Chapter 3

Equality of Domination and Inverse Domination numbers

3.1 Introduction

The concept of the inverse domination was introduced by V.R. Kulli and A. Sigarkanti in [24]. Actually, they attempted to prove that for any graph $G$, $\gamma'(G) \leq \beta_0(G)$, the vertex independence number equals the maximum order of set of vertices, in which no two vertices are adjacent. However, the proof given is incorrect. To date, no proof of this result is known, neither are any counter examples known. Thus this is known as the Kulli-Sigarkanti conjecture.

Since then, Gayle S. Domke, Jean E. Dunbar and Lisa R. Markus published a paper on the inverse domination number [13], in which they characterize the graphs for which
\( \gamma(G) + \gamma'(G) = n \). They also give a lower bound for the inverse domination number of a tree and give a constructive characterization of the class of trees which achieve this lower bound.

Since an inverse dominating set lies in the complement of a dominating set, a dominating set and an inverse dominating set are disjoint dominating sets. Researchers have for many years studied the existence of graphs having disjoint minimum dominating sets. Most notable among this research is the study of various kinds of domatic numbers, in which one seeks to partition the vertices of \( G \) into a maximum number of various kinds of dominating sets. For a survey of this literature, the reader is referred to B. Zelinka in [29].

Specifically the existence of two disjoint minimum dominating sets was first studied by D.W. Bange, A.E. Barkauskas and P.J. Slater [3] in 1978, who studied the existence of two disjoint minimum dominating sets in trees. In a related paper, T.W. Haynes and M.A. Henning [21] studied the existence of two disjoint minimum independent dominating sets in a tree.

It is well known by Ore’s Theorem [26] that if a graph \( G \) has
no isolated vertices, then the complement $V - D$ of every minimal dominating set $D$ contains a dominating set. Thus, every graph without isolated vertices contains an inverse dominating set with respect to a minimum dominating set and has an inverse domination number. In this chapter, therefore we will assume that all graphs have no isolated vertices and we will consider classes of graphs $G$ for which $\gamma(G) = \gamma'(G)$. Hereafter $G$ denotes a simple graph on $n$ vertices with no isolated vertices.

In the section 3.2, we give some graphs and results for which $\gamma(G) = \gamma'(G)$. In the section 3.3, we give some bounds for $\gamma'(G)$. In the section 3.4, we characterize the graphs with $\gamma(G) = \gamma'(G) = \frac{n}{2}$. In the section 3.5, we characterize the graphs with $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$. In the section 3.6, we give the inverse domination number for some classes of graphs. Some of the contents of this chapter are accepted for publication in “Ars Combinatoria”.

3.2 Graphs with $\gamma(G) = \gamma'(G)$

In this section, we give some graphs for which $\gamma(G) = \gamma'(G)$ and obtain some results on this aspect. The following result
can be proved without any difficulty.

**Proposition 3.2.1.** The following are true:

1. $\gamma(K_n) = \gamma'(K_n) = 1$, for $n \geq 2$.

2. Let $G$ be a complete bi-partite graph with bi-partition $(X,Y)$, $|X| = m$, $|Y| = n$ and $m, n \geq 2$. Then $\gamma(G) = \gamma'(G) = 2$.

3. Let $G$ be the complement of a connected bi-partite graph with bi-partition $(X,Y)$, $|X| = m$, $|Y| = n$ and $m, n \geq 2$. Then $\gamma(G) = \gamma'(G) = 2$.

4. For any graph $H$ with $|V(H)| \geq 2$, $\gamma(H) \geq 2$ and $k \geq 2$, $\gamma(kK_1 \circ H) = \gamma'(kK_1 \circ H) = 2$.

**Lemma 3.2.2.** Let $G$ be a graph with $\gamma(G) = \gamma'(G)$. Then $G$ has no $\gamma$-required vertex.

*Proof. Let $G$ be a graph with $\gamma(G) = \gamma'(G)$. Let $D$ be a $\gamma$-set and $D'$ be a $\gamma'$-set of $G$. Suppose $G$ contains a $\gamma$-required vertex $u$. Then $u$ lies in every $\gamma$-set of $G$. Hence $u \in D$ and $u \in D'$, which is a contradiction to $D' \subseteq (V(G) - D)$. □
**Proposition 3.2.3.** Let $x$ be a dominating vertex of a graph $G$. Then $\gamma'(G) = \gamma(G - x)$.

*Proof.* Since $x$ is a dominating vertex of $G$, $\{x\}$ is a $\gamma$-set of $G$. Hence any $\gamma'$-set of $G$ lies in $G - \{x\}$ and is a minimum dominating set of $G - \{x\}$. Therefore $\gamma'(G) = \gamma(G - x)$. \[\square\]

**Theorem 3.2.4.** Let $G$ be a graph such that $G$ and $\overline{G}$ are connected with at least two pendant vertices in $G$. Then $\gamma(G) = \gamma'(G) = 2$.

*Proof.* Since $G$ and $\overline{G}$ are connected graphs, $\Delta(G) \leq n - 2$ and $\Delta(\overline{G}) \leq n - 2$. Therefore $\gamma(\overline{G}) \geq 2$ and $\gamma'(\overline{G}) \geq 2$. Let $\{a, b\}$ be two pendant vertices in $G$. Let $a'$, $b'$ be the supports of $a$ and $b$ respectively.

**Case 1.** When $a' = b'$, $D = \{a, a'\}$ is a $\gamma$-set of $\overline{G}$. Since $\overline{G}$ is connected, $a'$ is adjacent to some other vertex $c$ in $\overline{G}$. Hence $D' = \{b, c\}$ is a $\gamma'$-set of $\overline{G}$. Thus $\gamma(\overline{G}) = \gamma'(\overline{G}) = 2$.

**Case 2.** When $a' \neq b'$, $D = \{a, a'\}$ is a $\gamma$-set of $\overline{G}$ and $D' = \{b, b'\}$ is a $\gamma'$-set of $\overline{G}$. Thus $\gamma(\overline{G}) = \gamma'(\overline{G}) = 2$. \[\square\]
Theorem 3.2.5. For any $n$, $\gamma(C_n) = \gamma'(C_n) = \lceil \frac{n}{3} \rceil$.

Proof. Assume that $V(C_n) = \{1, 2, \ldots, n\}$. When $n = 3k$, for some $k > 0$, the set $D = \{1, 4, 7, \ldots, 3k - 2\}$ is a $\gamma$-set and the set $D' = \{2, 5, 8, \ldots, 3k - 1\}$ is a $\gamma'$-set both containing $k = \frac{n}{3}$ elements. When $n = 3k + 1$, the set $D = \{1, 4, 7, \ldots, 3(k - 1) + 1, 3k\}$ is a $\gamma$-set and the set $D' = \{2, 5, 8, \ldots, 3k - 1, 3k + 1\}$ is a $\gamma'$-set both containing $k + 1 = \lceil \frac{n}{3} \rceil$ elements. When $n = 3k + 2$, the set $D = \{1, 4, 7, \ldots, 3k + 1\}$ is a $\gamma$-set with $k + 1 = \lceil \frac{n}{3} \rceil$ elements and $D' = \{2, 5, 8, \ldots, 3k + 2\}$ is a $\gamma'$-set with $k + 1 = \lceil \frac{n}{3} \rceil$ elements each. Thus in all the cases $\gamma(C_n) = \gamma'(C_n) = \lceil \frac{n}{3} \rceil$. □

Proposition 3.2.6. For $n \geq 4$, $\gamma(\overline{C_n}) = \gamma'(\overline{C_n}) = 2$.

