(2)} and then comparing the coefficient of \( T_{xs} \) on both sides of the relation thus obtained, we deduce that \( p_{x,w} + (v - v^{-1})p_{xs,w} = vp_{xs,w} \) which readily yields \( (1) \).

For \( x \in W \), we can find a finite sequence of elements \( s_1, \ldots, s_r \in S \) such that \( x < xs_1 < \ldots < xs_1 \cdots s_r = w_0 \). Since the longest element \( w_0 \) satisfies the condition \( w_0 s < w_0 \) for all \( s \in S \), we apply \( (1) \) repeatedly to get \( p_{x,w_0} = v^{-1} p_{xs_1,w_0} = \cdots = (v^{-1})^r p_{xs_1 \cdots s_r,w_0} = (v^{-1})^r \) (by Theorem 2.2.6 \textbf{(2)}, \( p_{w_0,w_0} = 1 \)). Note that \( r = l(w_0) - l(x) \), proving \( (2) \). \( \square \)

Kazhdan-Lusztig \( C' \)-basis

Consider the ring involution \( j : \mathcal{H} \to \mathcal{H} \), defined by
\[
\bar{f}(\sum a_u T_u) := \sum c_w \overline{a_w T_w}
\]

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where \( \epsilon_w := (-1)^{l(w)} \), and the element \( \overline{a} \in A \) is as defined in (2.4). Then define,

\[
C'_w := \epsilon_w j(C_w)
\]

(2.7)

Since \( j \) is an involution on \( \mathcal{H} \), the elements \( C'_w, w \in W \), also form a basis for \( \mathcal{H} \). It is called the \( C' \)-basis. By Theorem 2.2.6 and the fact that the bar involution commutes with the involution \( j \) defined above, it is clear that the elements \( C'_w, w \in W \), are also determined uniquely by the two properties

\[
\overline{C'_w} = C'_w \quad \text{and} \quad C'_w \equiv T_w \mod \mathcal{H}_{<0}
\]

All the other properties of the \( C \)-basis can also be carried over to the \( C' \)-basis via the involution \( j \). An instance of this, which we shall use later, is the following:

**Lemma 2.2.10** Let \( s \in S \) and \( ws < w \). Then \( C'_wT_s = vC'_w \).

**Proof:** Applying the involution to both sides of the relation \( C_wT_s = -v^{-1}C_w \), we get the required relation for \( C'_w \).

Before ending this subsection, in the light of Corollary 2.2.7(2) and Theorem 2.2.6(2), we note that if \( w_0 \) denotes the longest element in \( W \), then

\[
C_{w_0} = \epsilon_{w_0} v_{w_0} \sum_{w \in W} \epsilon_w v_w^{-1}T_w, \quad C'_{w_0} = v_{w_0}^{-1} \sum_{w \in W} v_w T_w
\]

(2.8)

The above expressions together with Lemma 2.2.8 and Lemma 2.2.10 gives

\[
C^2_{w_0} = \left( \epsilon_{w_0} v_{w_0}^{-1} \sum_{w \in W} v_{w_0}^2 \right) C_{w_0}
\]

(2.9)

\[
C'^2_{w_0} = \left( v_{w_0} \sum_{w \in W} v_w^{-2} \right) C'_{w_0}
\]

(2.10)

\[\]

**2.2.2 Kazhdan-Lusztig orders and cells**

The central goal of introducing the Kazhdan-Lusztig bases is to understand the representations of the Hecke algebra \( \mathcal{H} \). The advantage of the \( C \)-basis (analogously \( C' \)-basis) is that it leads to a systematic construction of certain representations. This is done by defining a pre-order on \( W \) the equivalence classes of which gives a partition of \( W \) into cells (left, right, two-sided). Later in §3.3.5 we construct representations of \( W \) associated to these cells. We now give the definition of these cells of \( W \).

Let \( y \) and \( w \) be elements in \( W \). Write \( y \leftarrow w \) if, for some element \( s \) in \( S \), the coefficient of \( C_y \) is non-zero in the expression of \( C_y C_w \) as an \( A \)-linear combination of the basis elements \( C_x \). Replacing all occurrences of \( 'C' \) by \( 'C'' \) in this definition would make no difference. The Kazhdan-Lusztig left pre-order is defined by: \( y \leq_L w \) if there exists a chain \( y = y_0 \leftarrow_L \cdots \leftarrow_L y_k = w \); the left equivalence relation by: \( y \sim_L w \) if \( y \leq_L w \) and
\( w \leq_L y \). Left equivalence classes are called left cells. Note that by the above definition the \( A \)-module \( \sum_{x \leq_L w} AC_x \) is a left ideal of \( \mathcal{H} \) containing the left ideal \( \mathcal{H}C_w \).

Right pre-order, equivalence, and cells are defined similarly. The two sided pre-order is defined by: \( y \leq_{LR} w \) if there exists a chain \( y = y_0, \ldots, y_k = w \) such that, for \( 0 \leq j < k \), either \( y_j \leq_L y_{j+1} \) or \( y_j \leq_R y_{j+1} \). Two sided equivalence classes are called two sided cells.

