Chapter 4

Inverse Domination in Grid graphs

This chapter deals about the inverse domination in grid graphs. Actually in the sections of this chapter we obtain the inverse domination number for grid graphs of low order.

4.1 Introduction

This chapter concerns the inverse domination in grid graphs. The domination number of $k \times n$ grid graphs $P_k \times P_n$, for $1 \leq k \leq 10; n \geq 1$ have been previously established by Jacobson and Kinch [22, 23], Tony Yu Chang, W. Edwin Clark and E. O. Hare [7]. In tune with the methodology established [7], in this chapter, we obtain the inverse domination number and an inverse dominating set for the graphs $P_k \times P_n; 1 \leq k \leq 7; n \geq 1$. Though $\gamma(P_n)$ is known, $\gamma(P_k \times P_n)$ could not be determined using $\gamma(P_k)$ and $\gamma(P_n)$. Hence Jacobson and
Kinch [22], attempted to find $\gamma(P_k \times P_n)$ and in their attempt they have obtained the domination number for the grid graphs $P_k \times P_n$ for $1 \leq k \leq 4; n \geq 1$.

Further T.Y. Chang, W. Edwin Clark, and E. O. Hare [7] give the dominating sets for the grid graphs $P_k \times P_n$ for $5 \leq k \leq 10$ and $n \geq 1$, through smaller grids $P_k \times P_m, m < n$. Without loss of generality one may assume that $n \geq k$. It may be noted that $P_1 \times P_n$ is nothing but $P_n$. T. Tamizh Chelvam and G.S. Grace Prema [28] have established that $\gamma(P_n) = \gamma'(P_n) = \lceil \frac{n}{3} \rceil$ if and only if $n \not\equiv 0(\text{mod } 3)$.

Consider a $2 \times 3$ grid graph. It may be worth while to note that $\gamma'(P_2 \times P_3) = \gamma'(P_2) \times \gamma'(P_3) = 2$. However this is not the case always. Note that $\gamma(P_2 \times P_4) = \gamma'(P_2 \times P_4) = 3$, and $\gamma'(P_2) = 1, \gamma'(P_4) = 2$ and so $\gamma'(P_2 \times P_4) \neq \gamma'(P_2) \times \gamma'(P_4)$.

As mentioned earlier, even though $\gamma'(P_n)$ is obtained by T. Tamizh Chelvam and G.S. Grace Prema [28], $\gamma'(P_k \times P_n)$ could not be determined. Hence, in this chapter, we attempt to find the inverse domination number for the grid graphs $P_k \times P_n$, for $1 \leq k \leq 7$ and $n \geq 1$ and an inverse dominating set for the
grid graphs through the smaller grids of $P_k \times P_m$, $(m < n)$. In
Section 3.2, we give an inverse dominating set and hence obtain
inverse domination number for $P_1 \times P_n$ grid graphs. In sequel
in the subsequent sections, we determine inverse domination
number and an inverse dominating set for grid graphs
$P_k \times P_n$ for $2 \leq k \leq 7$. We denote $\gamma(P_k \times P_n)$ by $\gamma_{k,n}$ and $\gamma'(P_k \times P_n)$
by $\gamma'_{k,n}$. Finally we determine the inverse domination number
for cylinder graphs $P_2 \times C_n$.

4.2 Inverse Domination for $P_n$

Let $n = 3k + 1$ and $V(P_n) = \{1, 2, \ldots, n\}$. Then $S = \{2, 5, 8, \ldots, 3k - 1, 3k + 1\}$ and $S' = \{1, 4, 7, \ldots, 3k - 2, 3k\}$
are the $\gamma$-set and $\gamma'$-set of $P_n$ respectively with $k + 1$ vertices
each [28]. When $n = 3k + 2$, $S = \{2, 5, 8, \ldots, 3k + 2\}$, and
$S' = \{1, 4, 7, \ldots, 3k + 1\}$ are the $\gamma$-set and $\gamma'$-sets respectively
of $P_n$ [28] with $k + 1$ vertices.