Proof. Let $V(C_n) = \{1, 2, \ldots, n\}$. Let $G = C_n$. Then each vertex $i$ in $G$ is adjacent to $i - 1$ and $i + 1$ modulo $n$. Therefore each vertex $i$ in $\overline{G}$ is adjacent to the remaining $n - 3$ vertices. Also $i - 1$ and $i + 1$ are adjacent $\overline{G}$. Hence $D = \{i, i + 1\}$ is a $\gamma$-set of $\overline{G}$ and $D' = \{j, j + 1\}$ is a $\gamma'$-set of $\overline{G}$. Thus $\gamma(\overline{G}) = \gamma'(\overline{G}) = 2$. □
Corollary 3.2.7. \( \gamma'(C_n) = \gamma'(\overline{C}_n) \) if and only if \( n = 4, 5, 6 \).

Proof. From the Proposition 3.2.6 and by Theorem 3.2.5, 
\( \gamma'(C_n) = \gamma'(\overline{C}_n) = \lceil \frac{n}{3} \rceil = 2 \). Hence \( \gamma'(C_n) = \gamma'(\overline{C}_n) \) if and only if \( n = 4, 5, 6 \).

Theorem 3.2.8. For \( n > 3 \), \( \gamma(P_n) = \gamma'(P_n) = \lceil \frac{n}{3} \rceil \) if and only if \( n \not\equiv 0(\text{mod } 3) \).

Proof. Let \( V(P_n) = \{1, 2, \ldots, n\} \). Assume that \( n > 3 \) and \( n \not\equiv 0(\text{mod } 3) \). Then \( n = 3k + 1 \) or \( n = 3k + 2 \) for some positive integer \( k > 0 \). When \( n = 3k + 1 \), the set \( D = \{2, 5, 8, \ldots, 3(k - 1) + 2, 3k\} \) is a \( \gamma \)-set with \( k + 1 = \lceil \frac{n}{3} \rceil \) elements and \( D' = \{1, 4, 7, \ldots, 3k + 1\} \) is a \( \gamma' \)-set with \( k + 1 = \lceil \frac{n}{3} \rceil \) elements. When \( n = 3k + 2 \), the set \( D = \{2, 5, 8, \ldots, 3k + 2\} \) is a \( \gamma \)-set with \( k + 1 = \lceil \frac{n}{3} \rceil \) elements and the set \( D' = \{1, 4, 7, \ldots, 3k + 1\} \) is a \( \gamma' \)-set with \( k + 1 = \lceil \frac{n}{3} \rceil \) elements. On the converse, assume that \( \gamma(P_n) = \gamma'(P_n) = \lceil \frac{n}{3} \rceil \). Suppose \( n \equiv 0(\text{mod } 3) \), then \( n = 3k \) for some positive integer \( k \). The set \( D = \{2, 5, 8, \ldots, 3k - 1\} \) is a \( \gamma \)-set with \( k = \frac{n}{3} \) elements and \( D \) is the only \( \gamma \)-set in
Proposition 3.2.9. For $n \geq 4$, $\gamma(P_n) = \gamma'(P_n) = 2$.

Proof. Since $P_n$ has two pendant vertices, by Theorem 3.2.4 we have $\gamma(P_n) = \gamma'(P_n) = 2$. □

Corollary 3.2.10. $\gamma'(P_n) = \gamma'(P_n)$ if and only if $n = 4, 5$.

Proof. We have by Proposition 3.2.9, $\gamma'(P_n) = 2$. Also by Theorem 3.2.8, $\gamma'(P_n) = \lceil \frac{n}{3} \rceil$ for $n \not\equiv 0 \mod 3$ and $\gamma'(P_n) = \lceil \frac{n}{3} \rceil + 1$ for $n \equiv 0 \mod 3$. Thus $\gamma'(P_n) = 2$ if and only if $n = 4, 5$. Hence $\gamma'(P_n) = \gamma'(P_n)$ if and only if $n = 4, 5$. □

Theorem 3.2.11. Let $T$ be a tree in which all the vertices are either pendant vertices or their supports. Then $T$ is a wounded spider if and only if $\gamma'(T) = \Delta(T)$.  

$P_n$. The set $D' = \{1, 4, 7, \ldots, 3k - 2, 3k\}$ is a $\gamma'$-set with $k + 1 = \frac{n}{3} + 1$ elements, which contradicts $\gamma = \gamma' = \lceil \frac{n}{3} \rceil$. Hence $n \not\equiv 0 \mod 3$. Therefore $\gamma(P_n) = \gamma'(P_n) = \lceil \frac{n}{3} \rceil$ if and only if $n \not\equiv 0 \mod 3$. □
Proof. Let $T$ be a tree with $n$ vertices and whose vertices are either pendant vertices or their supports and let $\gamma'(T) = \Delta(T)$. Let $L$ be the set of all pendant vertices of $T$ and $S$ be its neighbor set. Then $|S| \leq |L|$. By the assumption of $T$, $V(T) = S \cup L$. Hence $S$ is a $\gamma$-set of $T$ and $L$ is a $\gamma'$-set of $T$. Since $|S| + |L| = n$, we get that $\gamma(T) + \gamma'(T) = n$. This implies that $\gamma(T) = n - \Delta(T)$. By Theorem 2.3.21, $T$ is a wounded spider.

Conversely, assume that $T$ is a wounded spider. Then by Theorem 2.3.21, $\gamma(T) = n - \Delta(T)$. As discussed above, we have $V(T) = S \cup L$ and so by Theorem 2.3.10, $\gamma(T) + \gamma'(T) = n$. Hence $\gamma'(T) = \Delta(T)$.

Corollary 3.2.12. Let $G$ be a wounded spider on $n$ vertices. Then $\gamma(G) = \gamma'(G)$ if and only if $\Delta(G) = \frac{n}{2}$.

Proof. Assume that $\gamma(G) = \gamma'(G)$. By Theorem 3.2.11, for a wounded spider $G$, we have $\gamma(G) + \gamma'(G) = n$ and $\gamma'(G) = \Delta(G)$. Hence $\Delta(G) = \frac{n}{2}$. On the reverse assume that $\Delta(G) = \frac{n}{2}$. We have $\gamma'(G) = \Delta(G)$. Then by Theorem 2.3.21, $\gamma(G) = n - \Delta(G)$. Hence $\gamma(G) = \gamma'(G)$.

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Theorem 3.2.13. Let $d$ be a positive divisor of a positive integer $n$. Then there exists a regular graph $G$ on $n$ vertices, for which $\gamma(G) = \gamma'(G) = d$.

Proof. Assume that $d$ is a positive integer and divides $n \geq 1$. Therefore let $n = kd$. Let $V = \bigcup_{i=1}^{k} V_i$, where $V_i$ is a set of $d$ independent vertices. Let each vertex of $V_i$ be adjacent exactly to one vertex each of $V_j$, $j \neq i$. Then $d(v) = k - 1$ for all $v \in V$. Therefore $G$ is $k - 1$ regular. Now since each $v \in V_i$ is adjacent exactly to one vertex of $V_j$, $j \neq i$, each $V_i$ is a $\gamma$-set of $G$. Hence $\gamma(G) = |V_i| = d$ and $\gamma'(G) = |V_j| = d$. \hfill \Box

Proposition 3.2.14. If a $\gamma$-set of a connected graph $G$ of order $n \geq 4$ is a status, then $\gamma(G) = \gamma'(G) = 2$.