**Remark 2.2.11** Since \( C \) is defined as \( T_a - vT_1 \), it may be easily seen that, if for \( y, w \in W \) we set \( y \leftarrow_L w \) whenever there exists an element \( s \in S \) such that the coefficient of \( C_y \) is non-zero in the expression of \( T_aC_w \) as a \( A \)-linear combination of the \( C \)-basis elements, then the pre-order \( \leq_L \) obtained from the relation \( \leftarrow_L \) is the same as the left pre-order defined above (using the \( C \)-basis instead of \( T \)).

**Lemma 2.2.12** Let \( w_0 \) be the longest element in \( W \). Then for elements \( w, w' \in W \) we have, \( w \leq_L w' \) if and only if \( w_0w' \leq_L w_0w \). Similarly for \( \leq_R \). Moreover, \( \nu(w, w') = \nu(w_0w', w_0w) \). (See [KL70] Corollary 3.2 for proof; [Sh90] Lemma 1.4.6(ii))

**Remark 2.2.13** By Remark 2.2.1(ii), we have \( p_{y,w} = p_{y^{-1},w^{-1}} \) and hence \( \nu(y, w) = \nu(y^{-1}, w^{-1}) \). By definition, the latter term is \( \mu(y, w) \). Thus, by Lemma 2.2.12 we conclude that \( \mu(w, w') = \mu(w_0w', w_0w) \) for \( w, w' \in W \).

For any \( y \in W \), we associate to it two sets defined as follows:

\[
\mathcal{R}(y) := \{ s \in S | ys < y \} \quad \mathcal{L}(y) := \{ s \in S | sy < y \}
\]

Then we have,

**Lemma 2.2.14** Let \( w, w' \in W \).

1. If \( w \leq_R w' \) then \( \mathcal{L}(w') \subset \mathcal{L}(w) \).

2. If \( w \leq_L w' \) then \( \mathcal{R}(w') \subset \mathcal{R}(w) \).

(See [Lus03] Lemma 8.6] for a proof)

### 2.3 Combinatorics of cells in \( \mathfrak{S}_n \)

From now on, we fix \( A \) to be the Laurent polynomial ring in one indeterminate \( \mathbb{Z}[v, v^{-1}] \). Let \( n \) be a fixed integer and \( \mathfrak{S}_n \) the symmetric group on \( n \) letters. Let \( S \) denote the subset consisting of simple transpositions \( (1, 2), (2, 3) \ldots (n - 1, n) \). Then \( (\mathfrak{S}_n, S) \) is a Coxeter system and its Hecke algebra defined over \( A \) is denoted as \( \mathcal{H} \).

In \( \mathfrak{S}_n \), there are certain combinatorial descriptions for cells as defined in the earlier section. Before beginning with this description we introduce some basic definitions.
2.3.1 Basic notions

Partitions and shapes

By a partition \( \lambda \) of \( n \), written \( \lambda \vdash n \), is meant a sequence \( \lambda_1 \geq \ldots \geq \lambda_r \) of positive integers such that \( \lambda_1 + \ldots + \lambda_r = n \). The integer \( r \) is the number of parts in \( \lambda \). We often write \( \lambda = (\lambda_1, \ldots, \lambda_r) \); sometimes even \( \lambda = (\lambda_1, \lambda_2, \ldots) \). When the latter notation is used, it is to be understood that \( \lambda_t = 0 \) for \( t > r \).

Partitions of \( n \) are in bijection with shapes of Young diagrams (or simply shapes) with \( n \) boxes: the partition \( \lambda_1 \geq \ldots \geq \lambda_r \) corresponds to the shape with \( \lambda_1 \) boxes in the first row, \( \lambda_2 \) in the second row, and so on, the boxes being arranged left- and top-justified. This diagram of boxes is sometimes also called the Young diagram of shape \( \lambda \) denoted as \( \lambda \). Here for example is the shape corresponding to the partition \((3,3,2)\) of 8:

\[
\begin{array}{ccc}
\cdot & \cdot & \\
\cdot & & \\
\cdot & \cdot & \\
\cdot & & \\
\end{array}
\]

Partitions are thus identified with shapes and the two terms are used interchangeably.

Dominance order on partitions

Given partitions \( \mu = (\mu_1, \mu_2, \ldots) \) and \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of \( n \), we say \( \mu \) dominates \( \lambda \), and write \( \mu \succeq \lambda \), if

\[
\mu_1 \geq \lambda_1, \quad \mu_1 + \mu_2 \geq \lambda_1 + \lambda_2, \quad \mu_1 + \mu_2 + \mu_3 \geq \lambda_1 + \lambda_2 + \lambda_3, \quad \ldots
\]

We write \( \mu \succ \lambda \) if \( \mu \succeq \lambda \) and \( \mu \neq \lambda \). The partial order \( \succeq \) on the set of partitions (or shapes) of \( n \) will be referred to as the dominance order.