In contrast to the values of $\gamma(P_n)$ and $\gamma'(P_n)$ for $n \neq 0$
mod 3), it may be noted that $\gamma(P_{3k}) = \frac{n}{3}$ and $\gamma'(P_{3k}) = \frac{n}{3} + 1$. Further $D = \{2, 5, 8, \ldots, 3k - 1\}$ is the $\gamma$-set and
$D' = \{1, 4, 7, \ldots, 3k - 2, 3k\}$ is a $\gamma'$-set for $P_{3k}$. Hence in
this case $\gamma(P_{3k}) \neq \gamma'(P_{3k})$. In general one can conclude that
Figure 4.1:

\(\gamma'_{1,n} = \left\lceil \frac{n}{3} \right\rceil + 1\), for all \(n\).

4.3 Inverse Domination in \(2 \times n\) grid graphs

In this section, we give the inverse domination number and an inverse dominating set for the \(2 \times n\) grid graphs. It is established in [22] that \(\gamma_{2,n} = \left\lceil \frac{n+1}{2} \right\rceil\) for \(n \geq 1\) and is restated in [7] as \(\gamma_{2,n} = \left\lfloor \frac{n+2}{2} \right\rfloor\) for \(n \geq 1\). As obtained in [7], we make use of the blocks \(A, B_1, B_2, B_3\) and \(B_4\) given in Figure 4.1 to construct a dominating set \(S\) and an inverse dominating set \(S'\) for \(P_2 \times P_n; n \geq 1\). Note that the vertices of the grid graphs \(P_k \times P_n\), are precisely the points of intersection of the lines and hence \(|V(P_k \times P_n)| = k \cdot n|.

We construct \(S\) and \(S'\) by concatenating suitable blocks given in Figure 4.1. For example, if we concatenate \(A\) and \(B_3\) we get the block \(AB_3\) as in Figure 4.2. Note that \(AB_3\) is the \(2 \times 7\) grid graph.
The vertices with the symbol ‘•’ in each of the blocks in the figures represent the vertices to be included for a dominating set $S$ and the vertices with the symbol ‘×’ represent the vertices to be included for an inverse dominating set $S'$. The vertices with symbol ‘o’ (or ‘⊗’) in the blocks indicate those vertices that are not dominated by a dominating set $S$ (or by $S'$) constructed up to this stage and to be considered while concatenation. However a dominating set or an inverse dominating set constructed after concatenation shall dominate all the vertices of the newly constructed block. The concatenated blocks $A^2, AB_1, AB_2, AB_3$ and $AB_4$ are given in Figure 4.3.
After concatenation the set of vertices with the symbol ‘●’ give a dominating set and the set of vertices with the symbol ‘×’ give an inverse dominating set.

Now $B_1, B_2, B_3$ and $B_4$ shall give $S$ and $S'$ for $P_2 \times P_n$ for $1 \leq n \leq 4$. Now for $n \geq 5$, let $n = 4q + r$ with $1 \leq r \leq 4$ and $q \geq 1$. Then $A^qB_r$ gives a dominating set $S$ and an inverse dominating set $S'$ for $P_2 \times P_n$. By $A^q$, we mean the concatenation of $A$ with itself for $q$ times. Let $a = |V(A) \cap S|$, $a' = |V(A) \cap S'|$, $b_r = |V(B_r) \cap S|$ and $b'_r = |V(B_r) \cap S'|$. Note that $a = a' = 2$, $b_1 = b_1' = 1$, $b_2 = b_2' = 2$, $b_3 = b_3' = 2$ and $b_4 = b_4' = 3$. Note that the dominating set $S$ as well as an inverse dominating set $S'$ in $A^qB_r$ contain $2q + b_r$ vertices. Hence

$$|S'| = |S| = 2q + b_r = \left\lfloor \frac{n + 2}{2} \right\rfloor.$$ 

In view of $\gamma_{2,n} = \left\lfloor \frac{n+2}{2} \right\rfloor$ [7], we see that $S'$ is a $\gamma'$-set and hence $\gamma_{2,n} = \gamma'_{2,n}$, for all $n$.

Thus we have proved the following theorem:

**Theorem 4.3.1.**

$$\gamma'_{2,n} = \left\lfloor \frac{n+2}{2} \right\rfloor$$ for all $n \geq 1$. 