Proof. Let $S$ be a $\gamma$-set, a status in $G$. Trivially $|S| \geq 2$. Then any vertex of $S$ is adjacent to every other vertex in $V - S$. Let $S = \{u_1, u_2, \ldots, u_k\}$ and let $V - S = \{v_1, v_2, \ldots, v_{n-k}\}$. Then $D = \{u_1, v_1\}$ is a dominating set of $G$ and hence a $\gamma$-set of $G$. Now $D' = \{u_2, v_2\}$ is an inverse dominating set of $G$ with respect to $D$ with $|D'| = \gamma(G)$. Hence $D'$ is a $\gamma'$-set of $G$. Hence $\gamma(G) = \gamma'(G) = 2$. \hfill \Box
3.3 Some bounds for $\gamma'(G)$

In this section, we give some bounds for $\gamma'(G)$. Here we give a very important result that $\gamma'(G + e) \leq \gamma'(G)$ for $e \in E(G)$.

**Theorem 3.3.1.** Let $G$ be a connected graph with $n$ vertices, $m$ edges and $\gamma(G) = \gamma'(G)$. Then $\gamma'(G) \geq \frac{1}{3}(2n - m)$.

**Proof.** Let $D$ be a $\gamma$-set and $D'$ be a $\gamma'$-set of $G$. Then every vertex in $V - (D \cup D')$ has at least one neighbor in $D$ and one neighbor in $D'$, every vertex in $D$ has a neighbor in $D'$, and every vertex in $D'$ has a neighbor in $D$. Thus the number of edges in $G$ is at least $2(V - (D \cup D')) + |D'|$. That is $m \geq 2(n - 2\gamma') + \gamma'$ and so $m \geq 2n - 3\gamma'(G)$. Therefore $\gamma'(G) \geq \frac{1}{3}(2n - m)$.

**Corollary 3.3.2.** Let $T$ be a tree. Then $\gamma'(T) \geq \frac{n+1}{3}$.

**Proof.** When $T$ is a tree, we have $m = n - 1$. By Theorem 3.3.1, $\gamma'(G) \geq \frac{1}{3}(2n - m)$ which implies that $\gamma'(T) \geq \frac{n+1}{3}$. 


Proposition 3.3.3. For any graph $G$, $n - m \leq \gamma'(G) \leq m$.

Proof. We have, for any $(m, n)$ graph $G$, $n - m \leq \gamma(G) \leq n - \Delta$. by Theorem 2.3.5. Since $\gamma(G) \leq \gamma'(G)$, we get $n - m \leq \gamma'(G)$. Since $\gamma'(G) \leq n - \gamma(G)$, we get $\gamma'(G) \leq m$. \qed

Proposition 3.3.4. For any graph $G$,

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma'(G) \leq \left\lfloor \frac{n\Delta(G)}{1+\Delta(G)} \right\rfloor.$$  

Proof. By Theorem 2.3.5, $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq \gamma'(G)$. On the other hand $\gamma'(G) \leq n - \gamma(G)$, implies $\gamma'(G) \leq \left\lfloor \frac{n\Delta(G)}{1+\Delta(G)} \right\rfloor$. \qed

Proposition 3.3.5. If $G$ has degree sequence $(d_1, d_2, \ldots d_n)$ with $d_i \geq d_{i+1}$, then $\gamma'(G) \leq \max \{n - k : k + (d_1 + d_2 + \ldots + d_k) \geq n \}$.

Proof. Let $G$ be a graph with degree sequence $(d_1, d_2, \ldots d_n)$ with $d_i \geq d_{i+1}$. Then by Theorem 2.12 [20], we have $\gamma(G) \geq \min \{k : k + (d_1, d_2, \ldots d_n) \geq n \}$. Hence $\gamma'(G) \leq n - \gamma(G)$ implies that $\gamma'(G) \leq \max \{n - k : k + (d_1 + d_2, \ldots d_k) \geq n \}$. \qed
Theorem 3.3.6. Let $uv = e \in E(G)$. If $\gamma(G + e) < \gamma(G)$, then there exists a $\gamma$-set $D$ of $G$ containing both $u$ and $v$ such that at least one of them has no private neighbor other than itself.

Proof. Assume that there exists no $\gamma$-set $G$ containing both $u$ and $v$. Let $D$ be a $\gamma$-set such that $u \notin D$. Then there exists a vertex $u_1 \in N(u)$ which has a private neighbor other than $u$ that lies in $D$ to dominate $u$. Hence $u_1$ lies in a $\gamma$-set of $G + e$ also. Therefore $\gamma(G + e)$ is in no way smaller than $\gamma(G)$, which is a contradiction. Hence there exists a $\gamma$-set containing both $u$ and $v$.

Suppose both of them have a private neighbor other than itself with respect to any $\gamma$-set $D$, then neither $D - \{u\}$ nor $D - \{v\}$ is a $\gamma$-set of $G + e$. Also no set with fewer than $\gamma(G)$ elements can dominate $G + e$. Therefore $\gamma(G + e) \geq \gamma(G)$, which is a contradiction. □

Theorem 3.3.7. Let $uv = e \in E(G)$. Then $\gamma'(G + e) \leq \gamma'(G)$.
Proof. Let $D$ be a $\gamma$-set and $D'$ be a $\gamma'$-set of $G$, which is an inverse dominating set of $G$ with respect to $D$. We have $\gamma(G + e) \leq \gamma(G)$ for any $e \in E(G)$.

Case 1. Let $uv = e \in E(G)$ be such that $\gamma(G + e) = \gamma(G)$. Then $D$ is a $\gamma$-set of $G + e$ also. Hence $D'$ is an inverse dominating set of $G + e$ with respect to $D$. Therefore $\gamma'(G + e) \leq |D'| = \gamma'(G)$.

Case 2. Let $xy = e \in E(G)$ be such that $\gamma(G + e) < \gamma(G)$. Then by Theorem 3.3.6, there exists a $\gamma$-set $D$ of $G$ containing both $x$ and $y$ such that at least one of them say $x$ has no private neighbor other than itself. Therefore $D_1 = D - \{x\}$ is a $\gamma$-set of $G + e$ and $D_1' = D'$ is an inverse dominating set of $G + e$ with respect to $D_1$. Therefore $\gamma'(G + e) \leq |D'| = \gamma'(G)$.  

Remark 3.3.8. (i). Let $uv = e \in E(G)$. Then $\gamma'(G) \leq \gamma'(G - e)$.

(ii). Let $H$ be a spanning sub-graph of a graph $G$. Then $\gamma'(G) \leq \gamma'(H)$.

Proposition 3.3.9. Let $G$ be a hamiltonian graph on $n$ vertices. Then $\gamma'(G) \leq \lceil \frac{n}{3} \rceil$.  

