Chapter 2

Preliminaries

In this chapter, we collect some basic definitions and theorems which are needed for the subsequent chapters. For basic graph theoretic terminology, we refer to Harary [18].

In the first section we present the notations followed in this dissertation. In the second section of this chapter, we give some basic definitions in graph theory. Also we present some fundamental results of domination in graphs and they are listed in the third section for use in the subsequent chapters.

2.1 Notations

$\lfloor x \rfloor$ - the largest integer less than or equal to $x$.

$\lceil x \rceil$ - the smallest integer greater than or equal to $x$.

$G$ - graph
$n$ - number of vertices in $G$

$V = V(G)$ - vertex set of $G$

$E = E(G)$ - edge set of $G$

$< S >$ - sub-graph induced by $S \subseteq V(G)$

$\text{deg}(v) = d(v)$ - degree of the vertex $v$

$K_n$ - complete graph on $n$ vertices

$K_{m,n}$ - complete bipartite graph

$C_n$ - cycle on $n$ vertices

$P_n$ - path on $n$ vertices

$\overline{G}$ - complement of the graph $G$

$N(S)$ - open neighborhood of $S$

$N[S]$ - closed neighborhood of $S$

$G_1oG_2$ - corona of the graphs $G_1$ and $G_2$

$\gamma(G)$ - domination number of $G$

$\gamma'(G)$ - inverse domination number of $G$

$i(G)$ - independent domination number of $G$

$\beta_0(G)$ - independence number of $G$

$G_1 \cup G_2$ - union of the graphs $G_1$ and $G_2$

$G_1 + G_2$ - join of the graphs $G_1$ and $G_2$. 
2.2 Definitions

In this section, we present certain basic definitions in graph theory.

Definition 2.2.1. A graph $G = (V, E)$ is a finite nonempty set $V$ of objects called vertices together with a set $E$ of unordered pairs of distinct vertices of $G$ called edges. We write $V = V(G)$ and $E = E(G)$ when there are more than one graph under consideration. The edge $e = uv$ is said to join the vertices $u$ and $v$. We write $e = uv$, and say that $u$ and $v$ are adjacent vertices, $v$ and the edge $e$ are incident with each other, as are $u$ and $e$. If $e_1$ and $e_2$ are two distinct edges of $G$ incident with a common vertex then they are said to be adjacent edges.

Definition 2.2.2. The edge $e = uu$ is called a loop. The edges $e_i = e_j = uv$ are called multiple edges or parallel edges. A graph without loops and parallel edges is called a simple graph.

Definition 2.2.3. The cardinality of the vertex set of a graph $G$ is called the order of $G$ and is denoted by $n$ where as the
cardinality of the edge set is called the size of $G$ and is denoted by $m$. A graph with $n$ vertices and $m$ edges is called a $(n, m)$ graph.

**Definition 2.2.4.** A graph $H$ is said to be a sub-graph of $G$, if its vertex set $V(H) \subseteq V(G)$ and edge set $E(H) \subseteq E(G)$ respectively. Let $S$ be a non-empty set of the vertex set $V(G)$. The sub-graph of $G$ whose vertex set is $S$ and whose edge set is the set of those edges of $G$ which have both ends in $S$ is called the sub-graph of $G$ induced by $S$ and is denoted by $< S >$.

**Definition 2.2.5.** Let $v \in V(G)$. Then the induced sub-graph $V - \{v\}$ is denoted by $G - v$ and it is the sub-graph obtained from $G$ by the removal of $v$.

**Definition 2.2.6.** Let $G$ be graph. A sub-graph $H$ of $G$ with $V(H) = V(G)$ is called a spanning sub-graph of $G$. A cycle that spans a graph $G$ is called a spanning cycle of $G$.

**Definition 2.2.7.** The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $deg(v)$. A vertex of degree 0 in $G$ is called an isolated vertex.
and a vertex of degree 1 is called a pendant vertex. A vertex that is adjacent to a pendant vertex \( v \) is called a support of \( v \).

**Definition 2.2.8.** The minimum and maximum degrees of vertices of \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \) respectively.

**Definition 2.2.9.** A graph is said to be regular if all its vertices are of the same degree \( k \) and it is said to be \( k \)-regular.

**Definition 2.2.10.** A graph is said to be a complete graph if every two of its vertices are adjacent. The complete graph on \( n \) vertices is denoted by \( K_n \).