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4.4 Inverse Domination in $3 \times n$ grid graphs

In this section, we give the inverse domination number and an inverse dominating set for $3 \times n$ grid graphs. In [7], it is proved that,

$$\gamma_{3,n} = \left\lfloor \frac{3n + 4}{4} \right\rfloor, \quad n \geq 1.$$ 

To construct a dominating set and an inverse dominating set of $P_3 \times P_n$, we use the grid blocks $A, B_1, B_2, B_3$ and $B_4$ given in Figure 4.4.

We give the concatenation of $A$ with $A, B_1, B_2, B_3$, and $B_4$ as $A^2, AB_1, AB_2, AB_3$ and $AB_4$ in Figure 4.5.

Note that for $n = 1, 2, 3$ and 4 the dominating and inverse dominating sets are as given by the grid blocks $B_1, B_2, B_3$ and $B_4$ respectively. Note that $a = |V(A) \cap S| = 3$ and $a' = |V(A) \cap S'| = 3$. As mentioned earlier, let $b_r = |V(B_r) \cap S|$
and $b_r' = |V(B_r) \cap S'|$. Note that $b_1 = 1$, $b_1' = 2$, $b_2 = 2$, $b_2' = 3$, $b_3 = b_3' = 3$ and $b_4 = b_4' = 4$. For $n \geq 3$, let $n = 4q + r$ with $1 \leq r \leq 4$ and $q \geq 0$. Then $A^qB_r$ for $q \geq 0$ and $1 \leq r \leq 4$ gives a dominating set $S$ and an inverse dominating set $S'$ with $3q + b_r$ and $3q + b_r'$ vertices respectively.

Also, note that for $n \geq 3,$

$$3q + b_r' = \begin{cases} 3q + b_r & \text{if } r = 3, 4 \\ 3q + b_r + 1 & \text{if } r = 1, 2. \end{cases}$$

Hence, for $n = 4q + r; 1 \leq r \leq 4$, we get $|S'| = |S|$, when $r = 3, 4$ and $|S'| \leq |S| + 1$, when $r = 1, 2$. Thus we obtain the following result:
Theorem 4.4.1.

\[
\gamma'_{3,n} \begin{cases} 
\left\lfloor \frac{3n+4}{4} \right\rfloor & \text{if } n \equiv 0, 3 \pmod{4} \\
\leq \left\lfloor \frac{3n+4}{4} \right\rfloor + 1 & \text{if } n \equiv 1, 2 \pmod{4}.
\end{cases}
\]

4.5 Inverse Domination in $4 \times n$ grid graphs

This section gives the inverse domination number and an inverse dominating set for $4 \times n$ grid graphs. It is proved that [7],

\[
\gamma_{4,n} = \begin{cases} 
n + 1 & \text{if } n = 1, 2, 3, 5, 6, 9 \\
n & \text{otherwise}
\end{cases}
\]

The blocks $A_1, A_2, A_3, A_4$ and $A_5$ in Figure 4.6 gives a dominating set and an inverse dominating set for $P_4 \times P_k$, for $k = 1, 2, 3, 4$ and 7, respectively.

One can see that $A_4A_1, A_4A_2, A_4^2$ and $A_5A_2$ give a dominating set and an inverse dominating set for $P_4 \times P_k$, for $k = 5, 6, 8$ and 9, respectively.

We use the blocks $A, B$ and $B_1$ given in Figure 4.7 to construct a dominating set and an inverse dominating set for $P_4 \times P_n; n \geq 10$. The blocks $AB$ and $BB_1$ in Figure 4.8
Figure 4.6:

Figure 4.7:
Figure 4.8:

gives a $\gamma$-set and a $\gamma'$-set for $P_4 \times P_6$ and $P_4 \times P_4$ grids respectively. Let $n = 3q + r; 1 \leq r \leq 3$. Since $n \geq 10$, we have $q \geq 3$. Consider $q - r$. There are two possibilities. They are either $q - r$ is even or odd.

**Case 1.** When $q - r$ is even say $2t$, then $(BA)^t(BB_1)^r$ gives dominating and inverse dominating sets, for $P_4 \times P_n$ grid graph.