Tableaux and standard tableaux

A Young tableau, or just tableau, of shape \( \lambda \vdash n \) is an arrangement of the numbers 1, \ldots, \( n \) in the boxes of shape \( \lambda \). There are, evidently, \( n! \) tableaux of shape \( \lambda \). A tableau is row standard (respectively, column standard) if in every row (respectively, column) the entries are increasing left to right (respectively, top to bottom). A tableau is standard if it is both row standard and column standard. An example of a standard tableau of shape \((3,3,2)\):

\[
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 6 & 8 \\
4 & 7 & \\
\end{array}
\]

The number of standard tableaux: The number of standard tableaux of a given shape \( \lambda \vdash n \) is denoted \( d(\lambda) \). There is a well-known ‘hook length formula’ for it (see
\[ \text{Sag01 p.124; FR154]: } d(\lambda) = n! / \prod_\beta h_\beta, \text{ where } \beta \text{ runs over all boxes of shape } \lambda \text{ and } h_\beta \text{ is the hook length of the box } \beta \text{ which is defined as one more than the sum of the number of boxes to the right of } \beta \text{ and the number of boxes below } \beta. \]

The hook lengths for the shape (3, 3, 2) are shown below:

\[
\begin{array}{ccc}
5 & 4 & 2 \\
4 & 3 & 1 \\
2 & 1 & \\
\end{array}
\]

Thus \( d(3, 3, 2) = 8! / (5.4.2.4.3.1.2.1) = 42. \)

Row and Column stabilizers:

Given a tableau \( T \) of shape \( \lambda \vdash n \), we obtain two collections of subsets of the set \( \{1, 2, \ldots, n\} \):

\( \mathbf{R} \) : Each subset consists of numbers appearing along each row of \( T \).

\( \mathbf{C} \) : Each subset consists of numbers appearing along each column of \( T \).

The row stabilizer of \( T \) is the subgroup of \( \mathfrak{S}_n \) which leaves each subset in the collection \( \mathbf{R} \) invariant. It is the set of permutations in \( \mathfrak{S}_n \) that permute the numbers appearing in each row of \( T \) among themselves. Similarly, the column stabilizer of \( T \) is the subgroup of \( \mathfrak{S}_n \) which leaves each subset in the collection \( \mathbf{C} \), invariant. It is the set of permutations in \( \mathfrak{S}_n \) that permute the numbers in each column of \( T \) among themselves.

For example, let

\[
T = \begin{bmatrix}
1 & 3 \\
2 & 5 \\
4 & \\
\end{bmatrix}
\]

Then the row stabilizer of \( T \) is the subgroup \( \mathfrak{S}_{\{1,3\}} \times \mathfrak{S}_{\{2,5\}} \times \mathfrak{S}_{\{4\}} \) and its column stabilizer is \( \mathfrak{S}_{\{1,2,4\}} \times \mathfrak{S}_{\{3,5\}} \).

2.3.2 RSK-correspondence

We recall, in this subsection, the combinatorial algorithm which goes under the name of Robinson-Schensted-Knuth. It is a well-known procedure that sets up a bijection between the symmetric group \( \mathfrak{S}_n \) and ordered pairs of standard tableaux of the same shape with \( n \) boxes. The aim of the algorithm was to provide a purely combinatorial proof that the number of elements in \( \mathfrak{S}_n \) is equal to the number of pairs of standard tableaux of the same shape \( \lambda \), as \( \lambda \) varies over all partitions of \( n \), i.e.,

\[
\sum_{\lambda \vdash n} d(\lambda)^2 = n!
\]

The genesis of the above relation lies in the representation theory of \( \mathfrak{S}_n \) over \( \mathbb{C} \): the partitions, \( \lambda \vdash n \), parametrize all the non-isomorphic irreducible representations of \( \mathbb{C} \mathfrak{S}_n \);
the number \(d(\lambda)\) of standard tableaux of shape \(\lambda\) is the same as the dimension of the irreducible representation of \(\mathfrak{S}_n\) associated to \(\lambda\) (see [3.2]).

In order to describe the algorithm, which sets up the bijection as mentioned earlier, we will need the insertion algorithm which is described as follows: Let \(P\) be a tableau consisting of an arbitrary set of distinct numbers. If all the numbers from 1 to \(n\) appear then it is a tableau in the sense defined in [2.3]. Let \(x\) be a number not appearing in \(P\). Then the insertion algorithm to insert \(x\) in \(P\), denoted as \(P \leftarrow x\), is given as follows:

**I1** Set \(R\) to be the first row of \(P\).

**I2** While \(x\) is less than some element in \(R\), do

**I2a** Let \(y\) be the smallest number in \(R\) greater than \(x\) and replace \(y\) by \(x\).

**I2b** Set \(x := y\) and \(R\) as the next row down.

**I3** Now \(x\) is greater than every element in \(R\), so place \(x\) at the end of the row and stop.

The Robinson-Schensted-Knuth correspondence (or RSK-correspondence, for short) is a bijection between \(\mathfrak{S}_n\) and pairs of standard tableaux of the same shape with \(n\) boxes. The bijection is given by an algorithm that takes a permutation \(\pi\) and produces from it a pair \((P(\pi), Q(\pi))\) of standard tableaux of a certain shape. This is done as follows: Let \(\pi\) be written in two-line notation as

\[
\pi = \begin{array}{cccc}
1 & 2 & \cdots & n \\
x_1 & x_2 & \cdots & x_n
\end{array}
\]

We construct \(Q(\pi)\) as

\[
((\cdots((x_1 \leftarrow x_2) \leftarrow x_3)\cdots) \leftarrow x_{n-1}) \leftarrow x_n
\]

The tableaux \(P(\pi)\), called the recording tableau, is obtained by simply placing the integer \(k\) in the box that is added at the \(k\)-th step of the construction of \(Q(\pi)\).

