Chapter 2

\((G, D)\)-Number of a graph

In this Chapter, we introduce the concept of \((G, D)\)-sets, \((G, D)\)-number of a graph \(G\) and obtain \((G, D)\)-number for various graphs. Its bounds are found and its relationship with various other parameters are established. Graphs with \((G, D)\)-number equal to 2, \(p\) and equal to \(p - 1\) are characterized. The relation connecting diameter and \((G, D)\)-number of a graph is studied. Also, it is found that how the \((G, D)\)-number of a non-complete connected graph is affected by the addition of a single vertex and removal of the vertices or edges. Further, the change in \((G, D)\)-number of a complete graph due to the removal of the edges of a standard subgraph in \(K_n\), such as, path, cycle, clique and star are discussed.

2.1 Introduction

The concept of domination in graphs was introduced by Ore[13]. Let \(G = (V, E)\) be a finite undirected graph with neither loops nor
multiple edges. A subset $D$ of $V(G)$ is a dominating set of $G$ if every vertex in $V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. The concept of geodominating (or geodetic) set was introduced by Buckley and Harary in [1] and Chartrand, Harary and Zhang in [4]. A $u - v$ geodesic is a $u - v$ path of length $d(u, v)$. A set $S$ of vertices of $G$ is a geodominating (or geodetic) set of $G$ if every vertex of $G$ lies in an $x - y$ geodesic for some $x, y \in S$. Equivalently, a set $S \subseteq V(G)$ is said to be a geodetic set of $G$ if every vertex in $V - S$ lies in a geodesic joining two vertices of $S$. The minimum cardinality of a geodominating (or geodetic) set of $G$ is called the geodomination (or geodetic) number of $G$ and is denoted by $g(G)$.

For a graph $G = (V, E)$, some subsets of $V(G)$ are dominating but not geodetic in $G$ and some others are geodetic but not dominating in $G$. Also, some subsets of $V(G)$ are both dominating and geodetic in $G$.

In figure 2.1, let $DS(G)$ be the collection of all dominating sets of $G$ and let $GS(G)$ be the collection of all geodetic sets of $G$. 
Here, we study about those subsets of $V(G)$ which are both dominating and geodetic in $G$. We call such sets as $(G,D)$-sets of $G$. Suppose $S$ is a subset of $V(G)$ and $v \in V$. We say that, $v$ is dominated by the vertices of $S$ if either $v \in S$ or $v$ is adjacent to some vertex of $S$. Similarly, if $S \subseteq V(G)$ with $|S| \geq 2$ and $v \in V$, then we say that, $v$ is geodominated by the vertices of $S$ if either $v \in S$ or $v$ is an internal vertex of an $x - y$ geodesic for some $x, y \in S$. Throughout this Chapter, we consider only connected graphs with at least two vertices.

### 2.2 $(G, D)$-number of graphs:

**Definition 2.2.1.** Let $G = (V, E)$ be any connected graph with at least two vertices. A subset $S$ of $V(G)$ which is both dominating and geodetic set of $G$ is called a $(G, D)$-set of $G$. 

\[
\text{Figure 2.1}
\]
A \((G, D)\)-set \(S\) is said to be minimal if no proper subset of \(S\) is a \((G, D)\)-set of \(G\). A \((G, D)\)-set \(S\) is said to be a minimum \((G, D)\)-set of \(G\) if there exists no \((G, D)\)-set \(S'\) of \(G\) such that \(|S'| < |S|\). The cardinality of a minimum \((G, D)\)-set of \(G\) is called the \((G, D)\)-number of \(G\). It is denoted by \(\gamma_G(G)\). Any \((G, D)\)-set \(S\) of \(G\) of cardinality \(\gamma_G\) is called a \(\gamma_G\)-set of \(G\).

**Remark 2.2.2.** Let \(G\) be a connected graph with \(p(\geq 2)\) vertices. Then, \(\gamma(G) \leq \gamma_G(G) \leq i_2(G).\) (See 1.42). Strict inequality is also true in the above relation. For example, considering \(P_{11}\), let \(V(P_{11}) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}\). \(\gamma(P_{11}) = 4\), since \(\{v_2, v_5, v_8, v_{10}\}\) is a minimum dominating set of \(P_{11}\). \(\gamma_G(G) = 5\), for \(\{v_1, v_4, v_7, v_9, v_{11}\}\) is a minimum \((G, D)\)-set of \(P_{11}\) and \(i_2(G) = 6\), since \(\{v_1, v_3, v_5, v_7, v_9, v_{11}\}\) is a minimum independent 2-dominating set of \(P_{11}\).

**Example 2.2.3.** Considering the graph \(G\) as in figure 2.2, \(\{w, y\}\) is the unique \(g\)-set of \(G\). So, \(g(G) = 2\).
\(\{w, y\}\) is also the unique \(\gamma_G\)-set of \(G\). Therefore, \(\gamma_G(G) = 2\).

Also, \(\gamma(G) = 2\), though \(G\) has more than one \(\gamma\)-set.

Therefore, in this example, \(\gamma_G(G) = g(G) = \gamma(G)\).
Remark 2.2.4. In general, \( \gamma_G(G) \), \( g(G) \) and \( \gamma(G) \), all need not be equal. For example, \( \gamma_G(P_8) = 4 \), \( g(P_8) = 2 \) and \( \gamma(P_8) = 3 \). Further, \( \gamma_G(P_7) = \gamma(P_7) = 3 \). But, \( g(P_7) = 2 \). Also, \( \gamma_G(P_3) = g(P_3) = 2 \), whereas, \( \gamma(P_3) = 1 \).

Remark 2.2.5. Let \( G = (V, E) \) be any connected graph with atleast two vertices. Then,

1. \( \gamma_G(G) \geq g(G) \) and \( \gamma_G(G) \geq \gamma(G) \).
2. Every \((G, D)\)-set of \( G \) contains all the pendant vertices of \( G \).
3. If \( G \) is a graph with atleast one pendant vertex, then for every \((G, D)\)-set \( D \) of \( G \), \( V - D \) is not a \((G, D)\)-set of \( G \).
4. Every super set of a \((G, D)\)-set of \( G \) is a \((G, D)\)-set of \( G \).
5. \( \gamma_G(K_n) = n \), for, \( n = g(K_n) \leq \gamma_G(K_n) \leq n \).

Proposition 2.2.6. For a star graph \( G \), \( \gamma_G(G) = p - 1 \).
Proof. Let $G = K_{1,n}$ with $V(K_{1,n}) = \{v, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$. Let $S$ be a minimum $(G, D)$-set of $K_{1,n}$. By Remark 2.2.5, $\{v_1, v_2, \ldots, v_n\} \subseteq S$. Since $\{v_1, v_2, \ldots, v_n\}$ is itself a $(G, D)$-set of $G$, $S = \{v_1, v_2, \ldots, v_n\}$. Therefore, $\gamma_G(G) = p - 1$. \qed

Proposition 2.2.7. If $G$ is a bi-star, then $\gamma_G(G) = p - 2$.

Proof. Let $G = B(r, s)$ where $r, s \geq 1$. Suppose $V(B(r, s)) = \{u, v, u_i, v_j : 1 \leq i \leq r \text{ and } 1 \leq j \leq s\}$ and $E(B(r, s)) = \{uv, uu_i, vv_j : 1 \leq i \leq r, 1 \leq j \leq s\}$. Let $S$ be a minimum $(G, D)$-set of $B(r, s)$. By Remark 2.2.5, $\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\} \subseteq S$. Since $\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}$ is itself a $(G, D)$-set of $G$, $S = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}$. Therefore, $\gamma_G(G) = p - 2$. \qed

Remark 2.2.8.

$$\lceil \frac{n-4}{3} \rceil + 2 = \begin{cases} 
\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod{3} \\
\lceil \frac{n}{3} \rceil + 1 & \text{otherwise}
\end{cases}$$

Theorem 2.2.9.

$$\gamma_G(P_n) = \begin{cases} 
\lceil \frac{n-4}{3} \rceil + 2 & \text{if } n \geq 5 \\
2 & \text{if } n = 2, 3 \text{ or } 4
\end{cases}$$
Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. Let $S$ be a minimum $(G, D)$-set of $P_n$. We observe that every $(G, D)$-set of $P_n$ is a dominating set containing the end vertices of $P_n$. Let $D_1$ be a minimum dominating set containing $v_1, v_n$. Therefore, $|D_1| \leq |S|$. As $D_1$ is also a $(G, D)$-set of $G$, $|S| \leq |D_1|$. So, we have, $\gamma_G(P_n) = |S| = |D_1|$. Let $D$ be a minimum dominating set of $P_n$. Then,

$$|D_1| = \begin{cases} 
|D| = \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1 \pmod{3} \\
|D| + 1 = \left\lceil \frac{n}{3} \right\rceil + 1 & \text{otherwise} 
\end{cases}$$

$$= \left\lceil \frac{n-4}{3} \right\rceil + 2. \ (\text{By Remark 2.2.8})$$

Therefore, $\gamma_G(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$. \qed

Theorem 2.2.10. For $n \geq 4 (n \neq 5)$, $\gamma_G(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$

Proof. Clearly, $\gamma_G(C_4) = \gamma(C_4) = 2 = \left\lceil \frac{n}{3} \right\rceil$. For a cycle $C_n$ of length $n > 5$, every dominating set is a $(G, D)$-set. Therefore, $\gamma_G(C_n) \leq \gamma(C_n)$. By Remark 2.2.5, $\gamma_G(C_n) \geq \gamma(C_n)$.

Hence, by Theorem 1.38, $\gamma_G(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$. \qed

Theorem 2.2.11. Let $W_p = C_{p-1} + K_1$, $p \geq 5$ denote the wheel graph on $p$ vertices. Then, $\gamma_G(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

Proof. Let $V(W_p) = \{v, v_1, v_2, \ldots, v_{p-1}\}$ with $v$ as its central vertex. The central vertex $v$ lies in a geodesic joining every pair of
non-adjacent vertices of $C_{p-1}$ and so $d(v_i, v_j) = 2$ for every pair of non-adjacent vertices of $C_{p-1}$. Define $S = \{v_1, v_3, \ldots, v_{p-1}\}$, when $p$ is even and $S = \{v_1, v_3, \ldots, v_{p-4}, v_{p-2}\}$, when $p$ is odd. Clearly, $S$ is a minimum $(G, D)$-set of $W_p$. Hence, $\gamma_G(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

\begin{theorem}
\end{theorem}
Clearly, there is no \((G, D)\)-set of \(G\) with two elements.

**Claim:** There exists no \((G, D)\)-set \(S\) of \(G\) such that \(|S| = 3\).

**Case i:** If \(X\) is a proper subset of \(V_1\) (or \(V_2\)), then \(X\) is not dominating the vertices of \(V_1 - X\) (or \(V_2 - X\)).