**Case 2.** When $q - r$ is odd, say $2t + 1$, then $A(BA)^t(BB_1)^r$ gives dominating and inverse dominating sets, for $P_4 \times P_n$ grid graph.

Let $a = |V(A) \cap S|,$ $a' = |V(A) \cap S'|,$ $b = |V(B) \cap S|$ and $b' = |V(B) \cap S'|,$ $b_1 = |V(B_1) \cap S|$ and $b'_1 = |V(B_1) \cap S'|$. Note that $a = a' = 3,$ $b = b' = 3$ and $b_1 = b'_1 = 1$. 

Note that $S$ and $S'$ contain only one vertex from each column. Hence we see that $\gamma_{4,n} = \gamma'_{4,n} = n$ for all $n \geq 10$. As seen earlier, $\gamma_{4,n} = \gamma'_{4,n}$ for all $n \leq 10$. Therefore $\gamma_{4,n} = \gamma'_{4,n}$.
for all $n$. Thus we proved the following theorem.

**Theorem 4.5.1.**

$$
\gamma'_{4,n} = \begin{cases} 
  n + 1 & \text{if } n = 1, 2, 3, 5, 6, 9 \\
  n & \text{otherwise}
\end{cases}
$$

### 4.6 Inverse Domination in $5 \times n$ grid graphs

This section concerns the inverse domination number and inverse dominating set for $5 \times n$ grid graphs. We have [7],

$$
\gamma_{5,n} = \begin{cases} 
  \left\lfloor \frac{6n+6}{5} \right\rfloor & n = 2, 3, 7 \\
  \left\lfloor \frac{6n+8}{5} \right\rfloor & \text{otherwise}
\end{cases}
$$

Tony Yu Chang, Edwin Clark, E.O. Hare [7] constructed the blocks $A, B, B_1, B_2, B_3, B_4$ and $B_5$ for constructing a dominating set for a $5 \times n$ grid graph. The blocks are given in Figure 4.9.

Further, we use the blocks $C, D, D_1, D_2, D_3, D_4, D_5, F_2, F_3, F_4$ and $F_5$ for constructing an inverse dominating set for a $5 \times n$ grid graph and the blocks are given in Figure 4.10.

In contrast to the procedure mentioned in the previous sec-
Figure 4.9:

tions, we use different sets of blocks to determine a dominating set and an inverse dominating set. For $n \leq 4$ dominating and inverse dominating sets are obtained from $1 \times 5$, $2 \times 5$, $3 \times 5$ and $4 \times 5$ grids. Even though we make use of different sets of blocks, the final concatenation gives same graph.

Let $a = |V(A) \cap S|$ and $b = |V(B) \cap S'|, c = |V(C) \cap S|$ and $d = |V(D) \cap S'|, b_r = |V(B_r) \cap S|$ and $b'_r = |V(B_r) \cap S'|$. Also let $d_r = |V(D_r) \cap S|, d'_r = |V(D_r) \cap S'|, f_r = |V(F_r) \cap S|$ and $f'_r = |V(F_r) \cap S'|$. Note that $a = b = c = d = 6, b_1 = d_1 = 2, b_2 = d_2 = f_2 = 4, b_3 = d_3 = f_3 = 5, b_4 = d_4 = f_4 = 6$ and
Figure 4.10:
\[ b_5 = d_5 = f_5 = 7. \]

Let \( n = 5q + r; 1 \leq r \leq 5. \)

**Case 1.** When \( n \) is even, we deal with 3 cases.

(1) \( n \equiv 1(\text{mod } 5). \)

(2) \( n \not\equiv 1(\text{mod } 5) \) and \( q \) is even.

(3) \( n \not\equiv 1(\text{mod } 5) \) and \( q \) is odd.