**Example 2.3.1** Consider the permutation \((14253) \in \mathfrak{S}_5\), written in two-line notation as

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 5 & 1 & 2 & 3
\end{pmatrix}
\]

Applying the above algorithm, we get the pair

\[
P = \begin{pmatrix}
1 & 2 & 5 \\
3 & 4
\end{pmatrix} \quad Q = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5
\end{pmatrix}
\]

That the map defined by the above algorithm is a bijection is proved by reversing the algorithm step by step to recover the permutation associated to a pair of standard tableaux.
Denote by \((P_k, Q_k)\) the tableaux obtained at the \(k\)-th step. Then, to go from \((P_k, Q_k)\) to \((P_{k-1}, Q_{k-1})\) consider the number appearing in \(Q_k\) in the box which contains the largest number in \(P_k\) and apply the reverse row-insertion algorithm to \(Q_k\) with this number. The element that is bumped out is the image of \(k\) under the resulting permutation. And the tableau obtained by removing the originating box in \(Q_k\) is the tableau \(Q_{k-1}\). Removing the largest number in \(P_k\) we get \(P_{k-1}\).

It should be noted that the algorithm presented above is slightly different from those given in standard texts \([En97, Sag01]\). For reasons that will be explained later (see Remark 2.3.5), we have modified the procedure by associating to a permutation the same pair of standard tableaux as obtained by the standard procedure but with their positions interchanged, \(i.e.,\) if \((A(w), B(w))\) is the pair associated to \(w \in \mathcal{S}_n\) by the algorithm as in \([En97]\) or \([Sag01]\), then by the RSK-correspondence we mean the bijection that associates the pair \((B(w), A(w))\) to the permutation \(w\). In the light of the following result of Schützenberger, the modification amounts to associating the permutation \(w^{-1}\) to the pair \((A(w), B(w))\).

**Theorem 2.3.2** (Schützenberger) (see \([Sag01]\) Theorem 3.6.6) If \(w \in \mathcal{S}_n\) then \(A(w^{-1}) = B(w)\) and \(B(w^{-1}) = A(w)\). \(\square\)

**Notation 2.3.3** We write \((P(w), Q(w))\) for the ordered pair of standard Young tableaux associated to the permutation \(w\) by the RSK-correspondence. Call \(P(w)\) the \(P\)-symbol and \(Q(w)\) the \(Q\)-symbol of \(w\). It will be convenient also to use \((P(w), Q(w))\) for the permutation \(w\), \(C(P(w), Q(w))\) or \(C(P(w), Q(w))\) for the Kazhdan-Lusztig \(C\)-basis element \(C_w\).

**Definition 2.3.4** The RSK shape of a permutation \(w\) is defined to be the shape of the tableau \(P(w)\) (which is the same as that of \(Q(w)\)).

An example

The permutation \((1542)(36)\) (written as a product of disjoint cycles) has RSK-shape \((3, 2, 1)\). Indeed it is mapped under the RSK correspondence in our sense to the ordered pair \((A, B)\) of standard tableaux, where:

\[
A = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
6
\end{array} \quad B = \begin{array}{cc}
1 & 2 & 3 \\
4 & 6 \\
5
\end{array}
\]

**2.3.3 Cells and RSK-correspondence**

We now recall the combinatorial characterizations of left, right and two sided cells in terms of the RSK correspondence and the dominance order on partitions \((2.3.1)\). These statements are the foundation on which most of our arguments rest.
The following statement characterizes the relation $\leq_{LR}$. (Also see comments in [Gec06] about [Ari00]).

**Proposition 2.3.5** ([Gec06] Theorem 5.1) Let $w, w' \in S_n$. Then $w \leq_{LR} w'$ if and only if RSK-shape$(w) \leq$ RSK-shape$(w')$. $\square$

The next statement establishes the “unrelatedness” of distinct one-sided (left/right) cells within a two-sided cell. Though stated only for $\leq_L$ and left cells, the analogous statement is true also for $\leq_R$ and right cells.

**Proposition 2.3.6** ([Gec06] Theorem 5.3) Let $w, w' \in S_n$. If $w \leq_L w'$ and $w \sim_{LR} w'$ then $w \sim_L w'$. (See also [Lus81] Lemma 4.1) $\square$

Finally, the following proposition gives a combinatorial characterization of the left, right and two-sided equivalence. Statements (1) and (2) of the Proposition can be found also in [KL79] or [Ari00].

**Proposition 2.3.7** ([Gec06] Corollary 5.6) Let $w, w' \in S_n$. Then the following hold:

1. $w \sim_L w' \iff Q(w) = Q(w')$.
2. $w \sim_R w' \iff P(w) = P(w')$.
3. $w \sim_{LR} w' \iff \text{RSK-shape}(w) = \text{RSK-shape}(w')$. (This follows easily from Proposition 2.3.5 and the definition of $\sim_{LR}$). $\square$

**Remark 2.3.8** The proofs of the above statements can be found in [Gec06]. However, it should be noted that in [Gec06] permutations act from the left while for us permutations always act from the right. Also, in [Gec06], the RSK-algorithm as given in [Knu94] (or [Sag01]) is used. So, in the light of Theorem 2.3.2, the statements of the above propositions hold verbatim even in our setup, under the assumption that the (modified) RSK-correspondence as described here is used to obtain the $P$, $Q$ symbols of a permutation.