**Case ii:** Suppose \(X \cap V_1 = \{a_i, a_j\}\) and \(X \cap V_2 = \{b_k\}\), then every \(v \in V_1 - X\) is not covered by any geodesic path joining pair of vertices of \(X\). Therefore, \(X\) is not a geodetic set of \(G\) and so \(X\) is not a \((G, D)\)-set of \(G\).

**Case iii:** Similarly, if \(X \cap V_1 = \{a_i\}\) and \(X \cap V_2 = \{b_j, b_k\}\), then in this case also \(X\) is not a \((G, D)\)-set of \(G\).

Therefore, no 3-element subset of \(V\) is a \((G, D)\)-set of \(K_{m,n}\).

Therefore, \(\gamma_G(K_{m,n}) \geq 4 - (2)\).

Hence, by (1) and (2), \(\gamma_G(K_{m,n}) = 4\) if \(m, n > 3\). \(\square\)

**Theorem 2.2.13.** If \(k \geq 3\) is an integer, then

\[
\gamma_G(K_{n_1, n_2, \ldots, n_k}) = \begin{cases} 
  k & \text{if } n_i = 1 \text{ for every } i. \\
  n_t & \text{if } n_i \neq 1 \text{ for exactly one } i = t \\
  \min_{n_i \geq 2} \{n_i, 4\} & \text{otherwise}
\end{cases}
\]

**Proof.** Let \(G = K_{n_1, n_2, \ldots, n_k}\) with the vertex partition sets \(U_1, U_2, \ldots, U_k\) and \(|U_i| = n_i\), \(1 \leq i \leq k\). First, we observe the
following: If $X$ is a proper subset of $U_i$ ( $|U_i| > 1$) for some $i$, then the members of $U_i - X$ are not dominated by $X$.

**Case 1:** $n_i = 1$ for every $i$, $1 \leq i \leq k$.

Then, $G$ is a complete graph on $k$ vertices and so by 2.2.5, $\gamma_G(K_{n_1,n_2,\ldots,n_k}) = k$.

**Case 2:** $n_i \neq 1$ for exactly one $i$, $n_i = t > 1$(say).

Obviously, $V(U_t)$ is the only unique minimum $(G,D)$-set of $G$ and so $\gamma_G(K_{n_1,n_2,\ldots,n_k}) = |U_t| = n_t$.

**Case 3:** $n_i \neq 1$ for at least two $i$.

**Subcase 3a:** Let $\min_{n_i \geq 2} n_i = n_t < 4$.

Clearly, $n_t$ vertices of $U_t$ forms a minimum $(G,D)$-set of $G$ and so $\gamma_G(K_{n_1,n_2,\ldots,n_k}) = |U_t| = n_t = \min_{n_i \geq 2} n_i < 4$.

**Subcase 3b:** Let $\min_{n_i \geq 2} n_i \geq 4$.

Let $U_1$ and $U_2$ be two components of $V(G)$ such that $|U_1| \neq 1$ and $|U_2| \neq 1$. Let $S = \{x_1,y_1,x_2,y_2 : x_1,y_1 \in U_1$ and $x_2,y_2 \in U_2\}$. Clearly, $S$ is a minimal $(G,D)$-set of $K_{n_1,n_2,\ldots,n_k}$ and so $\gamma_G(G) \leq 4 - - - (I)$.

Let $X$ be a 3-element subset of $V(K_{n_1,n_2,\ldots,n_k})$

**Claim:** $X$ is not a $(G,D)$-set of $V(K_{n_1,n_2,\ldots,n_k})$.

We observe that $X \notin U_i$, $1 \leq i \leq k$.

**Case i:** Suppose $X \cap U_i = \{u_1,u_2\}$ and $X \cap U_j = \{v_1\}$ for some $i$ and $j$. Then, as in Theorem 2.2.12, $X$ is not a geodetic set of $G$. 

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Case ii: Suppose $X$ contains exactly one element from 3 different sets $U_x, U_y, U_z$, ($1 \leq x, y, z \leq k$), say, $u_x, u_y, u_z$ respectively. Clearly, $d(u_x, u_y) = d(u_x, u_z) = d(u_y, u_z) = 1$. Therefore, $X = \{u_x, u_y, u_z\}$ forms a complete subgraph and so no element of $V - X$ lies in any geodesic joining the vertices of $X$. Thus, $X$ is not a $(G, D)$-set of $K_{n_1, n_2, \ldots, n_k}$. Therefore, no 3-element subset of $V$ is a $(G, D)$-set of $K_{n_1, n_2, \ldots, n_k}$. By Remark 4 of 2.2.5, no subset of $V$ of cardinality $\leq 3$ is a $(G, D)$-set of $G$. Hence, by (I), $\gamma_G(K_{n_1, n_2, \ldots, n_k}) = 4$. □

Theorem 2.2.14. If $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$, $r \geq 2$ such that $n_1 + n_2 + \cdots + n_r = p - 1$. Then, $\gamma_G(G) = p - 1$.

Proof. Let $v \in V(K_1)$. Then, $v$ lies in the $xvy$ geodesic joining $x$ and $y$ where $x \in K_{n_i}$ and $y \in K_{n_j}$ for $1 \leq i, j \leq r$ and $i \neq j$. Therefore, $v$ is both dominated and geodominated by the vertices of $S = V - \{v\}$ and so $S$ is a $(G, D)$-set of $G$. Thus, $\gamma_G(G) \leq |S| = p - 1$. By Theorem 1.57 and Remark 2.2.5, $\gamma_G(G) \geq g(G) = p - 1$. Hence, $\gamma_G(G) = p - 1$. □

2.3 Bounds on $(G, D)$-number.

Theorem 2.3.1. For any graph $G$, $2 \leq \gamma_G(G) \leq p$. 

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**Proof.** From Remark 2.2.5 and from theorem 1.53, $\gamma_G(G) \geq 2$. Also, any $(G, D)$-set is a subset of $V(G)$. Hence, $2 \leq \gamma_G(G) \leq p$. \hfill $\square$

In the above proposition, upper bound is sharp as $\gamma_G(K_p) = p$ and the lower bound is sharp. (See Example. 2.2.3.)

**Theorem 2.3.2.** Let $G$ be any graph with $k$ support vertices and $l$ end vertices. Then, $l \leq \gamma_G(G) \leq p - k$.

**Proof.** Let $L$ and $K$ denote the set of all end and support vertices of $G$ respectively and $|L| = l; |K| = k$. Clearly, $l \geq k$. By Remark 2.2.5, $L$ is a subset of every $(G, D)$-set of $G$ and so $\gamma_G(G) \geq l$. Further, every vertex of $K$ lies in a geodesic joining two vertices of $L$ as well as dominated by the vertices of $L$. Therefore, $V - K$ is a $(G, D)$-set of $G$ and so $\gamma_G(G) \leq |V - K| = |V| - |K| = p - k$. \hfill $\square$

**Corollary 2.3.3.** Let $T$ be any tree with $k$ support vertices and $l$ end vertices such that $l + k = p$. Then, $\gamma_G(G) = p - k$.

The following example shows even if $l + k < p$, $\gamma_G(G) = p - k$.

**Example 2.3.4.** Considering the graph $G$ in figure 2.3, $\{v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is a minimum $(G, D)$-set of $G$ and so $\gamma_G(G) = 8 = |V - k|$. 

\[ \text{27} \]
Remark 2.3.5. In the above theorem, the lower bound is sharp for a comb graph.

Proposition 2.3.6. Let $G = (V, E)$ be a connected graph of order $p$ and diameter $d$. Then, $\gamma_G(G) \leq p - d + \lceil \frac{d}{3} \rceil$.

Proof. Let $P : (u = v_0, v_1, \ldots, v_d = v)$ be a path of length $d$ in $G$. If $S = \{v_1, v_2, \ldots, v_{d-1}\}$, then $V - S$ is a geodetic set of $G$. Also, it is a dominating set of $V - (S - \{v_1, v_{d-1}\})$. Let $P' = (v_2, v_3, \ldots, v_{d-2})$. $|V(P')| = d - 3$. Let $D'$ be a minimum dominating set of $P'$. By Theorem 1.38, $|D'| = \lceil \frac{d-3}{3} \rceil$. Let $D = (V - S) \cup D'$. Clearly, $D$ is a $(G, D)$-set of $G$. Therefore, $\gamma_G(G) \leq |D| = |(V - S) \cup D'| \leq p - d + 1 + \left\lceil \frac{(d-3)}{3} \right\rceil = p - d + 1 + \left\lceil \frac{d}{3} \right\rceil - 1 = p - d + \left\lceil \frac{d}{3} \right\rceil$. □

Remark 2.3.7. Equality holds in the above theorem. For example, considering the graph $G$ as in Figure 2.4,
Figure 2.4

\[ p = 14, \, d = 8 \text{ and so } p - d + \left\lceil \frac{d}{3} \right\rceil = 9. \]

\[ S = \{v_0, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_3, v_5, v_8\} \text{ is a minimum } (G, D)\text{-set of } G \]

and \( \gamma_G(G) = |S| = 9 = p - d + \left\lceil \frac{d}{3} \right\rceil. \)

**Example 2.3.8.** Consider the graph consisting of a complete graph \( H \) on \( p - d + 1 \) vertices attached with a path of length \( d - 1 \).

Figure 2.5

Number of vertices of \( G = p - d + 1 + d - 1 = p \) and diam \( G = d. \)

Let \( P' = \{v_2, v_3, \ldots, v_{d-2}\} \) and let \( D \) be a minimum dominating set of \( P' \). Clearly, \( S = (V(H) - \{v_1\}) \cup \{v_d\} \cup D \) is a minimum \((G, D)\text{-set of } G, \text{ where } H \cong K_{p-d+1}. \) Then, \( \gamma_G(G) = p - d + 1 + \gamma(P') = p - d + 1 + \left\lceil \frac{d-3}{3} \right\rceil = p - d + \left\lceil \frac{d}{3} \right\rceil. \)

□
Corollary 2.3.9. Let $G$ be a connected graph of order $p$ and diameter $d$. If $\delta \geq 3$, then $\gamma_G(G) \leq p - d + 1$.

Proof. As $\text{diam}(G)=d$, there exists a shortest path $P = (u = v_0, v_1, \ldots, v_d = v)$ of length $d$ in $G$. If $S = \{v_1, v_2, \ldots, v_{d-1}\}$, then $V - S$ is a geodetic set of $G$. Since $\delta \geq 3$, each vertex of $S$ is adjacent to at least one vertex of $V - S$. Therefore, $V - S$ is a $(G, D)$-set of $G$ and so $\gamma_G(G) \leq |V - S| = p - (d - 1) = p - d + 1$. □

2.4 Minimal $(G, D)$ - sets.

Observation 2.4.1. Let $G$ be a connected graph with $p(\geq 2)$ vertices. Then,
1. A minimal geodetic set of $G$ which is also a dominating set of $G$ is a minimal $(G, D)$-set of $G$.
2. A minimum geodetic set of $G$ which is also a dominating set of $G$ is a minimum $(G, D)$-set of $G$.
3. Any minimal dominating set of $G$ which is also a geodetic set of $G$ is a minimal $(G, D)$ - set of $G$.
4. Any minimum dominating set of $G$ which is also a geodetic set of $G$ is a minimum $(G, D)$-set of $G$.