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Proof. Let $G$ be hamiltonian. Then $G$ contains a spanning cycle say $C$. By Theorem 3.2.5, we have $\gamma(C_n) = \gamma'(C_n) = \lceil \frac{n}{3} \rceil$. Therefore by Remark 3.3.8, we have $\gamma'(G) \leq \gamma'(C_n) = \lceil \frac{n}{3} \rceil$.

**Corollary 3.3.10.** Let $G$ be a graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then $\gamma'(G) \leq \lceil \frac{n}{3} \rceil$.

Proof. Let $G$ be a connected graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then by Theorem 4.3 [5], we have $G$ is hamiltonian. Hence by Remark 3.3.8, $\gamma'(G) \leq \lceil \frac{n}{3} \rceil$.

**Corollary 3.3.11.** Let $G$ be a connected graph on $n \geq 3$ vertices. If $C(G)$ is complete, then $\gamma'(G) \leq \lceil \frac{n}{3} \rceil$.

Proof. Let $G$ be a connected graph on $n \geq 3$ vertices. If $C(G)$ is complete, then by corollary 4.4[5], we have $G$ is hamiltonian. Hence by Remark 3.3.8, $\gamma'(G) \leq \lceil \frac{n}{3} \rceil$.

**Proposition 3.3.12.** For any graph $G$ with no isolated vertices, $\gamma'(G) \leq \epsilon_f(G)$, where $\epsilon_f(G)$ denotes the maximum number of pendant edges in a spanning forest of $G$. 

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Proof. We have for any graph $G$, by Theorem 2.3.6, $\gamma(G) + \epsilon_f(G) = n$, where $\epsilon_f(G)$ denotes the maximum number of pendant edges in a spanning forest of $G$. Then $\epsilon_f(G) = n - \gamma(G)$. Hence we get $\gamma'(G) \leq \epsilon_f(G)$.

\[\square\]

3.4 Graphs with $\gamma(G) = \gamma'(G) = \frac{n}{2}$

In this section, we characterize the graphs with $\gamma(G) = \gamma'(G) = \frac{n}{2}$. Recall that T.W Haynes and others, in [20] have characterized the class of graphs with $\gamma(G) = \frac{n}{2}$.

**Theorem 3.4.1.** Let $G$ be a connected graph with $n$ vertices. Assume that $n$ is even and $\delta(G) = 1$. Then $\gamma(G) = \gamma'(G) = \frac{n}{2}$ if and only if $G = H \circ K_1$ for some connected graph $H$ on $\frac{n}{2}$ vertices.

Proof. Let $G = H \circ K_1$, where $H$ is a connected graph on $\frac{n}{2}$ vertices. Then trivially $V(H)$ is a $\gamma$-set for $G$ and hence $\gamma(G) = \frac{n}{2}$. Note that the set of all pendant vertices in $G$ forms an inverse dominating set of $G$, with minimum cardinality. Hence $\gamma'(G) = \frac{n}{2}$. On the converse, assume that $\gamma(G) = \gamma'(G) = \frac{n}{2}$. Let $L = \{v/d(v) = 1\}$. Since $\delta(G) = 1$, we get that $L \neq \phi$ and let $S = N(L)$. Since $\gamma(G) + \gamma'(G) = n$, by
Theorem 2.3.10, we have (i) $V - S$ independent and (ii) for every $x \in V - (S \cup L)$, each vertex in $N(x)$ is adjacent to at least two pendant vertices.

Let $S_1$ be the set of all vertices in $S$, which are adjacent to exactly one pendant vertex in $L$ and let $S_2 = S - S_1$.

**Case 1.** $V - (S \cup L) = \emptyset$. Then every vertex of $G$ is either a pendant vertex or adjacent to a pendant vertex. Thus $S$ is a dominating set of $G$ and no set with fewer than $|S|$ elements can dominate $S$. Also $|S| \leq |L|$. Hence $S$ is a $\gamma$-set of $G$. Since $\gamma(G) = \gamma'(G) = \frac{n}{2}$, we have $|S| = \frac{n}{2}$ and $|L| = \frac{n}{2}$. Hence each vertex of $S$ is adjacent to exactly one pendant vertex of $L$. Let $H = \langle S \rangle$. Then $G = H \circ K_1$.

**Case 2.** $V - (S \cup L) \neq \emptyset$. Let $N_1 = N(S) - (S \cup L)$ and $N_2 = V - (N_1 \cup (S \cup L))$.

**Claim 1.** $N_2 = \emptyset$. For by Theorem 2.3.10, we have $V - S$ is independent. Hence every vertex of $G$ is either in $S$ or adjacent to some vertex in $S$. Hence there is no vertex in $N_2$ and so $S$ is a $\gamma$-set of $G$.

**Claim 2.** $N_1 = \emptyset$. For let $L_1 = N(S_1) \cap L$ and $L_2 =$
$N(S_2) \cap L$. Then $|S_1| = |L_1|$ and $|S_2| < |L_2|$. If $N_1 \neq \phi$, by Theorem 2.3.10, for each $x \in N_1$, each stem in $N(x)$ is adjacent to at least two pendant vertices. If $S_2 \neq \phi$, since $S$ is a $\gamma$-set and $V - S$ is a $\gamma'$-set, we have $L$ is contained a $\gamma'$-set with $|L| > |S|$. This is a contradiction to $\gamma(G') = \gamma'(G)$. Hence $S_2 = \phi$ and $N_1 = \phi$. Thus $V - (S \cup L) = \phi$. Hence by Case (i) we get that $G = H \circ K_1$ where $H = \langle S \rangle$. \hfill $\square$

**Theorem 3.4.2.** For a connected graph $G$ with even number of vertices $n$ and $\delta(G) \geq 2$, $\gamma(G) = \gamma'(G) = \frac{n}{2}$ if and only if $G = C_4$.

**Proof.** When $G = C_4$, trivially we have $\gamma(G) = \gamma'(G) = \frac{n}{2}$. On the converse, $\gamma(G) = \gamma'(G) = \frac{n}{2}$ implies that $\gamma(G) + \gamma'(G) = n$, since $n$ is even. Also since $G$ is a graph with $\delta(G) \geq 2$, by Theorem 2.3.9, we get $G$ is $C_4$. \hfill $\square$

**Remark 3.4.3.** Let $G_1, G_2, \ldots G_k$ be the $k$ connected components of a graph $G$. Let $D_1, D_2, \ldots D_k$ be the $\gamma$-sets and $D_1', D_2', \ldots D_k'$ be the $\gamma'$-sets of the $k$ connected components of the graph $G$. Then $D_1 \cup D_2 \cup \ldots \cup D_k$ is a $\gamma$-set of
\( G \) and \( D_1' \cup D_2' \cup \ldots \cup D_k' \) is a \( \gamma' \)-set of \( G \). Therefore
\[
\gamma(G) = \sum_{i=1}^{k} \gamma(G_i) \quad \text{and} \quad \gamma'(G) = \sum_{i=1}^{k} \gamma'(G_i).
\]

**Proposition 3.4.4.** Let \( G_1, G_2, \ldots, G_k \) be the \( k \) connected components of a graph \( G \). Then \( \gamma(G) = \gamma'(G) \) if and only if \( \gamma(G_i) = \gamma'(G_i) \) for \( i = 1 \) to \( k \).