2.3.4 Some notes and notations

For $\lambda$ a partition of $n$,

- $\lambda'$ denotes the transpose of $\lambda$ which is defined to be the shape obtained by taking the transpose of the Young diagram of shape $\lambda$. E.g., $\lambda' = (3, 2, 2, 1)$ for $\lambda = (4, 3, 1)$.

- $t^\lambda$ denotes the standard tableau of shape $\lambda$ in which the numbers $1, 2, \ldots, n$ appear in order along successive rows; $t_\lambda$ is defined similarly, with ‘columns’ replacing ‘rows’. E.g., for $\lambda = (4, 3, 1)$, we have:

$$t^\lambda = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & \\
8 & \\
\end{array} \quad \quad t_\lambda = \begin{array}{ccc}
1 & 4 & 6 \\
2 & 5 & 7 \\
3 & \\
\end{array}$$
• Permutations in $\mathfrak{S}_n$ act naturally on tableaux of shape $\lambda \vdash n$ by acting entry-wise.

Denote by $w_\lambda$ the permutation in $\mathfrak{S}_n$ such that $t^\lambda w_\lambda = t_\lambda$.

The parabolic subgroup $W_\lambda$ and its coset representatives

Let $W_\lambda$ denote the row stabilizer of $t^\lambda$. It is a parabolic subgroup of $\mathfrak{S}_n$ generated by $W_\lambda \cap S$. Let $w_{0,\lambda}$ denote the longest element of $W_\lambda$. E.g., for $\lambda = (4, 3, 1)$, $W_\lambda$ is isomorphic to the product $\mathfrak{S}_4 \times \mathfrak{S}_3 \times \mathfrak{S}_1$. The longest element of $W_\lambda$ is given by the sequence $(1w_{0,\lambda}, \ldots, nw_{0,\lambda}) = (4, 3, 2, 1, 7, 6, 5, 8)$.

**Remark 2.3.9** The longest element $w_{0,\lambda}$ of $W_\lambda$ has RSK-shape $\lambda'$; by the definition of the RSK-correspondence it is obvious that $w_{0,\lambda}$ corresponds to the pair $(t_\lambda', t_\lambda')$.

Define $\mathfrak{D}_\lambda := \{ w \in \mathfrak{S}_n \mid t^\lambda w \text{ is row standard} \}$. Clearly, $w_\lambda$ as defined above, is an element of $\mathfrak{D}_\lambda$. The next proposition lists out a few properties of the elements in $\mathfrak{D}_\lambda$.

**Proposition 2.3.10** ([BB05] Lemma 1.1) For $\lambda$ a partition of $n$,

1. $\mathfrak{D}_\lambda$ is a set of right coset representatives of $W_\lambda$ in $\mathfrak{S}_n$.
2. The element $d \in \mathfrak{D}_\lambda$ is the unique element of minimal length in $W_\lambda d$.
3. $l(wd) = l(w) + l(d)$, for $w \in W_\lambda$ and $d \in \mathfrak{D}_\lambda$.
4. $\mathfrak{D}_\lambda = \{ d \in \mathfrak{S}_n \mid l(sd) > l(d) \text{ for all } s \in W_\lambda \cap S \}$.

**Proof:** If for each element $w \in \mathfrak{S}_n$ we associate the tableau $t^\lambda w$ then under this association the elements of the coset $W_\lambda d$, $d \in \mathfrak{D}_\lambda$ correspond to the collection of tableaux which vary from each other by a permutation of the row-wise entries. Now noting that $d \in \mathfrak{D}_\lambda$ corresponds, under the above association, to the unique tableau in the collection of $t^\lambda w$, $w \in W_\lambda d$, which is row-standard (i.e., increasing along rows but not necessarily increasing along columns), the bijection as in (1) is immediate.

To prove (2) we use the fact that the length $l$ counts the number of inversions, so $l(d) = \#\{(i, j) | 1 \leq i < j \leq n, i.d > j.d\}$ (see [BB05] Proposition 1.5.2]). Let $s \in W_\lambda$ such that $s = (i, i + 1)$. Let $a, b$ appear in $t^\lambda d$ in the positions where $i, i + 1$ appear in $t^\lambda$, then $t^\lambda sd$ is obtained by inverting the positions of $a$ and $b$ in $t^\lambda d$. Also, since $t^\lambda d$ is row-standard we have $a < b$. Thus, we see that the number of inversions in $sd$ is $> \text{ number of inversions in } d$ i.e., $l(sd) > l(d)$. Since this is true for all $s \in W_\lambda \cap S$ the uniqueness in (2) can be deduced by method of contradiction, as outlined in the proof of item (1) given below.

For $w \in W_\lambda$, $d \in \mathfrak{D}_\lambda$, we know that $l(wd) \leq l(w) + l(d)$. Suppose that $l(wd) < l(w) + l(d)$ then by the deletion condition (see (2)), we get $w' < w$, $d' < d$ such that $w'd' = wd$. Hence we get $d \in W_\lambda d' = W_\lambda d$ but $l(d') < l(d)$ which contradicts (2). Thus $l(wd) = l(w) + l(d)$, as claimed in (4).
For establishing the equality in (1), we first notice using (2) that $\mathcal{D}_\lambda$ is contained in the set on the right-hand side of (1). To prove the other way inclusion, suppose $d \in \mathfrak{S}_n$ such that $l(sd) > l(d)$ for all $s \in S \cap W_\lambda$, we claim that $d$ is of minimal length in the coset $W_\lambda d$, which then by (2) implies that $d \in \mathcal{D}_\lambda$. We prove the claim by assuming the contrary, as follows:

Suppose $d$ is not of minimal length in the coset $W_\lambda d$, then let $x \in W_\lambda d$ be one such. We can express $d$ as a product of the form $wx$ for some $w \in W_\lambda$ so that $l(d) \leq l(w) + l(x)$. If $l(d) < l(w) + l(x)$ then by deletion condition, we find elements $w', x'$ such that $w' < w$ and $x' < x$ with $d = w' x'$, which will contradict the minimality of $l(x)$ unless $x' = x$. Thus, we conclude that $l(d) = l(w) + l(x)$. Now since $x \neq d$, we get $w \neq 1$, which implies there is a $u \in S \cap W_\lambda$ such that $uw < w$, so that $uw x < wx$ which contradicts the hypothesis that $sd > d$ for all $s \in W_\lambda \cap S$.

We now prove a useful lemma that enables us to characterize elements $xs$, for $x \in \mathcal{D}_\lambda$ and $s \in S$. The lemma is true more generally for “distinguished” coset representatives of an arbitrary parabolic subgroup where the distinguished coset representatives are defined by property (1) in Proposition 2.3.10. The proof presented here holds true verbatim even in this general setup.

**Lemma 2.3.11** (Deodhar’s lemma) Let $x \in \mathcal{D}_\lambda$, $s \in S$. Then either $xs \in \mathcal{D}_\lambda$ or $xs = ux$ for $u \in W_\lambda \cap S$.

**Proof:** Suppose $xs \notin \mathcal{D}_\lambda$, then property (1) in Proposition 2.3.10 does not hold, i.e., there exists an element $u \in W_\lambda \cap S$ such that $l(uxs) < l(xs)$. On the other hand, $x \in \mathcal{D}_\lambda$ implies that $l(ux) > l(x)$. Thus we have,

$$l(x) < l(ux) = l(uxs) + 1 \leq l(xs)$$

Let $s_1 \cdots s_r$ be a reduced expression for $uxs$. Then $us_1 \cdots s_r$ is a reduced expression for $xs$ since $l(uxs) + 1 = l(xs)$. As $l(xs) > l(x)$, we should be able find a reduced expression for $x$ as a suitable subexpression of $us_1 \cdots s_r$. If $s_i$ is dropped then $x = us_1 \cdots \hat{s}_i \cdots s_r$ leading us to the contradiction that $ux < x$. Hence $x = s_1 \cdots s_r$ so that $xs = us_1 \cdots s_r = ux$, as required.

The set $\mathcal{D}_\lambda$ can be described entirely based on just one element in it, namely the longest coset representative. We have,

**Proposition 2.3.12** ([DJ86] Lemma 1.4) Let $d_\lambda$ be an element of maximal length in $\mathcal{D}_\lambda$. Then,

1. $d_\lambda$ is unique.

2. $\mathcal{D}_\lambda$ is precisely the set of all the prefixes of $d_\lambda$.

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Proof: Let \( w_0 \) be the longest element in \( \mathcal{S}_n \). Let \( x \in \mathfrak{D}_\lambda \) such that \( w_0 \in W_\lambda x \). Then by maximality of length and Proposition 2.3.10(3), we have \( w_0 = w_{0,\lambda} x \) and \( x \) should be of maximal length in \( \mathfrak{D}_\lambda \). By uniqueness of the longest element in \( \mathcal{S}_n \), we have \( w_{0,\lambda} x = w_{0,\lambda} d_\lambda \), proving (1) above.

For proving (2), first we observe that if \( y \in \mathfrak{D}_\lambda \) and \( s \in S \) such that \( y s < y \) then \( y s \in \mathfrak{D}_\lambda \). This follows immediately by Deodhar’s lemma (Lemma 2.3.11), since if \( y s \notin \mathfrak{D}_\lambda \) then \( y s > y \). Thus, all prefixes of \( d_\lambda \) are in \( \mathfrak{D}_\lambda \). The other way inclusion is proved by reverse induction on \( l(y) \) where \( y \in \mathfrak{D}_\lambda \). If \( l(y) \) is maximal then \( y = d_\lambda \). Assume \( y \neq d_\lambda \). For \( s \in S \), if \( l(y s) > l(y) \) and \( y s \in \mathfrak{D}_\lambda \) then by induction \( y s \) is a prefix of \( d_\lambda \), so is \( y \) and we are done. So, assume for every \( s \in S \) either \( y s \notin \mathfrak{D}_\lambda \) or \( l(y s) < l(y) \). Suppose \( y s \notin \mathfrak{D}_\lambda \) where \( s \in S \) then by Deodhar’s lemma \( y s = w y \) for some \( u \in W_\lambda \cap S \). Then,

\[
l(w_{0,\lambda} y s) = l(w_{0,\lambda} u y) = l(w_{0,\lambda}) - 1 + l(y) = l(w_{0,\lambda} y) - 1
\]

On the other hand, if \( l(y s) < l(y) \) and \( y s \in \mathfrak{D}_\lambda \) then

\[
l(w_{0,\lambda} y s) = l(w_{0,\lambda}) + l(y s) = l(w_{0,\lambda}) + l(y) - 1 = l(w_{0,\lambda} y) - 1
\]

Thus, we obtain that for all \( s \in S \), we get \( w_{0,\lambda} y s < w_{0,\lambda} y \). This readily implies \( w_{0,\lambda} y = w_0 \), a contradiction to the assumption that \( y \neq d_\lambda \). Hence the proof is complete.