Remark 2.4.2. Let $D$ be a minimal $(G, D)$-set of $G$. Then, $D$
need not be a minimal dominating set of $G$ as well as a minimal geodetic set of $G$. For example, considering $P_8$, by Theorem 2.2.9, 
$\gamma_G(P_8) = 2 + \left\lceil \frac{8-4}{3} \right\rceil = 2 + 2 = 4$.

\[ D = \{v_1, v_4, v_7, v_8\} \] is a minimal $(G, D)$-set of $G$. But, it is not a minimal dominating set of $G$, since $D' = D - \{v_8\} \subset D$ is a dominating set of $G$. Further, $D$ is not a minimal geodetic set of $G$ since $D'' = D - \{v_7\} \subset D$ is a geodetic set of $G$.

**Theorem 2.4.3.** Let $G$ be any connected graph with at least two vertices. Let $L$ and $K$ denote the set of all end vertices and support vertices of $G$ respectively and let $D$ be a $(G, D)$-set of $G$ containing no support vertex of $G$. If for every $v \in D - L$, there exists at least one vertex $u \in V - ((D - \{v\}) \cup K)$ such that $N[u] \cap (D - L) = \{v\}$−−−(1), then $D$ is a minimal $(G, D)$-set of $G$.

**Proof.** Let $D$ be a $(G, D)$-set of $G$ satisfying the given condition.

**Case i:** Suppose $L = \phi$. Then, $K = \phi$ and condition (1) becomes
for every $v \in D$, there exists at least one vertex $u$ in $V - (D - \{v\})$ such that $N[u] \cap D = \{v\}$. Then, by Remark 1.43, $D$ is a minimal dominating set of $G$. As $D$ is a $(G, D)$-set of $G$, $D$ is a geodetic set of $G$. Hence, by Observation 2.4.1, $D$ is a minimal $(G, D)$-set of $G$.

**Case ii:** Suppose $L \neq \emptyset$. Then, every vertex of $L$ dominates all its support vertices and no other vertex of $V(G)$. Also, for the set $K \cup L$, $L$ is a minimal dominating set. If every vertex $v \in D - L$ satisfies condition (1), then $D$ becomes a minimal dominating set of $G$. Hence, as in case (i), $D$ is a minimal $(G, D)$-set of $G$. 

### 2.5 Further results on $(G, D)$-number of graphs.

**Theorem 2.5.1.** Let $G = (V, E)$ be any graph. Suppose $S$ is a proper subset of $V(G)$ such that the subgraph of $G$ induced by $S$ is complete, then $S$ is not a $(G, D)$-set of $G$.

**Proof.** Let $v \in V - S$. Since the subgraph of $G$ induced by $S$ is complete, $d(x, y) = 1$ for every $x, y \in S$. Therefore, $v$ does not lie on any geodesic joining any pair of vertices of $S$. Hence, $S$ is not a $(G, D)$-set of $G$. 

**Theorem 2.5.2.** Let $G = (V, E)$ be a graph. Then, every $(G, D)$-set of $G$ contains all the extreme vertices of $G$. 

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Proof. Let $S$ be the set of all extreme vertices of $G$. By 1.49, every geodetic set of $G$ contains $S$. As every $(G, D)$-set of $G$ is a geodetic set of $G$, every $(G, D)$-set of $G$ contains $S$. □

Theorem 2.5.3. Let $G = (V, E)$ be any graph. If $S$, the set of all extreme vertices of $G$, is a $(G, D)$-set of $G$, then $S$ is the unique minimum $(G, D)$-set of $G$.

Proof. Suppose $S$ is a $(G, D)$-set of $G$. Then, $\gamma_G(G) \leq |S|$. By Theorem 2.5.2, $\gamma_G(G) \geq |S|$. Therefore, $\gamma_G(G) = |S|$ and so $S$ is a minimum $(G, D)$-set of $G$. Again by Theorem 2.5.2, every minimum $(G, D)$-set of $G$ contains $S$ and so $S$ is the unique minimum $(G, D)$-set of $G$. □

Remark 2.5.4. If $S$, the set of all end vertices of $G$, is a $(G, D)$-set of $G$, then $S$ is the unique minimum $(G, D)$-set of $G$.

Theorem 2.5.5. For every positive integer $k \geq 2$, there exists a graph $G$ with $\gamma_G(G) = k$.

Proof. Let $k \geq 2$. Consider the complete bipartite graph $K_{1,k}$. Let $S = V(K_{1,k}) - \{v\}$ where $v$ is the central vertex. Obviously, $S$ is a $(G, D)$-set of $G$. Therefore, by Remark 2.5.4, $S$ is the unique minimum $(G, D)$-set of $K_{1,k}$ and so $\gamma_G(K_{1,k}) = k$. □
Theorem 2.5.6. For every pair \( k, p \) of integers such that \( 2 \leq k \leq p \), there exists a connected graph \( G \) of order \( p \) such that \( \gamma_G(G) = k \).

Proof. As \( \gamma_G(K_p) = p \), the result is true when \( k = p \). For \( k = 2 \), the result is true when \( G \cong P_3 \). Let \( 2 < k < p \). Consider the graph \( G \) as in figure 2.7,

![Figure 2.7](image)

where \( m = p - (k + 1) \). Here, \( u \) and \( v \) are vertices of \( G \) such that there are exactly \( m \) paths of length 2 connecting \( u \) and \( v \). Let \( x_1, x_2, \ldots, x_m \) be the internal vertices of these paths. Let \( s \) and \( t \) be positive integers such that \( 1 \leq s, t \leq k - 2 \) and \( s + t = k - 1 \). Attach \( s \) end vertices \( v_1, v_2, \ldots, v_s \) to \( u \) and \( t \) end vertices \( w_1, w_2, \ldots, w_t \) to \( v \). Then,

Number of vertices of \( G = s + t + 2 + m = k - 1 + 2 + p - k - 1 = p \).

Further, \( \{v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t, u\} \) is a minimum \( (G, D) \)-set of \( G \) and so \( \gamma_G(G) = k \). □
Observation 2.5.7. Let $G = (V, E)$ be any graph.

1. Let $S$ be a minimum geodetic set of $G$. If $S$ is not a dominating set of $G$, then $D = S \cup T$, where $T$ is a minimum dominating set of the subgraph of $G$ induced by $V - N[S]$, is a $(G, D)$-set of $G$.

2. The $(G, D)$-set $D$ constructed above need not be minimal. (See Example 2.5.8).

3. Let $D$ be the set as in (1). Then, $D$ is a minimal $(G, D)$-set of $G$ if and only if for every $x \in D$, either $pn[x, D] \neq \varnothing$ or $D - \{x\}$ is not a geodetic set of $G$.

Example 2.5.8. Considering the graph in figure 2.8,

![Figure 2.8](image)

$S = \{u, v, x, y\}$ is a minimum geodetic set of $G$ and $T = \{u', v'\}$ is a minimum dominating set of the subgraph of $G$ induced by $V - N[S]$.
Then, \( D = \{u, v, x, y, u', v'\} \) is a \((G, D)\)-set of \( G \). But, \( D \) is not a minimal \((G, D)\)-set of \( G \) since \( D - \{x\} \) is a \((G, D)\)-set of \( G \).

### 2.6 Graphs with given \((G, D)\)-number.

**Proposition 2.6.1.** Let \( G = (V, E) \) be a connected graph on \( p \) vertices. Then, \( \gamma_G(G) = p \) if and only if \( g(G) = p \).

**Proof.** Let \( \gamma_G(G) = p \). Assume that \( g(G) \neq p \). Then, there exist at least two vertices \( u \) and \( v \) such that \( d(u, v) \geq 2 \). Let \( u' \) and \( v' \) (not necessarily distinct) be the vertices adjacent to \( u \) and \( v \) respectively in a \( u - v \) geodesic. This implies \( V - \{u', v'\} \) is a \((G, D)\)-set of \( G \). It contradicts our hypothesis that \( \gamma_G(G) = p \). Thus, \( g(G) = p \). The converse follows from Remark 2.2.5.

**Corollary 2.6.2.** Let \( G = (V, E) \) be a connected graph on \( p \) vertices. Then, \( \gamma_G(G) = p \) if and only if \( G \) is complete.

**Proof.** The result follows from Theorem 1.52 and Proposition 2.6.1.

**Theorem 2.6.3.** Let \( G = (V, E) \) be a connected graph on \( p \) vertices. If \( g(G) = p - 1 \), then \( \gamma_G(G) = p - 1 \).

**Proof.** Suppose \( g(G) = p - 1 \). Then, by Remark 2.2.5, \( \gamma_G(G) = p \) or \( p - 1 \). Therefore, by using Theorem 2.6.1, \( \gamma_G(G) = p - 1 \).
Lemma 2.6.4. If $g(G) = p - 2$, then $\gamma_G(G) = p - 2$.

Proof. Let $S$ be a minimum geodetic set of $G$. As, $g(G) = p - 2$, $|V - S| = 2$. Let $V - S = \{u, v\}$. Since $S$ is geodetic, there exists $x, y \in S$ such that an $x - y$ geodesic $P$ of length atmost 3 contains $u$. So, $u$ is adjacent to either $x$ or $y$. Similarly, $v$ is adjacent to atleast one vertex of $S$. Therefore, $S$ is also a dominating set of $G$. By Observation 2.4.1, $S$ is a minimum $(G, D)$-set of $G$ and hence $\gamma_G(G) = p - 2$. □

Remark 2.6.5. Converse of the above lemma is not true. For example, $g(P_5) = 2 = p - 3$ and $\gamma_G(P_5) = 3 = p - 2$.

Lemma 2.6.6. Let $G$ be a $(p,q)$ graph and $k \geq 3$. If $g(G) = p - k$, then $\gamma_G(G) \leq p - 2$.

Proof. Let $S$ be a minimum $(G, D)$-set of $G$. Then, $|V - S| = k$. As $S$ is geodetic, every vertex of $V - S$ lies in a geodesic connecting two vertices of $S$ of length atmost $k + 1$. So, atleast two vertices of $V - S$ are dominated by the vertices of $S$. Now, the remaining $k - 2$ vertices of $V - S$ together with $S$ constitute a $(G, D)$-set of $G$. Therefore, $\gamma_G(G) \leq p - k + k - 2 = p - 2$. Hence, $\gamma_G(G) \leq p - 2$. □

Corollary 2.6.7. If $g(G) = p - 3$, then $\gamma_G(G) = p - 3$ or $p - 2$.

Proof. The result follows from Remark 2.2.5 and Lemma 2.6.6. □
Theorem 2.6.8. If $\gamma_G(G) = p - 1$, then $g(G) = p - 1$.