**Definition 2.2.11.** The complement \( \overline{G} \) of a graph \( G \) is the graph with vertex set \( V(G) \) such that two vertices are adjacent in \( \overline{G} \) if and only if they are not adjacent in \( G \).

**Definition 2.2.12.** A bipartite graph \( G \) is a graph whose vertex set \( V(G) \) can be partitioned into two subsets \( X \) and \( Y \) such that every edge of \( G \) has one end in \( X \) and the other end in \( Y \). The pair \((X, Y)\) is called a bipartition of \( G \). Further if \( G \) contains every edge joining any vertex of \( X \) to any vertex of
Y, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(X, Y)$ such that $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. The graph $K_{1,n-1}$ is called a star.

**Definition 2.2.13.** Let $u$ and $v$ (not necessarily distinct) be vertices of a graph $G$. A $u - v$ walk of $G$ is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots u_{n-1}, e_n, u_n = v$ of vertices and edges, beginning with vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}u_i$ for all $i = 1, 2, \ldots n$. The number $n$ is called the length of the walk. A $u - v$ walk is determined by the sequence $u = u_0, u_1, \ldots u_{n-1}, u_n = v$ of its vertices and hence we specify a walk simply by $u = u_0, u_1, \ldots u_{n-1}, u_n$. A walk in which all the vertices are distinct is called a path. A walk $u_0, u_1, \ldots u_{n-1}, u_n$ is called a closed walk if $u_0 = u_n$. A closed walk in which $u_0, u_1, \ldots u_{n-1}, u_n$ are distinct is called a cycle. A path on $n$ vertices is denoted by $P_n$ and a cycle $n$ on vertices is denoted by $C_n$.

**Definition 2.2.14.** A path that contains every vertex of $G$ is called a Hamiltonian path of $G$. A Hamiltonian cycle is a cycle that contains every vertex of $G$.  

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Definition 2.2.15. A graph $G$ is said to be connected if any two vertices of $G$ are joined by a path. A maximal connected subgraph of $G$ is called a component of $G$. Thus a disconnected graph has at least two components. The number of components of $G$ is denoted by $\omega(G)$.

Definition 2.2.16. A graph $G$ is called acyclic if it has no cycles. A connected acyclic graph is called a tree. A spanning subgraph of a graph which is also a tree is called a spanning tree of the graph.

Definition 2.2.17. A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle. A wheel with $n$ vertices is denoted by $W_n$.

Definition 2.2.18. A subset $S$ of $V$ in a graph $G$ is said to be independent if no two vertices in $S$ are adjacent. The maximum cardinality of an independent set is called the independence number of $G$ and is denoted by $\beta_0(G)$.

Definition 2.2.19. The open neighborhood $N(v)$ of a vertex $v$ is the set of all vertices adjacent to $v$ in $G$. $N[v] = N(v) \cup \{v\}$
is called the *closed neighborhood* of $v$. The open neighborhood $N(S)$ of a set $S$ of vertices is the set of all vertices adjacent to some vertex in $S$. $N[S] = N(S) \cup \{S\}$ is called the closed neighborhood of $S$.

**Definition 2.2.20.** A *status* is a set $S$ of vertices in a graph which has the property that for any two vertices $u, v \in S$, $N(u) \cap V - S = N(v) \cap V - S$. In other words, the set of vertices in $V - S$ dominated by $u$ equals the set of vertices in $V - S$ dominated by $v$.

**Definition 2.2.21.** Let $S$ be a set of vertices of a graph $G$ and let $u \in S$. We say that a vertex $v$ is a *private neighbor* of $u$ (with respect to $S$) if $N[v] \cap S = \{u\}$. The private neighbor set of $u$ with respect to $S$ is defined as $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. Notice that $u \in pn[u, S]$ if $u$ is an isolate in $< \langle S \rangle$, in which case we say that $u$ is its own private neighbor. The private neighbor set of a set $S$ is defined as $pn(S) = \{v : v \in pn[u, S]$ for every $u \in S\}$.

**Definition 2.2.22.** [20] A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$
is either an element of $S$ or adjacent to an element of $S$. The domination number of $G$ is the smallest cardinality of all minimal dominating sets in $G$ and is denoted by $\gamma(G)$ or simply by $\gamma$.