**Sub case 1.1.** When \( n \) is even and \( n \equiv 1(\text{mod } 5) \), then in this case \( r = 1 \) and \( n = 5q + 1. \) Since \( n \) is even, \( q \) must be odd and let \( q = 2k + 1 \) for some \( k \) with \( k \geq 0. \) Then \( A(BA)^kB_1 \) gives a dominating set \( S \) as in [22], hence a \( \gamma \)- set and \( D_5(CD)^kD_1 \) gives an inverse dominating set \( S' \) with respect to \( S. \) Also, one can note that \(|S| = a + k(b + a) + b_1 = 12k + 8 \) and similarly \(|S'| = 12k + 9. \) Thus \(|S'| = |S| + 1. \) Hence in this case, \( \gamma \leq \gamma' \leq \gamma + 1. \)

**Sub case 1.2.** When \( n \) is even, with \( n \not\equiv (1 \text{ mod } 5) \) and \( q \) is even, then \( n = 5q + r \) where \( 2 \leq r \leq 5. \) Let \( q = 2k. \) Note that \( k \geq 1. \) Now \( (BA)^kB_r \) gives a dominating set \( S \) and \( D_r(CD)^kD_1 \) gives an inverse dominating set \( S' \) with respect to \( S. \) Also note that \(|S'| = |S| = 12k + b_r \) and hence in this case \( \gamma' = \gamma. \)
**Sub case 1.3.** When \( n \) is even, with \( n \not\equiv (1 \mod 5) \) and \( q \) is odd, then in this case, \( n = 5q + r \) and \( 2 \leq r \leq 5 \). Let \( q = 2k + 1; \ k \geq 0 \). Then \( A(BA)^kB_r \) gives a dominating set \( S \) and \( D_r(CD)^kC \) gives an inverse dominating set \( S' \) with respect to \( S \). Note that \( |S'| = |S| = 12k + 6 + b_r \) and hence in this case also \( \gamma' = \gamma \).

**Case 2.** When \( n \) is odd, we deal with 3 cases.

(1) \( n \equiv 1 (mod 5) \).

(2) \( n \not\equiv 1 (mod 5) \) and \( q \) is even.

(3) \( n \not\equiv 1 (mod 5) \) and \( q \) is odd.

**Sub case 2.1.** When \( n \) is odd and \( n \equiv 1 (mod 5) \), then \( n = 5q + 1 \) with \( q \) is even. Let \( q = 2k \) for \( k \geq 1 \). Then \( (BA)^kB_1 \) gives a dominating set \( S \). And \( F_5(DC)^kDD_1 \) gives an inverse dominating set \( S' \) with respect to \( S \). We shall verify that \( |S| = 12k + 2 \) and that \( |S'| = 12k + 3 \). Hence in this case \( \gamma \leq \gamma' \leq \gamma + 1 \).

**Sub case 2.2.** When \( n \) is odd, with \( n \not\equiv 1 (mod 5) \) and \( q \) is even, then \( n = 5q + r \) and \( 2 \leq r \leq 5 \). Let \( q = 2k \). Note that \( k \geq 1 \). Then \( (BA)^kB_r \) gives a dominating set \( S \) and \( F_r(DC)^k \) gives an inverse dominating set \( S' \) with respect to \( S \). Also,
one can verify that $|S| = |S'| = 12k + b_r$. From this, again we have $\gamma' = \gamma$.

**Sub case 2.3.** When $n$ is odd, with $n \neq 1 \mod 5$ and $q$ is odd, then $n = 5q + r$ and $2 \leq r \leq 5$. Let $q = 2k + 1$. Note that $k \geq 0$. Then $A(BA)^kB_r$ gives a dominating set $S$ and $F_r(DC)^kD$ gives an inverse dominating set $S'$ with respect to $S$ and one can note that $|S| = |S'| = 12k + 6 + b_r$. Hence $\gamma' = \gamma$ in this case too. Thus

$$\gamma'_{5,n} = \begin{cases} 
\gamma_{5,n} \text{ or } \gamma_{5,n} + 1 & \text{if } n \equiv 1(mod5) \\
\gamma_{5,n} & \text{otherwise.}
\end{cases}$$

Hence we have proved the following result:

**Theorem 4.6.1.**

$$\gamma'_{5,n} = \begin{cases} 
\left\lfloor \frac{6n+6}{5} \right\rfloor & n = 2, 3, 7 \\
\left\lfloor \frac{6n+8}{5} \right\rfloor \text{ or } \left\lfloor \frac{6n+8}{5} \right\rfloor + 1 & \text{if } n \equiv 1(mod5) \\
\left\lfloor \frac{6n+8}{5} \right\rfloor & \text{otherwise}
\end{cases}$$