Proof. Suppose $\gamma_G(G) = p - 1$. Then, by Remark 2.2.5, $g(G) \leq p - 1$. Suppose $g(G) < p - 1$. Then, by Lemma 2.6.4 and Lemma 2.6.6, $\gamma_G(G) \leq p - 2$. This is a contradiction. Hence, $g(G) = p - 1$.

Corollary 2.6.9. Let $G$ be a $(p, q)$ graph. Then, $g(G) = p - 1$ if and only if $\gamma_G(G) = p - 1$.

Proof. The proof is immediate from Lemma 2.6.4 and by Theorem 2.6.8.

Corollary 2.6.10. A connected graph $G$ of order $p(\geq 3)$ has $(G, D)$-number $p - 1$ if and only if $G$ is the join of $K_1$ and the union of at least two disjoint complete graphs. That is, $G \cong (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$, $r \geq 2$ for positive integers $n_1, n_2, \ldots, n_r$ satisfying the condition $n_1 + n_2 + \cdots + n_r = p - 1$.

Proof. The result follows from corollary 2.6.9 and by Theorem 1.57.

2.7 Results connecting domination, geodetic and $(G, D)$-number of a graph.

In this section, we find graphs for which all the three parameters $\gamma, g$ and $\gamma_G$ are equal. Also, we prove for any given set of 3 positive
integers $a, b$ and $c$ with $b \geq 2$, and $\max\{a, b\} \leq c \leq a + b$, there exists a graph $G$ such that $\gamma(G) = a$, $g(G) = b$ and $\gamma_G(G) = c$.

**Lemma 2.7.1.** Given a positive integer $k \geq 2$, there exists a graph $G$ with $\gamma(G) = g(G) = \gamma_G(G) = k$.

**Proof.** Consider the graph $G$ in figure 2.9.

![Figure 2.9](image)

Clearly, $\{u_0, u_3, w_1, w_2, \ldots, w_{k-2}\}$ is a $\gamma$-set, $g$-set and $\gamma_G$-set of $G$ and so $\gamma(G) = g(G) = \gamma_G(G) = k$. \qed

**Lemma 2.7.2.** Given two positive integers $a$ and $b$ with $b > a$, there exists a graph $G$ such that $\gamma(G) = a$ and $\gamma_G(G) = g(G) = b$.

**Proof.** Let $a = 1$ and $b > a$. The star graph $K_{1, b}$ satisfies the required condition.

Let $a = 2$ and $b > a$. Let $s$ and $t$ be two positive integers such that
$s + t = b$. Considering $G$ in figure 2.10,

![Figure 2.10]

{$u, v$} is a minimum dominating set of $G$ and so $\gamma(G) = 2$. Further, 
{$v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t$} is a minimum geodetic and minimum 
($G, D$)-set of $G$ and hence $\gamma_G(G) = g(G) = s + t = b$.

![Figure 2.11]

Let $a > 2$ and $b > a$. Let $P_a : (u_1, u_2, \ldots, u_a)$. Choose positive 
integers $s_1, s_2, \ldots, s_a$ such that $s_1 + s_2 + \cdots + s_a = b$. Join $s_1, s_2, \ldots, s_a$
pendant vertices respectively to the vertices $u_1, u_2, \ldots, u_a$ of $P_a$. 

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The resulting graph appears as in figure 2.11. Clearly, \( \{u_1, u_2, \ldots, u_a\} \) is a minimum dominating set of \( G \) and so \( \gamma(G) = a \) and \( \{u_{11}, u_{12}, \ldots, u_{1s_1}, u_{21}, u_{22}, \ldots, u_{2s_2}, \ldots, u_{a1}, u_{a2}, \ldots, u_{as_a}\} \) is a \( g \)-set and \( \gamma_G \)-set of \( G \) and hence \( \gamma_G(G) = g(G) = b \).

**Remark 2.7.3.** From Remark 2.2.5, we observe the following result. Given 2 positive integers \( a, b \) with \( a > 2 \) and \( b < a \), there exists no graph with \( \gamma(G) = a \) and \( \gamma_G(G) = g(G) = b \).

**Lemma 2.7.4.** Given 2 positive integers \( a, b \) with \( 2 \leq b < a \), there exists a graph \( G \) with \( \gamma_G(G) = \gamma(G) = a \) and \( g(G) = b \).

**Proof.** For \( b = 2 \), \( P_{3a-2} \) satisfies the required condition since

1. The set of end vertices of \( P_{3a-2} \) is the unique minimum geodetic set of \( P_{3a-2} \) and so \( g(P_{3a-2}) = 2 = b \).

2. By Theorem 1.38, \( \gamma(P_{3a-2}) = \left\lceil \frac{3a-2}{3} \right\rceil = a \).

3. Further, \( 3a - 2 \geq 5 \) as \( a > 2 \). Therefore, by Theorem 2.2.9,
   \[ \gamma_G(P_{3a-2}) = \left\lceil \frac{(3a-2)-4}{3} \right\rceil + 2 = \left\lceil \frac{3a-6}{3} \right\rceil + 2 = a - 2 + 2 = a. \]

Let \( b > 2 \). Let \( P_{3a-2b+1} = (v_1, v_2, \ldots, v_{3a-2b+1}) \). As \( a > b \), \( 3a - 2b + 1 > b \). Let \( G \) be a graph as in figure 2.12 which is obtained by attaching \( b - 1 \) pendant vertices \( w_1, w_2, \ldots, w_{b-1} \) respectively to \( v_1, v_2, \ldots, v_{b-1} \) of the path \( P_{3a-2b+1} \).
\{w_1, w_2, \ldots, w_{b-1}, v_{3a-2b+1}\} \text{ is a minimum geodetic set of } G \text{ and so } g(G) = b.

The set \( S = \{v_1, v_2, \ldots, v_{b-1}\} \) together with any minimum dominating set of the path \( P' = (v_{b+1}, v_{b+2}, \ldots, v_{3a-2b+1}) \) forms a minimum dominating set of \( G \).

Number of vertices in \( P' = 3a - 2b + 1 - b = 3(a - b) + 1 \) and so by Theorem 1.38, \( \gamma(G) = b - 1 + \gamma(P') = b - 1 + \left\lceil \frac{3(a-b)+1}{3} \right\rceil = b - 1 + a - b + 1 = a. \)

![Figure 2.12](image)

Further, \( \{w_1, w_2, \ldots, w_{b-1}, v_{3a-2b+1}\} \) along with any minimum dominating set of the path \( P'' = (v_b, v_{b+1}, v_{b+2}, \ldots, v_{3a-2b-1}) \) forms a minimum \((G, D)\)-set of \( G \).

Number of vertices in \( P'' = 3a - 2b - 1 - (b - 1) = 3(a - b) \).

Therefore, \( \gamma_G(G) = b + \gamma(P'') = b + \left\lceil \frac{3(a-b)}{3} \right\rceil = b + (a - b) = a. \) \( \square \)

**Remark 2.7.5.** As in Remark 2.7.3, for any 2 positive integers \( a, b \)
with \( a < b \), there exists no graph \( G \) with \( g(G) = b \) and \( \gamma(G) = \gamma_G(G) = a \).

\[ \square \]

Theorem 2.7.6. For three positive integers \( a, b \) and \( c \) with \( b \geq 2 \) and \( \max\{a, b\} \leq c \leq a + b \), there exists a graph \( G \) with \( \gamma(G) = a \), \( g(G) = b \) and \( \gamma_G(G) = c \).

Proof. Let \( a, b \) and \( c \) be three positive integers satisfying the given condition.

Case 1: Suppose \( c = \max\{a, b\} \)

If \( b = a \), then the result follows from Lemma 2.7.1

If \( b > a \), then the result follows from Lemma 2.7.2.

If \( b < a \), then the result follows from Lemma 2.7.4.

Case 2: Let \( c > \max\{a, b\} \).

Subcase 2a: Suppose \( a = b \).

\( a + 1 \leq c \leq a + b = 2a \). Let \( r = c - a \). Then, \( 1 \leq r \leq a \).

Suppose \( 1 \leq r < a \). Considering the graph \( G \) as in figure 2.13,

\[
V(G) = \{v_1, v_2, \ldots, v_a, w_1, w_2, \ldots, w_a, x_{i1}, x_{i2} : 1 \leq i \leq r\} \quad \text{and}
E(G) = \{v_iw_i : 1 \leq i \leq a\} \cup \{x_{i1}x_{i2} : 1 \leq i \leq r\} \cup \{w_ix_{i1} : 1 \leq i \leq r\} \cup \{x_{i2}w_{i+1} : 1 \leq i \leq r\} \cup \{w_iw_{i+1} : r+1 \leq i \leq a-1\}.
\]

It is easy to see that \( \{w_1, w_2, \ldots, w_a\} \) and \( \{v_1, v_2, \ldots, v_a\} \) are the minimum dominating and minimum geodetic sets of \( G \) respectively.

Therefore, \( \gamma(G) = g(G) = a \). Further, \( \{v_1, v_2, \ldots, v_a, x_{11}, x_{21}, \ldots, x_{r1}\} \)
is a minimum \((G, D)\)-set of \(G\) and so \(\gamma_G(G) = a + r = c\).

![Figure 2.13](image)

Suppose \(r = a\). Considering the graph \(G\) as in figure 2.14, \(V(G) = \{v_1, v_2, \ldots, v_a, w_1, w_2, \ldots, w_a, x_{i1}, x_{i2}, y_{i1}, y_{i2} : 1 \leq i \leq a - 1\}\) and \(E(G) = \{v_iw_i : 1 \leq i \leq a\} \cup \{x_{i1}x_{i2}, y_{i1}y_{i2} : 1 \leq i \leq a - 1\} \cup \{w_ix_{i1}, w_iy_{i1} : 1 \leq i \leq a - 1\} \cup \{x_{i2}w_{i+1}, y_{i2}w_{i+1} : 1 \leq i \leq a - 1\}\).

![Figure 2.14](image)

As \(\{w_1, w_2, \ldots, w_a\}\) and \(\{v_1, v_2, \ldots, v_a\}\) are minimum dominating set and minimum geodetic set of \(G\) respectively, \(\gamma(G) = g(G) = a\).
Further, $\gamma_G(G) = 2a$, as $\{v_1, v_2, \ldots, v_a, w_1, w_2, \ldots, w_a\}$ is a minimum $(G, D)$-set of $G$.