**Proof.** Let \( G_1, G_2, \ldots, G_k \) be the \( k \) connected components of a graph \( G \). Then we have
\[
\gamma(G) = \sum_{i=1}^{k} \gamma(G_i) \quad \text{and} \quad \gamma'(G) = \sum_{i=1}^{k} \gamma'(G_i)
\]
for \( i = 1 \) to \( k \). Thus trivially, \( \gamma(G) = \gamma'(G) \) if \( \gamma(G_i) = \gamma'(G_i) \) for \( i = 1 \) to \( k \). Conversely assume that \( \gamma(G) = \gamma'(G) \). We have \( \gamma(G_i) \leq \gamma'(G_i) \) for \( i = 1 \) to \( k \). Suppose \( \gamma(G_i) < \gamma'(G_i) \) for some \( i \), then we must have \( \gamma(G_j) > \gamma'(G_j) \) for some \( j \neq i \), which is impossible. Hence \( \gamma(G_i) = \gamma'(G_i) \) for \( i = 1 \) to \( k \). \( \square \)

**Corollary 3.4.5.** Let \( G \) be a graph with \( n \) vertices. Let \( G_1, G_2, \ldots, G_k \) be the \( k \) connected components of \( G \) with \( |V(G_i)| = n_i \). Then \( \gamma(G) = \gamma'(G) = \left\lfloor \frac{n}{2} \right\rfloor \) if and only if \( \gamma(G_i) = \gamma'(G_i) = \left\lfloor \frac{n_i}{2} \right\rfloor \), for \( i = 1 \) to \( k \). In particular \( n \) is even if and only if each \( n_i \) is even.
Proof. First assertion follows from Proposition 3.4.4. Let \( n \) be even. Since \( \gamma(G) = \gamma'(G) = \lfloor \frac{n}{2} \rfloor \), we have \( \gamma(G) + \gamma'(G) = n \). and we must have \( \gamma(G_i) + \gamma'(G_i) = n_i \) and \( \gamma(G_i) = \gamma'(G_i) = \lfloor \frac{n_i}{2} \rfloor \), for each \( i \). This implies that each \( n_i \) must be even. Conversely if each \( n_i \) is even, then \( n \) is even. \( \square \)

**Corollary 3.4.6.** Let \( G \) be a graph with even number of vertices and no isolated vertices. Then \( \gamma(G) = \gamma'(G) = \frac{n}{2} \) if and only if the components of \( G \) are the cycle \( C_4 \) or the corona \( G = H \circ K_1 \) for some connected graph \( H \).

**Proof.** Assume that \( \gamma(G) = \gamma'(G) = \frac{n}{2} \). Let \( G_1, G_2, \ldots, G_k \) be the connected components of \( G \) and \( |V(G_i)| = n_i \) for \( 1 \leq i \leq k \). Since \( n \), the number of vertices in \( G \), is even and \( \gamma(G) = \gamma'(G) = \frac{n}{2} \), in view of Corollary 3.4.5, we get \( \gamma(G_i) = \gamma'(G_i) = \frac{n_i}{2} \). Now by Theorems 3.4.1 and 3.4.2, we see that \( \gamma(G) = \gamma'(G) \) if and only if the components of \( G \) are either the cycle \( C_4 \) or the corona \( G = H \circ K_1 \) for some connected graph \( H \). \( \square \)

**Remark 3.4.7.** Let \( G \) be a graph with even number of vertices and no isolated vertices. Let \( \gamma(G) = \gamma'(G) = \frac{n}{2} \). Then
the components of $G$, with $\delta(G_i) \geq 2$ are the cycle $C_4$ and the components of $G$, with $\delta(G_i) = 1$ are the corona $G_i = H \circ K_1$ for some connected graph $H$.

3.5 Graphs with $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$

We now turn our attention to graphs $G$ for which $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$. In this section, we give the complete characterization of graphs which satisfy the above condition.

Consider the 5 classes of graphs given in the figures. Let $\mathbb{A}$ be the set of all graphs given in Figure 3.1. Let $\mathbb{B}$ be the set of all graphs given in Figure 3.2. Let $\mathbb{Q}_1 = \mathbb{A} \cup \mathbb{B}$. For any graph $H$, let $S(H)$ denotes the set of all connected graphs, each of which can be formed from $H \circ K_1$ by adding a new vertex $x$ and edges joining $x$ to two or more vertices of $H$. Then define $\mathbb{Q}_2 = \bigcup_H S(H)$, where the union is taken over all graphs $H$. The graphs in $\mathbb{Q}_2$ are of the form given in Figure 3.3.

Consider a new vertex $a$ and a copy of $C_4$. For a graph $G \in \mathbb{Q}_2$, let $\theta(G)$ be the graph obtained by joining $G$ to $C_4$ with the single edge $xa$ where $x$ is the new vertex added in
Figure 3.1:

Figure 3.2:

Figure 3.3:
Let \( u, v, w \) be a vertex sequence of the path \( P_3 \). For any graph \( H \), let \( P(H) \) be the set of all connected graphs which are formed from \( H \circ K_1 \) by joining each of \( u \) and \( w \) to one or more vertices of \( H \). Then define \( Q_4 = \bigcup_H P(H) \). Any graph in \( Q_4 \) is of the form given in Figure 3.5.

Let \( H \) be a graph and \( X \in B \). Let \( R(H, X) \) be the set of connected graphs which may be formed from \( H \circ K_1 \) by joining forming \( G \). Then define \( Q_3 = \{ \theta(G) / G \in Q_2 \} \). The graphs in \( Q_3 \) are of the form given in Figure 3.4.
each vertex of $U \subseteq V(X)$ to one or more vertices of $H$, such that no set with fewer than $\gamma(X)$ vertices of $X$ dominates $V(X) - U$. Then define $\mathcal{Q}_5 = \bigcup_{H,X} R(H, X)$. Any graph $G$ in $\mathcal{Q}_5$ is of the form as given in Figure 3.6.

**Lemma 3.5.1.** For all graphs $G$ in $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5$, we have $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$.

**Proof.** $D = \{1, 3, 6\}$ and $D' = \{2, 5, 7\}$ form the $\gamma$-set and $\gamma'$-set respectively for all the graphs in $\mathcal{A}$ with $\frac{n-1}{2}$ vertices each. For $C_3$ in $\mathcal{B}$, the set $D = \{1\}$ and $D' = \{2\}$ are the $\gamma$ and $\gamma'$-sets with $\frac{n-1}{2}$ vertices each. $D = \{1, 4\}$ and $D' = \{3, 5\}$ are the $\gamma$ and $\gamma'$-sets for all the other graphs in $\mathcal{B}$ with $\frac{n-1}{2}$ vertices each. Thus $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ for all the graphs in $\mathcal{Q}_1$. 

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Consider the class \( Q_2 \) and \( G \in Q_2 \). Let \( x \) be adjacent to \( y \) and \( z \) in \( H \). Now, \( y \) and \( z \) are adjacent to the pendant vertices \( y_1 \) and \( z_1 \) respectively. Since there are \( \frac{n-1}{2} \) pendant vertices, which are adjacent to distinct \( \frac{n-1}{2} \) vertices of \( H \), any \( \gamma \)-set of \( G \) contains at least \( \frac{n-1}{2} \) vertices. Further, the set \( D = V(H) \cup \{y_1\} - \{y\} \) is a dominating set containing \( \frac{n-1}{2} \) elements and hence it is a \( \gamma \)-set. \( D' = V(G) - D \cup \{x\} \) is a dominating set in \( V - D \), containing \( \frac{n-1}{2} \) vertices. Hence \( D' \) is a \( \gamma' \)-set and thus \( \gamma(G) = \gamma'(G) = \frac{n-1}{2} \) for all \( G \in Q_2 \).