### 4.7 Inverse Domination in $6 \times n$ grid graphs

In this section, we give the inverse domination number and an inverse dominating set for $6 \times n$ grid graphs. It was proved
that \[7\],

\[
\gamma_{6,n} = \begin{cases} 
\left\lfloor \frac{10n + 10}{7} \right\rfloor & n \geq 6 \text{ and } n \equiv 1 (mod\ 7) \\
\left\lfloor \frac{10n + 12}{7} \right\rfloor & \text{otherwise for } n \geq 4.
\end{cases}
\]

T.Y. Chang, Edwin Clark and E. O. Hare \[7\] give a set of blocks to construct a dominating set for a \(6 \times n\) grid graph. By our assumption \(n \geq 6\). The case \(6 \times 6\) has been dealt by the block \(B_6\) of Figure 4.12. We give a set of blocks for the construction of dominating and inverse dominating sets for \(6 \times n\) (for \(n \geq 6\)) grid graphs in Figure 4.11 and in Figure 4.12.

Here we give a dominating and an inverse dominating set for \(6 \times n\), \(n \geq 6\) and those for \(n \leq 5\) are obtained from the \(k \times 6\) grids where \(1 \leq k \leq 5\). The blocks \(A, B, B_r, 1 \leq r \leq 7\) are used for finding the dominating and inverse dominating sets for the \(6 \times n\) grid graphs. Let \(n = 7q + r; 1 \leq r \leq 7\). Let \(b_r = |V(B_r) \cap S|\) and \(b'_r = |V(B_r) \cap S'|\). \(a = a' = |V(A) \cap S|\) and \(b = b' = |V(B) \cap S'|\). Then note that \(a = b = a' = b' = 10, b_1 = b'_1 = 2, b_2 = b'_2 = 4, b_3 = b'_3 = 6, b_4 = b'_4 = 7\) and
\[b_5 = 8 \text{ and } b'_5 = 9, \quad b_6 = b'_6 = 10 \text{ and } b_7 = b'_7 = 11.\]

We consider two cases according as \( q \) is even or \( q \) is odd.

**Case 1.** If \( q \) is even then \( q = 2k \). Note that \( k \geq 1 \). Then \((BA)^kB_r\) gives a dominating set \( S \) and an inverse dominating set \( S' \) with \(|S| = |S'| = 12k + b_r\).

**Case 2.** If \( q \) is odd, then \( q = 2k + 1 \) and note that \( k \geq 0 \). Then \( A(BA)^kB_r \) gives a dominating set \( S \) and an inverse dominating set \( S' \) with \(|S| = |S'| = 12k + 6 + b_r\).

Note that the number of vertices contributed by each block to a dominating set and an inverse dominating set are the same except for \( B_5 \), and that it differs by one for \( B_5 \). Hence we see that \( \gamma \leq \gamma' \leq \gamma + 1 \) when \( n = 7q + 5 \); and that \( \gamma = \gamma' \) otherwise. Thus we have proved the following theorem:

**Theorem 4.7.1.** For \( n \geq 6 \), \[\gamma'_{6,n} = \begin{cases} \left\lfloor \frac{10n + 10}{7} \right\rfloor & n \geq 6 \text{ and } n \equiv 1(\text{mod } 7) \\ \frac{10n + 12}{7}, \text{ or } \left\lfloor \frac{10n + 12}{7} \right\rfloor + 1 & \text{if } n \equiv 5(\text{mod } 7) \\ \left\lfloor \frac{10n + 12}{7} \right\rfloor & \text{otherwise for } n \geq 4. \end{cases}\]
4.8 Inverse Domination in $7 \times n$ grid graphs

In this section, we give the inverse domination number and an inverse dominating set for the $7 \times n$ grid graphs. We have \([7]\)

$$\gamma_{7,n} = \left\lfloor \frac{5n + 3}{3} \right\rfloor, 6 \leq n \leq 500.$$  