**Subcase 2b:** Suppose $a \neq b$. Then, $a, b$ and $c$ are distinct. Let $r = a + b - c$. As, $c \leq a + b$, $r \geq 0$. Suppose $r > 0$. Let $P$ be the path $(u_0, u_1, \ldots, u_{3(a-r)})$. Let $s$ and $t$ be positive integers such that $s + t = b - (r - 1)$. Add $s$ and $t$ pendant vertices $v_1, v_2, \ldots, v_s$ and $w_1, w_2, \ldots, w_t$ respectively to the vertices $u_0$ and $u_{3(a-r)}$ of the path $P$ to get the graph $G$. Add $r - 1$ paths of length 4 from $u_0$ to $u_2$, say, $(u_0, x_1, x_2, x_3, u_2), (u_0, x_4, x_5, x_6, u_2), \ldots, (u_0, x_{3(r-2)+1}, x_{3(r-2)+2}, x_{3(r-1)}, u_2)$.[See figure 2.15]

![Figure 2.15](image)

Let $D_1$ be any minimum dominating set of $P$ containing the end vertices $u_0$ and $u_{3(a-r)}$ and let $D$ be any minimum dominating set of $P$. 45
Number of vertices of the path $P = 3(a - r) + 1 \equiv 1 \pmod{3}$. Hence, $|D_1| = |D| = \gamma(P) = \left\lceil \frac{3(a-r)+1}{3} \right\rceil = a - r + 1$. It is easy to see that $D_1 \cup S_1$, where $S_1 = \{x_2, x_5, \ldots, x_{3(r-2)+2}\}$ forms a minimum dominating set of $G$. Therefore, $\gamma(G) = a - r + 1 + |S_1| = a - r + 1 + r - 1 = a$. Further, $S_2 = S_1 \cup \{v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t\}$ is a minimum geodetic set of $G$ and so $g(G) = s + t + r - 1 = b - (r - 1) + (r - 1) = b$. Also, $S_2$ together with any minimum dominating set of the path $(u_1, u_2, \ldots, u_{3(a-r)-1})$ forms a minimum $(G, D)$-set of $G$. Therefore, $\gamma_G(G) = b + \left\lceil \frac{(3(a-r)-1)}{3} \right\rceil = b + (a - r) = c$, as $a + b - c = r$.

Let $r = 0$. Then, $c = a + b$. Considering the graph $G$ in figure 2.16, $V(G) = \{v_1, v_2, \ldots, v_{3(a-1)}, v_{3(a-1)+1}\} \cup \{x_1, x_2, y_1, y_2\} \cup \{u_1, u_2, \ldots, u_s, w_1, w_2, \ldots, w_t\}$ and $E(G) = \{v_1u_i : 1 \leq i \leq s\} \cup \{v_3(a-1)+1w_i : 1 \leq i \leq t\} \cup \{v_iv_{i+1} : 1 \leq i \leq 3(a - 1)\} \cup \{v_1x_1, x_1x_2, x_2x_4\} \cup \{v_{3(a-1)-2}y_1, y_1y_2, y_2v_{3(a-1)+1}\}$, where $s + t = b$.

Let $S$ be a minimum dominating set of the path $(v_1, v_2, \ldots, v_{3(a-1)+1})$ containing the end vertices. $S$ is also a minimum dominating set of $G$ and so $\gamma(G) = |S|$. Further, $3(a - 1) + 1 \equiv 1 \pmod{3}$. Therefore, $\gamma(G) = |S| = \left\lceil \frac{3(a-1)+1}{3} \right\rceil = a - 1 + 1 = a$.

The set of end vertices of $G$ forms a minimum geodetic set of $G$ and so $g(G) = s + t = b$. 

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As the set of all end vertices together with \( \{v_1, v_4, v_7, \ldots, v_{3(a-1)+1}\} \) forms a minimum \((G, D)\)-set of \(G\), \(\gamma_G(G) = b + a = c\).

\[\text{Figure 2.16}\]

Hence, it is proved that for three positive integers \(a, b\) and \(c\) with \(b \geq 2\) and \(\max\{a, b\} \leq c \leq a + b\), there exists a graph \(G\) with \(\gamma(G) = a\), \(g(G) = b\) and \(\gamma_G(G) = c\). \[\square\]

### 2.8 Results connecting Diameter and \((G, D)\) number of a graph:

**Theorem 2.8.1.** If \(\text{diam } G = 1, 2\) or 3, then \(\gamma_G(G) = g(G)\).

**Proof.** If \(\text{diam } G = 1\), then \(G \cong K_p\) and \(\gamma_G(G) = p = g(G)\).

Suppose \(\text{diam } G = 2\) or 3, then every geodetic set of \(G\) is also a dominating set of \(G\). Therefore, every minimum geodetic set of \(G\) is
also a $(G, D)$-set of $G$ and so $\gamma_G(G) \leq g(G)$. Hence, by Remark 2.2.5, $\gamma_G(G) = g(G)$.

Remark 2.8.2. Converse of the above proposition need not be true since any caterpillar $T$ constructed from a path of length $n > 3$ in which every vertex is either a support vertex or a pendant vertex has diameter greater than 3 and $\gamma_G(T) = g(T)$.

Theorem 2.8.3. Let $G$ be a non-complete connected graph with $p(\geq 4)$ vertices. If $\text{diam} \ G \geq 3$, then $\gamma_G(G) \leq p - 2$.

Proof. Let $u, v$ be a pair of antipodal vertices of $G$ and let $P$ be a path of length $d$ between $u$ and $v$, where $d = \text{diam} \ G$. If $u'$ and $v'$ are the vertices adjacent to $u$ and $v$ respectively in $P$, then $V - \{u', v'\}$ is a $(G, D)$-set of $G$. Therefore, $\gamma_G(G) \leq p - 2$. □

Theorem 2.8.4. Let $G$ be a non-complete connected Graph with $p \geq 3$ vertices. Then, $\gamma_G(G) = 2$ if and only if $G$ satisfies the following two conditions.

(i) $\text{diam} \ G < 4$.

(ii) $G$ contains 2 antipodal vertices $u$ and $v$ such that every vertex of $V - \{u, v\}$ lies in some $u$-$v$ geodesic.

Proof. Every vertex of $V - \{u, v\}$ lies in some $u$-$v$ geodesic implies $S = \{u, v\}$ is a geodetic set of $G$. Also, $\text{diam} \ G < 4$ implies
$S$ dominates all the vertices of $V - S$. Hence, $S$ is a $(G, D)$-set of $G$ so that $\gamma_G(G) \leq 2$. But, always, $\gamma_G(G) \geq g(G) \geq 2$. Therefore, $\gamma_G(G) = 2$.

Conversely, suppose $\gamma_G(G) = 2$. Let $S = \{u, v\}$ be a minimum $(G, D)$-set of $G$. Now, $S$ is geodetic implies every vertex of $V - S$ must lie in some $u - v$ geodesic and further $S$ is a dominating set implies every vertex of $V - S$ is dominated by $S$. Hence, $u$ and $v$ are antipodal vertices of $G$ and so $d(u, v) = \text{diameter}$ of $G$.

If $\text{diam } G \geq 4$, every geodesic $P$ joining $u$ and $v$ must be of length greater than or equal to 4 and so there exists at least one vertex in $V - S$ which is not dominated by $S$. This is a contradiction to the fact that $S$ is a $\gamma_G$-set of $G$. Hence the result. \hfill \Box

**Theorem 2.8.5.** Let $G$ be a connected graph with $p(\geq 2)$ vertices. Then, $\gamma_G(G) = 2$ if and only if $G$ is isomorphic to one of the following types of graphs.

i. $P_2, P_3$ or $P_4$.

ii. $C_4$ or $C_6$.

iii $G$ contains $P_{2,p-2}$ or $P_{3,p-2}$ with $u, v$ as a pair of antipodal vertices of $V(G)$ such that the subgraph induced by $V(G) - \{u, v\}$ is isomorphic to a spanning subgraph of $K_{p-2}$. 

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Proof. If $G$ is one of the above three types of graphs then $\text{diam } G < 4$ and $G$ contains two antipodal vertices $u$ and $v$ such that every vertex of $V - \{u, v\}$ lies in some $u - v$ geodesic. Therefore by Theorem 2.8.4, $\gamma_G(G) = 2$.

Conversely, Suppose $\gamma_G(G) = 2$. Then, $G$ satisfies conditions (i) and (ii) of Theorem 2.8.4. If $\text{diam } G = 1$, then $G \cong K_2 = P_2$. If $\text{diam } G = 2$, then $G \cong P_3$ or $C_4$ or $G$ contains $P_{2p-2}$ with $u, v$ as a pair of antipodal vertices of $V(G)$ such that the subgraph induced by $V(G) - \{u, v\}$ is isomorphic to a spanning subgraph of $K_{p-2}$. If $\text{diam } G = 3$, then $G \cong P_4$ or $C_6$ or $G$ contains $P_{3, \frac{p-2}{2}}$ with $u, v$ as a pair of antipodal vertices of $V(G)$ such that the subgraph induced by $V(G) - \{u, v\}$ is isomorphic to a spanning subgraph of $K_{p-2}$. □

When a number of edges are added to a graph $G$, then there is a decrease in the diameter of $G$ and when a number of edges are deleted from a graph $G$, then there is an increase in the diameter of $G$. But, in both the cases, one could not predict the behavior of $(G, D)$-number of $G$.

For example, $\text{diam } (C_8) = 4$ and $\gamma_G(C_8) = 3$. The graphs $G_1, G_2$ and $G_3$ in figure 2.18 (a, b and c) are obtained from $C_8$ by addition of edges. All have diameters less than the diameter of $C_8$.  

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But, $\gamma_G(G_1) = 2 < \gamma_G(C_8)$, $\gamma_G(G_2) = 4 > \gamma_G(C_8)$ and $\gamma_G(G_3) = 3 = \gamma_G(C_8)$.

$C_8$:

$\gamma_G(C_8) = 2 + \lceil \frac{8-4}{3} \rceil = 4.$

$G_1$:

In Figure 2.18(a), $\text{Diam } (C_8) = 4$.

$\gamma_G(C_8) = 2 + \lceil \frac{8-4}{3} \rceil = 4.$

In Figure 2.18(a), $\text{Diam } (G_1) = 3 < \text{diam } G$.

As $\{v_1, v_5\}$ is a minimum $(G, D)$-set of $G_1$, $\gamma_G(G_1) = 2 < \gamma_G(G)$. 

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In Figure 2.18(b), Diam \((G_2) = 3 < \text{diam } G\).
As \(\{v_1, v_4, v_5, v_7\}\) is a minimum \((G, D)\)-set of \(G_2\), \(\gamma_G(G_2) = 4 > \gamma_G(G)\).

In Figure 2.18(c), Diam \((G_3) = 2 < \text{diam } G\).
As \(\{v_2, v_4, v_7\}\) is a minimum \((G, D)\)-set of \(G_3\), \(\gamma_G(G_3) = 3 = \gamma_G(G)\).