Similarly \( D = V(H) \cup \{b, d\} \) is a \( \gamma \)-set and \( D' = V(G) - (D \cup \{x\}) \) is a \( \gamma' \)-set for \( G \in Q_3 \). Thus, \( \gamma(G) = \gamma'(G) = \frac{n-1}{2} \) for all \( G \in Q_3 \).

Again \( D = V(H) \cup \{u\} \) is a \( \gamma \)-set with \( \frac{n-1}{2} \) vertices and \( D' = V(G) - (D \cup \{w\}) \) is a \( \gamma' \)-set containing \( \frac{n-1}{2} \) vertices. Thus \( \gamma = \gamma' = \frac{n-1}{2} \) for all \( G \in Q_4 \).

Let \( |V(H)| = m \) and \( |V(X)| = p \). Then \( |V(R(H, X))| = 2m + p = n \). Let \( D_X \) and \( D'_X \) be the \( \gamma \)-set and \( \gamma' \)-set of the graph \( X \) respectively. Then \( D = V(H) \cup D_X \) is a dominating set and no set with fewer vertices dominates \( G \) and hence \( D \) is a \( \gamma \)-set. \( D' = D'_X \) together with all the pendant vertices forms
a $\gamma'$-set of $R(H, X)$. Hence $|D| = m + \frac{p-1}{2} = \frac{2m+p-1}{2} = \frac{n-1}{2}$ and similarly $|D'| = \frac{n-1}{2}$. Hence $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ for all $G \in Q_5$. Thus $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ for all graphs in $G = \bigcup_{i=1}^{5} Q_i$. 

\[ \square \]

**Remark 3.5.2.** Note that the classes of graphs $Q_1, Q_3, Q_4$ and $Q_5$ are the same as the classes of graphs $G_2, G_4, G_5$ and $G_6$ respectively and $Q_2$ is a subclass of the class $G_3$ given in Theorem 2.3.17.

**Theorem 3.5.3.** A connected graph $G$ with $n$ vertices satisfies $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ if and only if $G \in Q = \bigcup_{i=1}^{5} Q_i$.

**Proof.** Let $G \in Q = \bigcup_{i=1}^{5} Q_i$. By Lemma 3.5.1 $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$. Conversely, suppose that $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$. Note that $n$ is odd. Now by Theorem 2.3.17, $G \in G = \bigcup_{i=1}^{6} G_i$. Since $n$ is odd, $G$ cannot be in $G_1$. In view of Remark 3.5.2, $G \in Q_1$ or $Q_3$ or $Q_4$ or $Q_5$ and $G \in G_3$. It is enough to prove that whenever $G \in G_3$, we get that $G \in Q_2$. If $G \in Q_3$, let $S$ be the set of end vertices in $G$ and $T$ be the set of neighbors of vertices in $S$. If $|T| = t$, then by Lemma
2.3.16, $|S| = t$ or $|S| = t + 1$. Hence there is a $\gamma$-set of $G$ containing $T$. Let $G' = G - (S \cup T)$.

**Case 1.** If $|S| = t + 1$, then one can show that $G' = \phi$ as in the course of proof of Theorem 2.3.17. Hence we get $\gamma = t$ and $\gamma' = t + 1$ a contradiction. Therefore $|S| \neq t + 1$.

**Case 2.** If $|S| = t$, then either $G'$ contains isolated vertices or $G'$ contains no isolated vertices.

**Sub Case 2.1.** If $G'$ contains isolated vertices, let $y$ be an isolated vertex of $G'$. As in the proof of Theorem 2.3.17, one can prove that $G' - y$ is empty. Since $y$ is not a pendant vertex of $G$, $y$ is adjacent to two or more vertices of $T$. Hence $G \in \mathbb{Q}_2$.

**Sub Case 2.2.** If $G'$ contains no isolated vertices as in the above mentioned proof $G \notin \mathbb{Q}_3$. If $G \in \mathbb{G}_4$ then $G \in \mathbb{Q}_3$. If $G \in \mathbb{G}_5$ then $G \in \mathbb{Q}_4$. If $G \in \mathbb{G}_6$, then $G \in \mathbb{Q}_5$. Thus $G \in \bigcup_{i=1}^{5} \mathbb{Q}_i$.

**Corollary 3.5.4.** Let $G$ be a graph with odd order $n$ and $\delta(G) = 1$, having no isolated vertices. Then $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ if and only if the components of $G$ are the cycle

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or the corona $G = H \circ K_1$ for any connected graph $H$ together with exactly one component $G_j \in \bigcup_{i=1}^{5} Q_i$.

**Proof.** Suppose the components of $G$ are either the cycle $C_4$ or the corona $G = H \circ K_1$ for some connected graph $H$ together with exactly one component of $G_j \in \bigcup_{i=1}^{5} Q_i$, then by Corollary 3.4.6, by Remark 3.4.7, and by Lemma 3.5.1, we get $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$. Conversely assume that $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$. Let $G_1, G_2, \ldots, G_m$ be the components of the graph $G$, with $|V(G_i)| = n_i$. Then each $G_i$ is connected. Since $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$, we see that $\gamma(G) + \gamma'(G) = n - 1$ and that there is exactly one vertex which is neither in the $\gamma$-set nor in the $\gamma'$-set of $G$. Since $n$ is odd, there exists at least one component say $G_k$ such that $n_k$ is odd. In view of Proposition 3.4.4, and $n_k$ being odd, we get $\gamma(G_k) = \gamma'(G_k) = \frac{n_k-1}{2}$ and in the component $G_k$, there is one vertex which is neither in the $\gamma$-set nor in the $\gamma'$-set of $G$. Suppose there is another odd component say $G_t$ then we get one more vertex which is neither in the $\gamma$-set nor in the $\gamma'$-set of $G$, which leaves at least two vertices that are neither in the $\gamma$-set nor in the $\gamma'$-set of $G$, a contradiction. Hence we see that each $n_j$ is even for $j \neq k$. Then $\gamma(G_k) = \gamma'(G_k) = \frac{n_k-1}{2}$ and
\(\gamma(G_j) = \gamma'(G_j) = \frac{n_j}{2}\), for \(j \neq k\). Thus the even components \(G_j, j \neq i\) are the cycle \(C_4\) or the corona \(H \circ K_1\), by the Corollary 3.4.6. And the only odd component is one of the graphs in \(\bigcup_{i=1}^{5} Q_i\), by Theorem 3.5.3. That is, we get \(G_j \in \bigcup_{i=1}^{5} Q_i\). Hence the components of \(G\) are the cycle \(C_4\) or the corona \(G = H \circ K_1\) for any connected graph \(H\) together with exactly one component \(G_j \in \bigcup_{i=1}^{5} Q_i\).

\[\square\]

**Remark 3.5.5.** Let \(G\) be a graph with odd order \(n\) and no isolated vertices. Let \(\gamma(G) = \gamma'(G) = \frac{n-1}{2}\) with \(\delta(G) \geq 2\). Then the components of \(G\) are the cycle \(C_4\) together with exactly one component \(G_j\) where, \(G_j \in Q_1\).