We use the blocks $A$, $C_1$, $C_3$, given in Figure 4.13 and $C_4$, $C_5$, $C_6$ and $B$ given in Figure 4.14 and in Figure 4.15 to construct a dominating set $S$ and an inverse dominating set $S'$ for a $7 \times n$ graph. Let $a = |V(A) \cap S|$ and $a' = |V(A) \cap S'|$. $c_r = |V(C_r) \cap S|$ and $b = |V(B) \cap S'|$. Note that $a = a' = b = 10$, $c_1 = c_1' = 2$, $c_3 = c_3' = 5$, $c_4 = c_4' = 7$ and $c_5 = c_5' = 9$, and $c_6 = c_6' = 11$. 
Let $n = 6q + r; 1 \leq r \leq 6$. As earlier we give a dominating set and an inverse dominating set for $7 \times n; n \geq 7$ and those for $n \leq 6$ are obtained from the $k \times 7$ grids where $1 \leq k \leq 6$. Since $n \geq 7$, we have $q \geq 1$. We handle the cases $r = 2$, $r = 3$ and $r = 6$ separately. When $r = 2$, $A^{q-1}C_4C_4$ gives
a dominating set $S$ and an inverse dominating set $S'$ and each has $10q + 4$ vertices. When $r = 3$, $A^{q-1}C_1C_3C_5$ gives a dominating set $S$ and an inverse dominating set $S'$ and each has $10q + 6$ vertices. When $r = 6$, $A^qC_6$ gives a dominating set $S$ and $C_6B^q$ gives an inverse dominating set $S'$ and each has $11q + 1$ vertices. $A^qC_r$ gives a dominating set $S$ and an inverse dominating set $S'$ for $r = 1, 4, 5$ with $10q + 2, 10q + 7$ and $10q + 9$ vertices respectively. Thus we see that $\gamma'_{7,n} = \gamma_{7,n}$ for all $n$. Therefore we have the following:

**Theorem 4.8.1.**

$$\gamma'_{7,n} = \gamma_{7,n} = \left\lfloor \frac{5n + 3}{3} \right\rfloor, 6 \leq n \leq 500.$$  

### 4.9 Inverse Domination in $P_2 \times C_n$ grid graphs

In this section, we give the domination number, dominating set, the inverse domination number and an inverse dominating set for the $P_2 \times C_n$; $n \geq 3$ cylinder grid graphs. We shall observe that $P_2 \times C_n$ can be obtained from $P_2 \times P_n$, by concatenating the ends of it. We make use of the basic blocks $A, B_1, B_2$ and $B_3$ given for the $P_2 \times P_n$ grid graphs in Figure 4.1. We now split into four cases according as $n \equiv i(mod\ 4), i = 0, 1, 2, 3.$
**Case 1** When \( n \equiv 0(\text{mod } 4) \), \( n = 4q \) for some integer \( q \geq 1 \). We observe that as in \( P_2 \times P_n \), \( A^q \) gives a dominating set with \( 2q \) vertices. Hence \( \gamma(P_2 \times C_n) \leq 2q \). By Theorem 2.3.5, we have \( \left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \). Here \( \Delta(G) = 3 \) and \( n = 2 \times 4q \). Therefore we get \( \left\lceil \frac{8q}{1+3} \right\rceil \leq \gamma(G) \leq 2q \). Thus \( \gamma(P_2 \times C_{4q}) = 2q \).

Note that the same block \( A^q \) gives an inverse dominating set with \( 2q \) vertices. Since \( \gamma(P_2 \times C_n) \leq \gamma'(P_2 \times C_n) \), we get \( \gamma'(P_2 \times C_{4q}) = 2q \).

**Case 2** When \( n \equiv 1(\text{mod } 4) \), \( n = 4q + 1 \) for some integer \( q \geq 1 \). Here, the block \( A^qB_1 \) gives a dominating set with \( 2q+1 \) vertices. Hence \( \gamma(P_2 \times C_{4q+1}) \leq 2q + 1 \). As in Case(i), we get \( \left\lceil \frac{8q+2}{1+3} \right\rceil \leq \gamma(G) \leq 2q + 1 \). Thus \( \gamma(P_2 \times C_{4q+1}) = 2q + 1 \). Note that the same block \( A^qB_1 \) gives an inverse dominating set with \( 2q + 1 \) vertices. As in Case 1, we get \( \gamma'(P_2 \times C_{4q+1}) = 2q + 1 \).