The graphs \(G_1, G_2\) and \(G_3\) in figure 2.20 \((a, b \text{ and } c)\) are obtained from \(G\) (figure 2.19) by removal of edges. Their diameters are greater than the diameter of \(G\). But, \(\gamma_G(G_1) = 2 < 3 = \gamma_G(G)\),
\[ \gamma_G(G_2) = 4 > \gamma_G(G) \] and \[ \gamma_G(G_3) = 3 = \gamma_G(G). \]

In Figure 2.19, Diam \((G) = 2.\)

As \(\{v_3, v_8, v_{10}\}\) is a minimum \((G, D)\)-set of \(G,\) \(\gamma_G(G) = 3.\)

In Figure 2.20(a), Diam \((G_1) = 5 > \text{diam } G.\)

As \(\{v_2, v_4, v_7, v_{10}\}\) is a minimum \((G, D)\)-set of \(G_1,\)
\[ \gamma_G(G_1) = 4 > \gamma_G(G). \]

In Figure 2.20(b), Diam \((G_2) = 3 > \text{diam } G.\)

As \(\{v_1, v_6\}\) is a minimum \((G, D)\)-set of \(G_2,\) \(\gamma_G(G_2) = 2 < \gamma_G(G).\)

In Figure 2.20(c), Diam \((G_3) = 4 > \text{diam } G.\)

As \(\{v_3, v_6, v_{10}\}\) is a minimum \((G, D)\)-set of \(G_3,\) \(\gamma_G(G_3) = 3 = \gamma_G(G).\)
Figure 2.20(a)

Figure 2.20(b)

Figure 2.20(c)
2.9 On \((G, D)\)-number of edge added graphs.

Here, it is studied that how the \((G, D)\)-number of a non-complete connected graph is affected by the addition of a single vertex with one edge incident with this vertex. Throughout this section, \(GoK_2\) represents the graph obtained from \(G\) by adjoining an edge with some vertex of \(G\).

**Theorem 2.9.1.** Let \(G\) be any non-complete connected graph. Let \(G' = G \circ K_2\) be a graph obtained from \(G\) by adjoining an edge with some vertex of \(G\). Then, \(\gamma_G(G) \leq \gamma_G(G') \leq \gamma_G(G) + 1\).

**Proof.** Let \(G' = G \circ K_2\). Let \(V(G'') = V(G) \cup \{u\}\) and \(E(G') = E(G) \cup \{uv\}\) for some \(v \in V(G)\). If \(S\) is any minimum \((G, D)\)-set of \(G\), then \(S \cup \{u\}\) is a \((G, D)\)-set of \(G'\). Therefore, \(\gamma_G(G') \leq |S \cup \{u\}| = |S| + 1 = \gamma_G(G) + 1\).

To prove the lower bound assume the contradiction that \(\gamma_G(G) > \gamma_G(G')\). Let \(S'\) be a \((G, D)\)-set of \(G'\) with \(|S'| \leq \gamma_G(G) - 1\). Since \(u\) is an end vertex of \(G'\), \(u \in S'\). Then, there are two cases.

If \(v \in S'\), then clearly, \(S'' = S' - \{u\}\) is a \((G, D)\)-set of \(G\) so that \(\gamma_G(G) \leq |S''| = |S' - \{u\}| \leq \gamma_G(G) - 2\) which is a contradiction.

If \(v \notin S'\), then \(S'' = (S' - \{u\}) \cup \{v\}\) is such that \(S'' \subseteq V(G)\) and \(|S''| = |S' - \{u\}| \leq \gamma_G(G) - 1\).

**Claim:** \(S''\) is a \((G, D)\)-set of \(G\).

As \(S'\) is a \((G, D)\)-set of \(G'\), \(S'\) is both a geodetic and a dominating set.
of $G'$. Let $x \in V(G) - S''$. Then, $x \in V(G) - S'$. Since $S'$ is a geodetic set of $G'$, $x$ lies in some $s - t$ geodesic $P$ in $G'$ for some $s, t \in S'$.

**Case 1:** Suppose neither $s$ nor $t$ is $u$.

In this case, $s, t \in S''$ and so $x$ lies on some $s - t$ geodesic $P$ in $G$ for some $s, t \in S''$.

**Case 2:** Suppose one of $s$ or $t$, say, $s$, is $u$.

Then, $x$ lies in the $v - t$ geodesic $P' = P - \{u\}$ in $G$.

In both cases, every $x \in V(G) - S''$ lies in an $s - t$ geodesic of $G$ for some $s, t \in S''$ and so $S''$ is a geodetic set of $G$. Also, by construction, $S''$ is a dominating set of $G$. So, $S''$ is a $(G, D)$-set of $G$ and so $\gamma_G(G) \leq |S''| = \gamma_G(G) - 1$, which is a contradiction.

Therefore, $\gamma_G(G) \leq \gamma_G(G')$.

Hence, it is proved that $\gamma_G(G) \leq \gamma_G(G') \leq \gamma_G(G) + 1$. \hfill \square

**Remark 2.9.2.** In the above theorem, both the upper and lower bounds for $\gamma_G(G')$ are sharp.

For example, considering $G = P_8$, $G, G'$ and $G''$ are as in figures 2.21(a), 2.21(b) and 2.21(c) respectively.

$\{v_1, v_4, v_6, v_8\}$ is a $(G, D)$-set of $P_8$.

Therefore, $\gamma_G(P_8) = \lceil \frac{8-4}{3} \rceil + 2 = 2 + 2 = 4$.  

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\( \{v, v_1, v_4, v_6, v_8\} \) is a \((G, D)\)-set of \(G'\).

Therefore, \( \gamma_G(G') = 5 = \gamma_G(P_8) + 1 \).

\( \{v, v_1, v_5, v_8\} \) is a \((G, D)\)-set of \(G''\).

Therefore, \( \gamma_G(G'') = 4 = \gamma_G(P_8) \).

**Theorem 2.9.3.** If a vertex is joined by an edge to any vertex of \(P_n\), where \(n = 3k + 1\) and \(k \geq 1\), then for the resulting graph \(G' = P_n \circ K_2\),

\( \gamma_G(G') = \gamma_G(P_n) + 1 \).

**Proof.**

**Case 1:** Suppose \(G'\) is the graph obtained from \(P_n\) by adding an
edge to one of the end vertices of \( P_n \).

In this case, \( G' \cong P_{n+1} \). Therefore,

\[
\gamma_G(G') = \gamma_G(P_{n+1}) = \left\lceil \frac{(n+1)-4}{3} \right\rceil + 2 = \left\lceil \frac{3k+2-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-2}{3} \right\rceil + 2 = k + 2
\]

and

\[
\gamma_G(G) = \gamma_G(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-3}{3} \right\rceil + 2 = k - 1 + 2 = k + 1.
\]

Hence, \( \gamma_G(G') = \gamma_G(G) + 1 \).

**Case 2:** Suppose \( G' \) is obtained by adding an edge to one of the internal vertices of \( P_n \).

In this case, the number of end vertices of \( G' \) is 3. Therefore, every minimum \((G,D)\)-set of \( G' \) contains these three end vertices. Clearly, any minimum dominating set of a path of \( n-4 \) or \( n-5 \) vertices along with these three end vertices forms a minimum \((G,D)\)-set of \( G' \) and so

\[
\gamma_G(G') = 3 + \left\lceil \frac{n-4}{3} \right\rceil = 3 + \left\lceil \frac{3k+1-4}{3} \right\rceil = 3 + k - 1 = k + 2 = \gamma_G(G) + 1,
\]

as \( \left\lceil \frac{n-4}{3} \right\rceil = \left\lceil \frac{n-5}{3} \right\rceil \) when \( n = 3k + 1 \).

\[\square\]

**Theorem 2.9.4.** Let \( G = P_n, \ n \geq 3, \) and let \( n = 3k \). If \( G' \) is obtained by adding an edge to one of the end vertices of \( G \), then \( \gamma_G(G') = \gamma_G(G) \) and if an edge is added to one of its internal vertices, then \( \gamma_G(G') = \gamma_G(G) + 1 \).

**Proof.**

**Case 1:** Suppose \( G' \) is obtained by adding an edge to one of the end vertices of \( G = P_n \).

In this case, \( G' \cong P_{n+1} \). Therefore,

\[
\gamma_G(G') = \gamma_G(P_{n+1}) = \left\lceil \frac{n+1-4}{3} \right\rceil + 2 = \left\lceil \frac{3k+1-4}{3} \right\rceil + 2 = k - 1 + 2 = k + 1,
\]

\[
\gamma_G(G) = \left\lceil \frac{n-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-4}{3} \right\rceil + 2 = \left\lceil \frac{3(k-1)-1}{3} \right\rceil + 2 = k - 1 + 2 = k + 1.
\]
Hence, \( \gamma_G(G') = \gamma_G(G) \).

**Case 2:** Suppose \( G' \) is obtained by adding an edge to any one of the internal vertices of \( G = P_n \).

As in Theorem 2.9.3, any minimum dominating set of a path on \( n-4 \) or \( n-5 \) vertices along with the three end vertices forms a minimum \((G, D)\)-set of \( G' \). Therefore, \( \gamma_G(G') = 3 + \left\lfloor \frac{n-4}{3} \right\rfloor = 3 + \left\lfloor \frac{3k-4}{3} \right\rfloor = 3 + \left\lfloor \frac{3(k-1)-1}{3} \right\rfloor = 3 + k - 1 = k + 2 = \gamma_G(G) + 1 \) as \( \left\lfloor \frac{n-4}{3} \right\rfloor = \left\lfloor \frac{n-5}{3} \right\rfloor \) when \( n = 3k \).

\[ \square \]

### 2.10 On \((G, D)\)-number of edge deleted graphs.

**Theorem 2.10.1.** For a complete graph \( K_p \), \( \gamma_G(K_p - \{e\}) = 2 \) for every edge \( e \) in \( K_p \).

**Proof.** Let \( e = uv \in E(K_p) \). Suppose \( S = \{u, v\} \), then for every vertex \( w \in V(K_p - e) - S \), there exists a \( u - v \) geodesic of length 2 containing \( w \). As \( \deg(u) = \deg(v) = p - 2 \) in \( K_p - e \), \( S \) is a \((G, D)\)-set of \( K_p - e \) and so \( \gamma_G(K_p - \{e\}) \leq 2 \). Hence, by Remark 2.2.5, \( \gamma_G(K_p - \{e\}) = 2 \).

\[ \square \]

**Theorem 2.10.2.** For \( p \geq 4 \),

\[ \gamma_G(K_p - \{e_1, e_2\}) = \begin{cases} 2 & \text{if } e_1 \text{ and } e_2 \text{ are non adjacent} \\ 3 & \text{otherwise} \end{cases} \]
Proof. Let \( e_1 = uv \) and \( e_2 = u'v' \).

Let \( G^* = K_p - \{e_1, e_2\} \)

**Case 1:** \( e_1 \) and \( e_2 \) are non-adjacent.

Let \( S = \{u, v\} \). In \( G^* \), \( \deg(u) = \deg(v) = p - 2 \). Proceeding as in Theorem 2.10.1, it is seen that \( S \) is a minimum \((G,D)\)-set of \( G \) and so \( \gamma_G(G^*) = 2 \).

**Case 2:** \( e_1 \) and \( e_2 \) are adjacent.

In this case, \( e_1 \) and \( e_2 \) have a common vertex, say, \( v = u' \). Let \( S = \{u, v, v'\} \). Then, every vertex in \( V(G^*) - S \) lies in a \( u - v \) geodesic of length 2. That is, every vertex in \( V(G^*) - S \) is both dominated and geodominated by the vertices of \( S \) and so \( S \) is a \((G,D)\)-set of \( G \).