### 3.6 Inverse Domination for Certain Graphs

In this section, we give the inverse domination number for some classes of graphs such as generalised corona, union and join of graphs. Recall that a \(k\)-corona \(kG \circ H\) contains \(k\) copies of \(G\) and \(|V(G)|\) copies of \(H\) with appropriate edges between each vertex \(x_i^j\) of the copy \(G^i\) and all of the vertices of the copy \(H_i\) as defined in [25].
Lemma 3.6.1. Let $H$ be any graph. Then the following are true.

(i) $\gamma(K_1 \circ H) = 1$ and $\gamma'(K_1 \circ H) = \gamma(H)$.

(ii) $\gamma(H \circ K_1) = \gamma'(H \circ K_1) = |V(H)|$.

(iii) $\gamma(kK_1 \circ H) = \gamma'(kK_1 \circ H) = 2$, for a positive integer $k \geq 2$ and for $|V(H)| \geq 2$.

(iv) For any integer $m \geq 1$, $\gamma(H \circ K_m) = \gamma'(H \circ K_m) = n$, where $|V(H)| = n$.

Proof. (i). Let $V(H) = \{h_1, h_2, \ldots, h_m\}$ and $V(K_1) = \{v\}$. Then $\{v\}$ is a $\gamma$- set of $K_1 \circ H$. Further a $\gamma'$- set of $K_1 \circ H$ is a minimum dominating set of $(K_1 \circ H) - \{v\}$, which is a $\gamma$-set of $H$. Hence $\gamma(K_1 \circ H) = 1$ and $\gamma'(K_1 \circ H) = \gamma(H)$.

(ii). Let $V(H \circ K_1) = \{h_1, h_2, \ldots, h_m, k_1, k_2, \ldots, k_m\}$ where $h_i \in H$ and $k_i$ denoted the vertex of $K_1$, adjacent to $h_i$. Since each vertex in $H \circ K_1$ is either an end vertex or its support. Hence $D = \{h_1, h_2, \ldots, h_m\}$ is a $\gamma$- set of $H \circ K_1$ and $\{k_1, k_2, \ldots, k_m\}$ is a $\gamma'$- set of $H \circ K_1$. Thus $\gamma(H \circ K_1) = \gamma'(H \circ K_1) = |V(H)|$.

(iii). Let $V(H) = \{h_1, h_2, \ldots, h_m\}$ and $\{v_1, v_2, \ldots, v_k\}$
be the vertices of $k$ copies of $K_1$. Each vertex $v_i$ is adjacent to every vertex of $H$ and no two vertices in \( \{v_1, v_2, \ldots, v_k\} \) are adjacent. Hence $D = \{h_1, v_1\}$ is a $\gamma$-set of $kK_1 \circ H$ and $D' = \{h_2, v_2\}$ is a $\gamma'$-set of $kK_1 \circ H$. Thus $\gamma(kK_1 \circ H) = \gamma'(kK_1 \circ H) = 2$. 

(iv). Let $V(H \circ K_m) = \{h_1, h_2, \ldots, h_n\} \cup (\bigcup_{i=1}^{i=n} \{k_{i1}, k_{i2}, \ldots, k_{im}\})$ where $h_i \in H$ and $k_{ji}$'s are the vertices of the $j^{th}$ copy of $K_m$. Note that the vertex $k_{ji}$ dominates the $j^{th}$ copy of $K_m$ and $h_j$, a vertex of $H$ for $j = 1$ to $n$. Hence $D = \{k_{i1}, k_{i2}, \ldots, k_{im}\}$ is a dominating set of $H \circ K_m$. Since $|D| = n$ and $|V(H)| = n \leq \gamma(H \circ K_m)$, $D$ is a $\gamma$-set of $G$. Similarly, $D' = \{k_{i1}, k_{i2}, \ldots, k_{i2}\}$ is an inverse dominating set of $H \circ K_m$ and a $\gamma'$-set of $G$. Thus $\gamma(H \circ K_m) = \gamma'(H \circ K_m) = n$, where $|V(H)| = n$. \qed

**Theorem 3.6.2.** Let $G$ be an arbitrary graph and $H$ be a graph with $\gamma(H) = 1$. Then for a positive integer $k$, we have $|V(G)| \leq \gamma'(kG \circ H) \leq 2|V(G)|$.

**Proof.** By Theorem 2.3.19, we have $\gamma(kG \circ H) = |V(G)| = n$. Let $\{h_1\}$ be a $\gamma$-set of $H$. Then $D_i = \{h_{i1}\}$ is a $\gamma$-set of $i^{th}$
copy of $H$ for $1 \leq i \leq n$. Then $D = \bigcup_{i=1}^{n} \{h_{i1}\}$ is a dominating set of $kG \circ H$. Since $\gamma(kG \circ H) = |V(G)|$, we get $D$ is a $\gamma$-set of $kG \circ H$. Since $\gamma(kG \circ H) \leq \gamma'(kG \circ H)$, we have $|V(G)| \leq \gamma'(kG \circ H)$. Now, for each $i$, choose $h_{i2} \in H_i$, and $x_i^j \in G^j$. Then $D' = (\bigcup_{i=1}^{n} \{h_{i2}\}) \cup (\bigcup_{i=1}^{n} \{x_i^j\})$ is an inverse dominating set with respect to $D$ with $2|V(G)|$ vertices and so $\gamma'(kG \circ H) \leq 2|V(G)|$. Thus $|V(G)| \leq \gamma'(kG \circ H) \leq 2|V(G)|$. \hfill \qed

**Remark 3.6.3.** Theorem 3.6.2 gives upper and lower bounds for $\gamma'(kG \circ H)$, for any graph $G$, a graph $H$ with $\gamma(H) = 1$ and for a positive integer $k$. We shall observe that both the bounds are sharp in the following results.

Also we shall note that the lower bound is obtained when $k = 1$. That is $\gamma(G \circ H) = \gamma'(G \circ H) = |V(G)|$. For, when $k = 1$, $D = (\bigcup_{i=1}^{n} \{h_{i1}\})$ is a $\gamma$- set of $G \circ H$ where $\{h_1\}$ is the $\gamma$- set of $H$, and $D' = V(G)$ is an inverse dominating set of $G \circ H$. Therefore $\gamma'(G \circ H) \leq |V(G)|$. Also we have $|V(G)| \leq \gamma'(G \circ H)$, and hence $\gamma'(G \circ H) = |V(G)|$. Therefore we get $D' = V(G)$ is the $\gamma'$- set of $G \circ H$. Hence $\gamma(G \circ H) = \gamma'(G \circ H) = |V(G)|$. 

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Theorem 3.6.4. Let $G, H$ be two connected graphs with $\gamma'(H) = 1$. Then $\gamma(kG \circ H) = \gamma'(kG \circ H) = |V(G)|$.