**Case 3** When \( n \equiv 2(\text{mod } 4) \), \( n = 4q + 2 \) for some integer \( q \geq 1 \). Here, the block \( A^qB_2 \) gives a dominating set with \( 2q + 2 \) vertices. Hence \( \gamma(P_2 \times C_{4q+2}) \leq 2q + 2 \). As in Case(i), we get \( \left\lceil \frac{8q+4}{1+3} \right\rceil \leq \gamma(G) \leq 2q + 2 \). Thus \( \gamma(P_2 \times C_{4q+2}) = 2q + 1 \) or \( 2q + 2 \).
We now claim that $\gamma(P_2 \times C_{4q+2}) = 2q + 2$. We prove by induction on $q$. When $q = 1$, we observe that $\gamma(P_2 \times C_{4q+2}) = 4 = 2q + 2$. Assume the result for $q - 1$. Thus we have $\gamma(P_2 \times C_{4(q-1)+2}) = 2(q - 1) + 2$. That is $\gamma(P_2 \times C_{4q-2}) = 2q$. Consider the vertices of the $\gamma$-set $D$ on the cylinder. Let $V(P_2 \times C_{4q-2}) = \{u_1, u_2, u_3, \ldots, u_{4q-2}, v_1, v_2, v_3, \ldots, v_{4q-2}\}$. Without loss of generality assume that $u_i \in D$. Now subdivide the edge $u_iu_{i+1}$ four times so as to get the new vertices $u'_1, u'_2, u'_3, u'_4$. Similarly sub-divide the edge $v_iv_{i+1}$ four times so as to get the new vertices $v'_1, v'_2, v'_3, v'_4$. Now join the vertices $u'_iv'_i$ for $i = 1, 2, 3, 4$. we shall note that the new graph obtained is nothing but $P_2 \times C_{4q+2}$. Let $D_1 = D \cup \{u'_4, v'_2\}$. The vertices $u'_4, v'_2$ dominates all the new vertices except $u'_1$, which is dominated by $u_i$ and the vertex $u'_4 \in D_1$ dominates $u_{i+1}$ which is dominated by $u_i$. Hence $D_1$ is a dominating set of $P_2 \times C_{4q+2}$ and no set with fewer vertices than $D_1$ can dominate $P_2 \times C_{4q+2}$. Thus $D_1$ is the required $\gamma$-set of $P_2 \times C_{4q+2}$ Hence $\gamma(P_2 \times C_{4q+2}) = 2q + 2$. Therefore $A^qB_2$ gives a $\gamma$-set.

Note that the same block gives an inverse dominating set with $2q + 2$ vertices. Hence $\gamma'(P_2 \times C_{4q+2}) = 2q + 2$. 

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**Case 4** When \( n \equiv 3 \pmod{4} \), \( n = 4q + 3 \) for some integer \( q \geq 0 \). Here, the block \( A^qB_3 \) gives a dominating set with \( 2q + 2 \) vertices. Hence \( \gamma(P_2 \times C_{4q+3}) \leq 2q + 2 \). As mentioned earlier, we get \( \lceil \frac{8q+6}{1+3} \rceil \leq \gamma(G) \leq 2q + 2 \). Thus \( \gamma(P_2 \times C_{4q+3}) = 2q + 2 \).

Here also the same block \( A^qB_3 \) gives an inverse dominating set with \( 2q + 2 \) vertices. Hence we get \( \gamma'(P_2 \times C_{4q+3}) = 2q + 2 \).

Thus in all the cases we see that \( \gamma(P_2 \times C_n) = \gamma'(P_2 \times C_n) \).

Hence we have proved the theorem

**Theorem 4.9.1.** \( \gamma'(P_2 \times C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0 \pmod{4} \\ \lceil \frac{n+1}{2} \rceil & \text{otherwise.} \end{cases} \)