Therefore, \( 2 \leq \gamma_G(G^*) \leq 3 \). \( \cdots \cdots \) (1)

**Claim:** \( \gamma_G(G^*) \neq 2 \).

Suppose \( S = \{x, y\} \) is a \((G,D)\)-set of \( G^* \). As \( p \geq 4 \), by the definition of \((G,D)\)-set, \( x \) and \( y \) are not adjacent in \( G^* \). Then, \( S \) is precisely either \( \{u, v\} \) or \( \{v, v'\} \). In both cases, there is a vertex in \( \{u, v, v'\} - S \), which is not geodominated by \( S \). Therefore, no two point set of \( G^* \) is a \((G,D)\)-set of \( G^* \).

Hence, \( \gamma_G(G^*) = 3 \). \[ \square \]

In the following, the \((G,D)\)-number of a complete graph after removing the edges of a path, cycle, clique or a star are obtained.

**Theorem 2.10.3.** Let \( p > 3 \). Suppose \( e_1, e_2, e_3 \in E(K_p) \) such that
they form a path in $K_p$. Then, $\gamma_G(G^*) = \begin{cases} 2 & \text{if } p = 4 \\ 3 & \text{otherwise} \end{cases}$

**Proof.** Let $P = (a, b, c, d)$ where $e_1 = ab$, $e_2 = bc$ and $e_3 = cd$.

**Case 1:** Let $p = 4$.

![Figure 2.22](image)

From the figure 2.22, it is easy to see that $S = \{b, c\}$ is a $(G, D)$-set of $G^*$ and so $\gamma_G(G^*) = 2$.

**Case 2:** Let $p > 4$.

Let $S = \{a, b, c\}$. In $G^*$, every vertex of $V(G^*) - S$ lies in some $a - b$ geodesic joining two vertices of $S$ of length 2 and so $S$ is a $(G, D)$-set of $G^*$. Therefore, $2 \leq \gamma_G(G^*) \leq 3$. –– (1)

**Claim:** $\gamma_G(G^*) \neq 2$.

The proof of this claim follows the same lines as in 2.10.2.

Hence, $\gamma_G(G^*) = 3$. □

**Theorem 2.10.4.** Let $G = K_p$, $p > 4$. Suppose $e_1, e_2, \ldots, e_k$ are in $E(G)$, where $4 \leq k < p - 1$, such that $\{e_1, e_2, \ldots, e_k\}$ forms a path of length $k$. Let $G^* = K_p - \{e_1, e_2, \ldots, e_k\}$. Then, $\gamma_G(G^*) = 3$.

**Proof.** Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Let $S = \{v_1, v_2, v_3\}$ and let
\[ P = (v_1, v_2, \ldots, v_{k+1}) \] with \( e_i = (v_i, v_{i+1}) \), \( 1 \leq i \leq k \). Every vertex of \( V(G^*) - S \) lies in a \( v_1 - v_2 \) geodesic of length 2 and so \( S \) is a \((G, D)\)-set of \( G^* \). Proceeding as in 2.10.2, we can prove that no two element subset of \( V(G^*) \) is a \((G, D)\)-set of \( G^* \). Hence, \( \gamma_{G^*}(G^*) = 3 \). \( \square \)

**Theorem 2.10.5.** Let \( p > 3 \) and \( 3 \leq k \leq p \). Let \( G^* = K_p - \{e_1, e_2, \ldots, e_k\} \), where \( \{e_1, e_2, \ldots, e_k\} \) forms a cycle of length \( k \) in \( K_p \). Then, the following are true.

1. If \( k = 3 \), then \( \gamma_G(G^*) = 3 \).
2. If \( k = 4 \), then \( \gamma_G(G^*) = 4 \).
3. If \( k \geq 5 \), then \( \gamma_G(G^*) = 3 \).

**Proof.**

Let \( C \) be a cycle of length \( k \) in \( K_p \).

**Case 1:** Let \( k = 3 \).

Let \( C = (a, b, c, a) \) where \( e_1 = ab \), \( e_2 = bc \) and \( e_3 = ca \). Suppose \( S = \{a, b, c\} \). Clearly, every vertex of \( V(G^*) - S \) is dominated by the elements of \( S \) and further it is covered by geodesics of length 2 in \( G^* \) joining two vertices of \( S \). Therefore, \( S \) is a \((G, D)\)-set of \( G^* \). Hence, \( 2 \leq \gamma_G(G^*) \leq 3 \).

**Claim:** \( \gamma_G(G^*) \neq 2 \).

Let \( S^* = \{x, y\} \) be a minimum \((G, D)\)-set of \( G^* \). As \( p > 3 \), \( S^* \) is a proper subset of \( V(G^*) \). By definition of \((G, D)\)-set, \( x \) and \( y \) are non-adjacent and clearly, \( S^* \) is precisely \( \{a, b\} \) or \( \{b, c\} \) or \( \{c, a\} \).
In all the three cases there is a vertex of \{a, b, c\} – \(S^*\) which is not geodominated by \(S^*\). Again, we get a contradiction. Therefore, \(\gamma_G(G^*) \neq 2\). Hence, \(\gamma_G(G^*) = 3\).

**Case 2:** Let \(k = 4\).

Let \(C = (a, b, c, d, a)\) where \(e_1 = ab, e_2 = bc, e_3 = cd\) and \(e_4 = da\).

If \(S = \{a, b, c, d\}\), then every vertex in \(V(G^*) - S\) is both dominated and geodominated by the vertices of \(S\) and so \(S\) is a \((G, D)\)-set of \(G^*\). Hence, \(2 \leq \gamma_G(G^*) \leq 4\).

**Claim:** There is no \((G, D)\)-set of \(G^*\) with 3 elements.

Let \(S^* = \{x, y, z\}\) be a \((G, D)\)-set of \(G^*\).

If \(S^*\) is a clique in \(G^*\), then \(S^*\) is not a \((G, D)\)-set of \(G^*\). If two elements of \(S^*\), say, \(x\) and \(y\) are non-adjacent, then \(\{x, y\}\) is either \(\{a, b\}\) or \(\{b, c\}\) or \(\{c, d\}\) or \(\{d, a\}\). Further, \(\{a, b, c, d\} - S^*\) is not geodominated by the vertices of \(S^*\). This is a contradiction.

If two pairs of the three vertices are adjacent, then \(S^*\) is either \(\{a, b, c\}\) or \(\{a, b, d\}\) or \(\{a, c, d\}\) or \(\{b, c, d\}\). Again, by the same reasoning, we get a contradiction.

Hence, there is no \((G, D)\)-set of \(G^*\) with 3 elements.

Further, as every super set of a \((G, D)\)-set is a \((G, D)\)-set, no two element subset of \(V(G^*)\) is a \((G, D)\)-set of \(G^*\).

Therefore, \(S\) is a minimum \((G, D)\)-set of \(G^*\) and so \(\gamma_G(G^*) = |S| = 4\).

**Case 3:** Let \(k \geq 5\).

Let \(C = (v_1, v_2, \ldots, v_k, v_1)\) where \(e_i = (v_i v_{i+1}), 1 \leq i \leq k - 1\) and
$e_k = (v_kv_1)$. If $S = \{v_1, v_2, v_3\}$, then every vertex in $V(G^*) - S$ is both dominated and geodominated by the vertices of $S$ and so $S$ is a $(G, D)$-set of $G^*$. Hence $2 \leq \gamma_G(G^*) \leq 3$. Proceeding as in case (i), we get $\gamma_G(G^*) = 3$. \hfill \Box$

**Theorem 2.10.6.** Let $G$ be a complete graph on $p(\geq 3)$ vertices. Let $G^*$ be a graph obtained from $G$ by removing the edges of a clique on $m (2 \leq m \leq p - 1)$ vertices in $G$. Then, $\gamma_G(G^*) = m$.

**Proof.** Let $H$ be a clique on $m$ vertices in $G$. If $S = V(H) = \{v_1, v_2, \ldots, v_m\}$, then the subgraph induced by $S$ in $G^*$ is totally disconnected and every vertex in $V(G^*) - S$ lies in a geodesic of length 2 joining two vertices of $S$ in $G^*$. Therefore, $S$ is a $(G, D)$-set of $G^*$ and so $2 \leq \gamma_G(G^*) \leq m$.

**Claim:** There is no $(G, D)$-set of $G^*$ with $m - 1$ elements.

Let $S^*$ be a $(G, D)$-set of $G^*$ with $m - 1 (< p)$ elements. If $S^*$ is a clique in $G^*$, then $S^*$ is not a $(G, D)$-set of $G^*$. So, $S^*$ contains at least 2 non-adjacent vertices. Since the subgraph induced by $S$ in $G^*$ is totally disconnected and $|S^*| = m - 1$, there is a vertex in $\{v_1, v_2, \ldots, v_m\} - S^*$ which is not geodominated by $S^*$. This is a contradiction to our assumption.

Therefore, there is no $(G, D)$-set of $G^*$ with $m - 1$ elements.

Since every super set of a $(G, D)$-set of $G^*$ is also a $(G, D)$-set of $G^*$, there is no $(G, D)$-set of $G^*$ with less than $m$ elements.

Hence, $\gamma_G(G^*) = m$. \hfill \Box
Theorem 2.10.7. If $G^*$ is the graph obtained from $K_p$ ($p \geq 3$) by removing the edges of a star with $k$ end vertices ($2 \leq k \leq p - 2$) in $K_p$, then $\gamma_G(G^*) = k + 1$.

Proof. Let $H$ be a star on $k + 1$ vertices in $G$. Let $S = V(H) = \{v, v_1, v_2, \ldots, v_k\} \subset V(K_p)$, where $v$ is the vertex of degree $k$ in $H$. In $G^*$, two vertices $x$ and $y$ are non-adjacent if and only if $\{x, y\} = \{v, v_i\}$ for some $i$, $1 \leq i \leq k$. Further, every vertex of $V(G^*) - S$ lies in a $v - v_i$ geodesic of length 2 in $G^*$ ($1 \leq i \leq k$). Hence, $S$ is a $(G, D)$-set of $G^*$ and so $2 \leq \gamma_G(G^*) \leq |S| = k + 1$.

Claim: There is no $(G, D)$-set of $G^*$ with $k$ elements. Suppose $S'$ is a $(G, D)$-set of $G^*$ with $k$ elements. If $S'$ is a clique, then $S'$ is not a $(G, D)$-set of $G^*$. Suppose $S'$ contains at least two non-adjacent vertices which is one of $\{v, v_i\}$, $1 \leq i \leq k$. Then, there is a vertex of $\{v, v_1, v_2, \ldots, v_k\} - S'$ which is not geodominated by $S'$. Therefore, $S'$ is not a $(G, D)$-set of $G^*$.