Proof. Since $\gamma(H) \leq \gamma'(H)$, we have $1 \leq \gamma(H) \leq \gamma'(H) = 1$, and we get $\gamma(H) = 1$. Let $D = \{h_1\}$ be a $\gamma$-set of $H$ and $D' = \{h_2\}$ be a $\gamma'$-set of $H$, where $V(H) = \{h_1, h_2, \ldots, h_m\}$ is the vertex set of $H$. For $i = 1$ to $n$, let $V(H_i) = \{h_{i1}, h_{i2}, \ldots, h_{im}\}$. Then $D = \bigcup_{i=1}^{n} \{h_{i1}\}$ is a dominating set of $(kG \circ H)$. Therefore $\gamma(kG \circ H) \leq |V(G)|$ and by Theorem 2.3.18, we have $|V(G)| \leq \gamma(kG \circ H)$. Hence we get $D$ is a $\gamma$-set of $(kG \circ H)$ and $D' = \bigcup_{i=1}^{n} \{h_{i2}\}$ is an inverse dominating set of $(kG \circ H)$. Therefore $\gamma'(kG \circ H) \leq |V(G)|$.

By Theorem 3.6.2, we have $|V(G)| \leq \gamma'(kG \circ H)$, we see that $D'$ is a $\gamma'$-set of $(kG \circ H)$, with $|V(G)|$ vertices. Hence $\gamma(kG \circ H) = \gamma'(kG \circ H) = |V(G)|$. \[\square\]

Theorem 3.6.5. Let $G = \overline{K}_n$ and let $H = K_{1,m}$. Then for any integer $k \geq 2$, $\gamma'(kG \circ H) = 2|V(G)| = 2n$.

Proof. Let $V(G) = \{x_1, x_2, \ldots, x_n\}, n \geq 2$ be the vertex set of $G$. Let $G^j$ be the $j^{th}$ copy of the graph $G$, with the vertex set $V(G^j) = \{x_1^j, x_2^j, \ldots, x_n^j\}$, Where $x_i^j$ corresponds to the vertex
$x_i \in V(G)$ and let $V(H) = \{v, h_1, h_2, \ldots h_m\}$ with $d(v) = m$. Here $\gamma(H) = 1$, and $\{v\}$ is the $\gamma-$ set of $H$. Then $D_i = \{v_i\}$ is a $\gamma$- set of $i^{th}$ copy of $H$ for $1 \leq i \leq n$. Then $D = \bigcup_{i=1}^{n} \{v_i\}$ is a dominating set of $kG \circ H$. By Theorem 2.3.19, we have $\gamma(kG \circ H) = |V(G)| = n$. Hence we get $D$ is a $\gamma$-set of $kG \circ H$. Therefore $D' = \{x_1^1, x_2^1, \ldots x_n^1\} \cup (\bigcup_{i=1}^{n} \{h_i^1\})$, is an inverse dominating set of $(kG \circ H)$. By definition of $kG \circ H$, and since $\gamma'(H) \neq 1$, we need at least $n$ vertices to dominate the vertices of the $H_i$'s and since $E(G^j) = \phi$, we need another set of at least $n$ vertices to dominate the vertices of the $G^j$'s. Hence $\gamma'(kG \circ H) \geq 2n$. So $D'$ is a $\gamma'$-set of $(kG \circ H)$. Hence $\gamma'(kG \circ H) = 2|V(G)| = 2n$. \hfill \Box

**Lemma 3.6.6.** Let $D$ be a $\gamma$-set and $D'$ be a $\gamma'$-set of $kG \circ H$, with $|D'| = |V(G)|$ and $\gamma(H) \geq 2$. Then $k \leq d(G)$ where $d(G)$ is the domatic number of the graph $G$.

**Proof.** We have $|V(G)| \leq |D|$ by Theorem 2.3.18 and given $|D'| = |V(G)|$. Also $|D| \leq |D'|$. Therefore $|V(G)| \leq |D| \leq |D'| = |V(G)|$, implies that $|D| = |V(G)|$. Therefore by Corollary 2.3.20, we get $k \leq d(G)$. \hfill \Box
**Theorem 3.6.7.** In usual notation, \( \gamma'(G_1 \cup G_2) = \gamma'(G_1) + \gamma'(G_2) \).

*Proof.* Let \( D \) be a \( \gamma \)-set of \( G_1 \cup G_2 \) and \( D' \) be a \( \gamma' \)-set of \( G_1 \cup G_2 \). Then \( D = D_1 \cup D_2 \) where \( D_1 \) and \( D_2 \) are the \( \gamma \)-sets of \( G_1 \) and \( G_2 \) and \( D' = D_1' \cup D_2' \) where \( D_1' \) and \( D_2' \) are the \( \gamma' \)-sets of \( G_1 \) and \( G_2 \) respectively. Hence \( \gamma'(G_1 \cup G_2) = \gamma'(G_1) + \gamma'(G_2) \). \( \square \)

**Proposition 3.6.8.** Let \( G \) be the join of two graphs \( G_1 \) and \( G_2 \). Then \( \gamma(G) = \gamma'(G) = 1 \), if at least one of the following holds good.

(i) \( \gamma(G_1) = \gamma(G_2) = 1 \).

(ii) \( \gamma(G_1) = \gamma'(G_1) = 1 \).

(iii) \( \gamma(G_2) = \gamma'(G_2) = 1 \).

*Proof.* Let \( \{u_1\} \) and \( \{u_2\} \) be the \( \gamma \) - sets of \( G_1 \) and \( G_2 \) respectively. Then \( \{u_1\} \) is a \( \gamma \)-set of \( G \) and \( \{u_2\} \) is the \( \gamma' \)-set of \( G \). Hence \( \gamma(G) = \gamma(G) = 1 \). Let \( \{u_1\} \) and \( \{v_1\} \) be the \( \gamma \) and \( \gamma' \) - sets of \( G_1 \). Then \( \{u_1\} \) is a \( \gamma \)-set of \( G \) and \( \{v_1\} \) is a \( \gamma' \)-set of \( G \). Hence \( \gamma(G) = \gamma'(G) = 1 \). Similar is the case for (iii). \( \square \)
Proposition 3.6.9. If $G$ is the join of $G_1$ and $G_2$ with $\gamma(G_i) \geq 2$ and $|V(G_1)|, |V(G_2)| \geq 2$, then $\gamma(G) = \gamma'(G) = 2$.

Proof. In $G$, each vertex of $G_1$ is joined to every vertex of $G_2$ and vice versa. Hence one vertex of $G_1$ and one vertex of $G_2$ form a $\gamma$-set of $G$ and also a $\gamma'$-set of $G$. Thus $\gamma(G) = \gamma'(G) = 2$. \qed

Corollary 3.6.10. Let $G$ be the join of $r$ graphs $G_1, G_2, \ldots, G_r$ and with $n_1, n_2, \ldots, n_r$ vertices respectively with $\gamma(G_i) \geq 2$, and $n_i \geq 2$, for $i = 1, 2, \ldots r$. Then $\gamma(G) = \gamma'(G) = 2$.

Theorem 3.6.11. Let $G$ be a complete $k$-partite graph with partition $V_1, V_2, \ldots, V_k$ with $|V_i| \geq 2$ for $i = 1$ to $k$. Then $\gamma(G) = \gamma'(G) = 2$.

Proof. Assume $|V_i| \geq 2$ for $i = 1$ to $k$. Then each $V_i$ contains at least two vertices. Let $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$. Then $D = \{x_1, x_2\}$ is a $\gamma$-set of $G$ and $D' = \{y_1, y_2\}$ is a $\gamma'$-set of $G$. Therefore $\gamma(G) = \gamma'(G) = 2$. \qed