We had already seen that \( w_\lambda \) is an element of \( \mathfrak{D}_\lambda \). By the above proposition, it is a prefix of \( d_\lambda \) and every prefix of \( w_\lambda \) is also in \( \mathfrak{D}_\lambda \). The next lemma characterizes these prefixes.

Lemma 2.3.13 ([DJ86, Lemma 1.5]) The set of \( w \in \mathcal{S}_n \) such that \( t^\lambda w \) is a standard tableau is the same as the set of prefixes of \( w_\lambda \).

Proof: We begin with an observation. Suppose \( w \in \mathcal{S}_n \) such that \( t^\lambda w \) is standard. If \( l(w(i, i + 1)) < l(w) \) then \( (i + 1).w^{-1} < i.w^{-1} \), since \( l \) counts the number of inversions. So, if \( i \) occurs in node \( (r, c) \) and \( (i + 1) \) occurs in node \( (r', c') \) of \( t^\lambda w \) then \( r > r' \) and \( c < c' \) (because \( t^\lambda w \) is standard). Now it is obvious that \( t^\lambda w(i, i + 1) \) is also standard.

The observation made in the last paragraph shows that for every prefix \( w \) of \( w_\lambda \), \( t^\lambda w \) is also standard. Conversely, let \( t^\lambda w \) be standard and assume \( w \neq w_\lambda \). Then there exists \( i, j \), with \( j > i + 1 \), occurring in consecutive boxes in some column of \( t^\lambda w \). It can be seen easily that there is a \( k, i \leq k < j \) such that \( k \) occurs in a node \( (a, b) \) and \( k + 1 \) occurs in node \( (a', b') \) such that \( a < a' \) and \( b > b' \) of \( t^\lambda w \). Then \( t^\lambda w(k, k + 1) \) is standard and \( l(w(k, k + 1)) = l(w) + 1 \). By reverse induction on \( l(w) \), \( w(k, k + 1) \) is a prefix of \( w_\lambda \), and so is \( w \).
More on coset representatives

After all this discussion about the right coset representatives of $W_\lambda$, it is easy to verify the following remark about the left coset representatives,

**Remark 2.3.14** The set $O^{-1}_\lambda$ is a set of minimal length left coset representatives of $W_\lambda$ in $G_n$. This set has similar properties as that of $O_\lambda$ as listed in Proposition 2.3.10

Let $\lambda, \mu \vdash n$. We now describe a set of double coset representatives for $W_\lambda W_\mu$ in the following lemma:

**Lemma 2.3.15** (see [GP00], Proposition 2.1.7) Let $\lambda, \mu \vdash n$. Then for each element $w \in G_n$ there is a $u \in W_\lambda, v \in W_\mu$ and a unique $d \in O_\lambda \cap O_\mu^{-1}$ such that $w = uvd$ and $l(w) = l(u) + l(d) + l(v)$. In particular, the set $O_\lambda \cap O_\mu^{-1}$ is a set of double coset representatives of $W_\lambda W_\mu$.

**Proof:** Let $w \in G_n$ and write $w = ux$, where $u \in W_\lambda$ and $x \in O_\lambda$, and $l(w) = l(u) + l(x)$ by Proposition 2.3.10. Write $x = dv$ where $d \in O_\mu^{-1}$ and $v \in W_\mu$ and $l(x) = l(d) + l(v)$. Since $d$ is a prefix of $x \in O_\lambda$, we get $d \in O_\lambda$. Thus, $w = uvd$ with $u \in W_\lambda$, $d \in O_\lambda \cap O_\mu^{-1}$, $v \in W_\mu$ and $l(w) = l(u) + l(d) + l(v)$, as required. The uniqueness follows by noting that $d$, as obtained above, is the unique element of minimal length in the double coset $W_\lambda W_\mu = W_\lambda w W_\mu$.

The following observation turns out to be useful later,

**Lemma 2.3.16** (see [GP00], Theorem 2.1.12) Let $\lambda, \mu \vdash n$. Let $d \in O_\lambda \cap O_\mu^{-1}$. Then the subgroup $d^{-1}W_\lambda d \cap W_\mu$ is generated by $d^{-1}W_\lambda d \cap W_\mu \cap S$.

**Proof:** Define $L := W_\lambda \cap S$, $K := W_\mu \cap S$ and $J := d^{-1}Ld \cap K$. Clearly, $W_J \subset d^{-1}W_\lambda d \cap W_\mu$. It therefore suffices to prove that $W_J d \cap dW_\mu \subset dW_J$. Let $w \in W_J d \cap dW_\mu$. Then $w = ud = dv$, where $u \in W_\lambda$ and $v \in W_\mu$ and $l(w) = l(u) + l(d) = l(d) + l(v)$. In particular, $l(u) = l(v)$. Let $v = v_0 \cdots v_r$ where $v_i \in K$. Set $d_0 = d$ and define $d_i \in O_\lambda$ using Deodar's Lemma, so that $d_{i-1}v_i = x_i d_i$ where $d_i \in O_\lambda$ and $x_i \in L$ if $d_{i-1}v_i \in O_\lambda$ then $x_i = 1$ otherwise $d_i = d_{i-1}$. Then we have $ud = dv = x_1 \cdots x_r d_r$ where $x_1 \cdots x_r \in W_\lambda$ and $d_r \in O_\lambda$. By uniqueness of the expression $ud$, we have $u = x_1 \cdots x_r$ and hence $r = l(u) = l(v)$. Therefore, $x_i \neq 1$ for all $i$ and so, $d_i = d$ for all $i$. This means $dv_i = x_i d$ for all $i$ equivalently, $v_i \in J$ from which we conclude that $w \in dW_J$.