By Remark 2.2.5, every super set of a $(G, D)$-set of $G^*$ is a $(G, D)$-set of $G^*$. Therefore, $G^*$ contains no $(G, D)$-set of order less than $k + 1$. Hence, $\gamma_G(G^*) = k + 1$.  

\[ \square \]

2.11 On $(G, D)$-number of join of two graphs.

Theorem 2.11.1. If $G_1$ and $G_2$ are $(p_1, q_1)$ and $(p_2, q_2)$ complete graphs respectively, then $\gamma_G(G_1 + G_2) = p_1 + p_2 = \gamma_G(G_1) + \gamma_G(G_2)$.

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Proof. Since $G_1$ and $G_2$ are complete graphs, $G_1 + G_2$ is again a complete graph with $p_1 + p_2$ vertices. Hence, by Theorem 2.6.2, $\gamma_G(G_1 + G_2) = p_1 + p_2 = \gamma_G(G_1) + \gamma_G(G_2)$. □

Theorem 2.11.2. Let $G_1$ and $G_2$ be non-complete graphs (not necessarily connected). Then, $\gamma_G(G_1 + G_2) \leq 4$.

Proof. Let $x_1, y_1$ and $x_2, y_2$ be the pairs of non-adjacent vertices of $G_1$ and $G_2$ respectively. Then, in $G_1 + G_2$, every vertex $u$ of $V(G_1)$ lies in a $x_2 - y_2$ geodesic and further dominated by $x_2$ and $y_2$. Similarly, every vertex $v$ of $V(G_2)$ lies in a $x_1 - y_1$ geodesic and also dominated by $x_1$ and $y_1$ in $G_1 + G_2$. Hence, $S = \{x_1, y_1, x_2, y_2\}$ forms a $(G, D)$-set of $G_1 + G_2$ and so $\gamma_G(G_1 + G_2) \leq 4$. □

Remark 2.11.3. Equality holds in the above proposition if $G_1$ and $G_2$ are totally disconnected with at least 4 vertices. In this case, $G_1 + G_2$ is a complete bipartite graph $K_{m,n}$ with $m, n \geq 4$. Hence, by theorem 2.2.12, $\gamma_G(G_1 + G_2) = 4$.

Theorem 2.11.4. Let $G_1$ and $G_2$ be $(p_1, q_1)$ and $(p_2, q_2)$ non-complete connected graphs with $p_1, p_2 \geq 3$. Then, $\gamma_G(G_1 + G_2) = 2$ if and only if at least one of $G_1, G_2$, say, $G_1$ is a graph of diameter 2 and $\gamma_G(G_1) = 2$.

Proof. Suppose $\text{diam } G_1 = \gamma_G(G_1) = 2$. Let $S = \{u, v\}$ be a minimum $(G, D)$ set of $G_1$. Then, in $G_1 + G_2$, every vertex $x$ of $V(G_1 + G_2) - \{u, v\}$ lies in a $u-v$ geodesic and dominated by $u$.
and $v$. Therefore, $S$ forms a minimum $(G, D)$-set of $G_1 + G_2$ and so $\gamma_G(G_1 + G_2) = 2$.

Conversely, Suppose $\gamma_G(G_1 + G_2) = 2$. Let $S = \{u, v\}$ be a minimum $(G, D)$-set of $G_1 + G_2$. Then, $p_1, p_2 \geq 3$ implies $u$ and $v$ are non adjacent in $G_1 + G_2$. By the definition of join of $G_1$ and $G_2$, $u, v$ are either in $G_1$ or in $G_2$. Without loss of generality, assume that $u, v \in V(G_1)$. It is easy to see that $d(u, v) = 2$ in $G_1 + G_2$. Therefore, every vertex of $V(G_1 + G_2) - \{u, v\}$ lies in a $u-v$ geodesic of length 2 in $G_1 + G_2$. In particular, every vertex of $V(G_1) - \{u, v\}$ lies in a $u-v$ geodesic of length 2 in $G_1$. So, $d(u, v) = 2$ in $G_1$ and every vertex of $V(G_1) - \{u, v\}$ is adjacent to both $u$ and $v$. This implies $S$ is a minimum $(G, D)$-set of $G_1$ and $d(x, y) \leq 2$ for every $x, y \in V(G_1) - \{u, v\}$. Hence, $diam G_1 = \gamma_G(G_1) = 2$. \qed

Remark 2.11.5. The above result holds good even if one of $G_1, G_2$ is complete. For, if $G_1 = P_3$ and $G_2 = K_n$, then $\gamma_G(G_1 + G_2) = 2$.

Observation 2.11.6.

1. If $G_1$ and $G_2$ are non-complete connected graphs, then, $G_1 + G_2$ contains no extreme vertices.

2. Let $G$ be any non complete connected graph. Then, $v \in K_p + G$ is an extreme vertex of $K_p + G$ if and only if it is an extreme vertex of $G$.

3. $\text{Diam}(G_1 + G_2) \leq 2$ for any two graphs $G_1$ and $G_2$. But
diam\( (G_1 + G_2) = 2 \) if and only if at least one of \( G_1 \) and \( G_2 \) is not complete.

**Theorem 2.11.7.** Let \( G_1(p_1, q_1) \) and \( G_2(p_2, q_2) \) be two non-complete connected graphs with \( p_1, p_2 > 3 \). Then, \( \gamma_G(G_1 + G_2) = 3 \) if and only if the following conditions are true.

i. If \( \gamma_G(G_i) = 2 \), then \( \text{diam } G_i \neq 2 \).

ii For at least one \( i \), \( 1 \leq i \leq 2 \), there exists a subset \( S \subseteq V(G_i) \) such that \( |S| = 3 \) and every \( x \in V(G_i) - S \) lies in a \( u-v \) geodesic of length 2 for some \( u, v \in S \).

**Proof.** Suppose (i) and (ii) are true. Then, (i) implies \( \gamma_G(G_1 + G_2) \neq 2 \). (by Theorem 2.11.4). (ii) implies \( S \) is a \((G,D)\)-set of \( G_1 + G_2 \) so that \( \gamma_G(G_1 + G_2) \leq 3 \).

Hence, \( \gamma_G(G_1 + G_2) = 3 \).

Conversely, Suppose \( \gamma_G(G_1 + G_2) = 3 \). Then, by Theorem 2.11.4, condition (i) is true. It remains to prove (ii). Let \( S = \{u_1, u_2, u_3\} \) be a minimum \((G,D)\)-set of \( G_1 + G_2 \).

**Claim:** \( S \) satisfies (ii). As \( p_1, p_2 > 3 \), \( S \) contains at least two non-adjacent vertices, say, \( u_1, u_2 \). Then, by definition of \( G_1 + G_2 \), \( u_1, u_2 \in V(G_i) \) for some \( i \). Without loss of generality, assume that \( u_1, u_2 \in V(G_1) \). Further, \( \text{diam } (G_1 + G_2) = 2 \) implies every \( x \in V(G_1 + G_2) - S \) lies in a \( u-v \) geodesic of length 2 for some \( u, v \in S \). In particular, every \( x \in V(G_1) - S \) lies in a \( u-v \) geodesic
of length 2 for some \( u,v \in S \) in \( G_1 + G_2 \). But, two non adjacent vertices of \( S \) belongs to \( V(G_1) \) and 2.11.6 imply that every \( x \in V(G_1) \) lies in a \( u-v \) geodesic of length 2 for some \( u,v \in S \) in \( V(G_1) \). If \( u_3 \in V(G_2) \), then every \( x \) in \( V(G_1) \) lies in a \( u_1-u_2 \) geodesic of length two. In this case, \( \gamma_G(G_1) = 2 \) and any two vertices of \( G_1 \) are adjacent to both \( u_1 \) and \( u_2 \). This implies \( d(x,y) \leq 2 \) for every \( x,y \in V(G_1) \) so that \( diamG_1 = 2 \). This is a contradiction to (i). Hence, \( u_3 \) also is in \( V(G_1) \) and so \( S \subseteq V(G_1) \). Therefore, it is established that \( S \) satisfies condition (ii).

\[ \square \]

**Remark 2.11.8.** Theorem 2.11.7 is true even if one of \( G_1,G_2 \) is complete. (See figure 2.23)

![Diagram](image)

**G_1** (Figure 2.23)

Considering \( G_1 \) in figure 2.23 and \( G_2 = K_n \) with \( n \geq 3 \), \( \gamma_G(G_1 + G_2) = 3 \).

**Theorem 2.11.9.** \( \gamma_G(K_p + P_n) = \gamma_G(P_n) \) if and only if \( n = 3 \) or 5.
Proof. Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \). If \( n = 3 \), then \( \{v_1, v_3\} \) forms a minimum \((G, D)\)-set of both \( P_n \) and \( K_p + P_n \). If \( n = 5 \), then \( \{v_1, v_3, v_5\} \) is a minimum \((G, D)\)-set of both \( P_n \) and \( K_p + P_n \). Hence, if \( n = 3 \) or \( 5 \), then \( \gamma_{G}(K_p + P_n) = \gamma_{G}(P_n) \). Conversely, \( \{v_1, v_3, \ldots, v_{n-2}, v_n\} \) or \( \{v_1, v_3, \ldots, v_{n-1}, v_n\} \) is a minimum \((G, D)\)-set of \( K_p + P_n \) when \( n \) is odd or even. Therefore, \( \gamma_{G}(K_p + P_n) = \left\lceil \frac{n}{2} \right\rceil + 1 \).

By Theorem 2.2.9, \( \gamma_{G}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2 \) for \( n \geq 5 \).

\[ < \left\lceil \frac{n}{2} \right\rceil + 1 \text{ for } n > 5. \]

Therefore, \( \gamma_{G}(P_n) < \gamma_{G}(K_p + P_n) \), \( n > 5 \).

By Theorem 2.11.1, \( \gamma_{G}(K_p + P_2) = p + 2 > \gamma_{G}(P_2) \).

Further, \( \gamma_{G}(P_4) = 2 < 3 = \gamma_{G}(K_p + P_4) \). Therefore, for all \( n \) except 3 and 5, \( \gamma_{G}(P_n) \neq \gamma_{G}(K_p + P_n) \).

Hence, \( \gamma_{G}(K_p + P_n) = \gamma_{G}(P_n) \) if and only if \( n = 3 \) or \( 5 \). \( \square \)

Theorem 2.11.10. \( \gamma_{G}(K_p + C_n) = \left\lceil \frac{n}{2} \right\rceil \) for all \( n \geq 4 \).

Proof. Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \). It is observed that \( D = \{v_1, v_3, \ldots, v_{n-2}, v_n\} \) or \( \{v_1, v_3, \ldots, v_{n-1}\} \) is a minimum \((G, D)\)-set of \( K_p + C_n \), accordingly when \( n \) is odd or even for every \( n \geq 4 \). It follows that \( \gamma_{G}(K_p + C_n) = |D| = \left\lceil \frac{n}{2} \right\rceil \) for all \( n \geq 4 \). \( \square \)