**Remark 2.3.17** Lemma 2.3.15, Lemma 2.3.16 and Remark 2.3.14 hold true more generally for $W_\lambda, W_\mu$ replaced by any parabolic subgroups of $G_n$ and $O_\lambda, O_\mu$ replaced by the respective sets of minimal length right coset representatives (defined as in Proposition 2.3.10) which implies it is the unique element of minimal length in the right coset containing it. The proofs are verbatim those given above.
Few more combinatorial results

We now gather together a few combinatorial results regarding the Kazhdan-Lusztig relations, which will be used in the sequel:

Lemma 2.3.18 (MP05 Lemma 3.3) Let $\lambda \vdash n$. Define $w = w_0w_{0,\lambda}w_\lambda$. Then,

1. $w_0,\lambda D_\lambda = \{ y \in S_n \mid y \leq_R w_{0,\lambda} \}$. Thus, $y \leq_R w_{0,\lambda}$ if and only if for every row of $t^\lambda y$ the entries are decreasing to the right.

2. $\{ w_{0,\lambda} d \mid d \text{ a prefix of } w_\lambda \} = \{ y \in S_n \mid y \sim_R w_{0,\lambda} \}$.

3. The element $w$, as defined above, is in the same left cell as $w_{0,\lambda}$.

4. $w$ is a prefix of every element in the right cell containing it.

Proof: (1) By Proposition 2.3.10(3) and (2.6), it is easy to check that $w_0,\lambda D_\lambda \subset \{ y \in S_n \mid y \leq_R w_{0,\lambda} \}$. To prove the other way inclusion, notice that if $y \leq_R w_{0,\lambda}$ then by Lemma 2.2.11 $\mathcal{L}(w_{0,\lambda}) \subset \mathcal{L}(y)$, so, $sy < y$ for all $s \in S \cap W_\lambda$. Expressing $y$ as $ud$ where $u \in W_\lambda$, $d \in D_\lambda$ we have for all $s \in S \cap W_\lambda$, $su < ud$, as we just observed, and $sd > d$, by Proposition 2.3.10(3). Therefore we conclude that $su < u$ for all $s \in S \cap W_\lambda$ and so, $u = w_{0,\lambda}$. Hence $y = w_{0,\lambda} d$ for some $d \in D_\lambda$, as required. The second part now follows immediately, noticing also that by its definition $w_{0,\lambda}$ reverses the entries in each row of $t^\lambda$.

(2) By (1), we have $w_{0,\lambda} d \leq_R w_{0,\lambda}$. Further, using (2.6) it can be seen that for any prefix $d$ of $w_\lambda$, $w_0,\lambda w_\lambda \leq_R w_0,\lambda d$. However, an easy verification shows that $w_0,\lambda w_\lambda$ corresponds under the RSK-correspondence to the pair $(t_0, t')$ (compare MP05 Lemma 3.2) while, $w_{0,\lambda} d$ corresponds to $(t_0, t')$. By Proposition 2.3.7(2) this means $w_{0,\lambda} w_\lambda \sim_R w_{0,\lambda}$. Thus, for each prefix $d$ of $w_\lambda$ we get, $w_0,\lambda d \sim_R w_{0,\lambda}$, thereby proving one-way inclusion. Now using Lemma 2.3.13 and the characterisation of right cells given by Proposition 2.3.7 a counting argument proves the equality of the two sets.

(3) Applying the RSK-correspondence to $w_0,\lambda w_\lambda$ we get the pair $(t_0, t')$ (compare MP05 Lemma 3.2). Notice that the pair corresponding under RSK to $w_0w_{0,\lambda}w_\lambda$ is just the transpose of the tableaux corresponding to $w_0,\lambda w_\lambda$. So, $w = w_0w_{0,\lambda}w_\lambda$ corresponds to $(t^\lambda, t_\lambda)$. Proposition 2.3.7 then proves (3), as the $Q$-symbols of both $w$ and $w_{0,\lambda}$ are the same.

(4) Putting together Lemma 2.2.12 and statement (2) above, we deduce that the right cell containing $w$ is given by $\{ w_0w_{0,\lambda} d \mid d \text{ a prefix of } w_\lambda \}$, which can then be easily identified with the set $\{ wb \mid b \text{ a prefix of } w_\lambda \}$ since $w_\lambda^{-1} = w_{\lambda'}$. Also, $l(w_0) = l(w) + l(w_0,\lambda) + l(w_\lambda)$ — observe that $l(w_0w) = l(w_0) - l(w)$. This fact along with the relation $w_0 = w_{w_{\lambda'}w_0,\lambda}$ implies that $l(wb) = l(w) + l(b)$ for all prefixes $b$ of $w_\lambda$. Hence the claim. □