**Definition 2.2.23.** [24] Let $D$ be a $\gamma$-set of a graph $G$. A dominating set $D' \subseteq V - D$ is called an inverse dominating set of $G$ with respect to $D$.

**Definition 2.2.24.** [24] The inverse domination number $\gamma'(G)$ of $G$ is the cardinality of a smallest inverse dominating set of $G$. An inverse dominating set $D'$ is called a $\gamma'$-set if $|D'| = \gamma'(G)$.

**Definition 2.2.25.** [25] If $D = \{x\}$ is a dominating set of $G$, then $x$ is called a dominating vertex of $G$.

**Definition 2.2.26.** A vertex $v \in V(G)$ is said to be a $\gamma$-required vertex of $G$ if $v$ lies in every $\gamma$-set of $G$.

**Definition 2.2.27.** [20] The domatic number $d(G)$ of a graph $G$ is defined to be the maximum number of elements in a partition of $V(G)$ into dominating sets.
2.3 Results on Domination

In this section, we collect certain fundamental results connecting domination number and inverse domination number. The following theorem which gives a characterization of minimal dominating sets is due to Ore [26].

**Theorem 2.3.1.** [26] A dominating set $S$ is a minimal dominating set if and only if for each vertex $u \in S$, one of the following two conditions holds.

(a) $u$ is an isolate of $S$.

(b) There exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.

**Theorem 2.3.2.** [26] If a graph $G$ has no isolated vertex and $D$ is a minimal dominating set of $G$, then $V - D$ is a dominating set of $G$.

**Corollary 2.3.3.** For any graph $G$ of order $n$, that has no isolated vertex, $\gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

**Theorem 2.3.4.** [4] For any $(m, n)$ graph $G$, $n - m \leq \gamma(G) \leq n - \Delta$. 

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Theorem 2.3.5. [20] For any \((m, n)\) graph \(G\), \(\lceil \frac{n}{1 + \Delta(G)} \rceil \leq \gamma(G) \leq n - \Delta(G)\).

Theorem 2.3.6. [20] For any graph \(G\), \(\gamma(G) + \epsilon_f(G) = n\) where \(\epsilon_f(G)\) denotes the maximum number of pendant edges in any spanning forest of \(G\).

Theorem 2.3.7. For any tree \(T\) with \(n \geq 2\), there exists a vertex \(v \in V\), such that \(\gamma(T - v) = \gamma(T)\).

Theorem 2.3.8. If \(G\) and \(\overline{G}\) have no isolated vertices, then \(\gamma(G) + \gamma(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2\).

Theorem 2.3.9. [10] let \(G\) be a connected graph with \(\delta(G) \geq 2\). Then \(\gamma(G) + \gamma'(G) = n\) if and only if \(G = C_4\).

Theorem 2.3.10. [10] Let \(G\) be a connected graph with \(n \geq 3\) and \(\delta(G) = 1\). Let \(L \subseteq V\) be the set of all degree one vertices and \(S = N(L)\). Then \(\gamma(G) + \gamma'(G) = n\) if and only if the following two conditions hold:
1. \(V - S\) is independent and
2. for every vertex \(x \in V - (S \cup L)\), every stem in \(N(x)\) is adjacent to at least two leaves.
Theorem 2.3.11. [10] For any tree $T$ of order $n \geq 2$, 
$\gamma'(T) \geq \frac{n+1}{3}$.

Lemma 2.3.12. [17] If $G$ has a unique $\gamma$-set $D$, then every vertex in $D$ that is not an isolated vertex has at least two private neighbors other than itself.

Lemma 2.3.13. [17] Let $D$ be a $\gamma$-set of a graph $G$. Suppose for every $x \in D$, $\gamma(G - x) > \gamma(G)$, then $D$ is the unique $\gamma$-set of $G$.

Lemma 2.3.14. [17] Let $G$ be a graph which has a unique $\gamma$-set $D$, then for any $x \in G - D$, $\gamma(G - x) = \gamma(G)$.

Lemma 2.3.15. [17] Let $G$ be a graph with a unique $\gamma$-set $D$. Then $\gamma(G - x) \geq \gamma(G)$ for all $x \in D$.

Lemma 2.3.16. [20] If $G$ is a connected graph and $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$, then there is at most one end vertex adjacent to each $v \in V$, except possibly for one vertex which may be adjacent to exactly two end vertices when $n$ is odd.
Theorem 2.3.17. [20] A connected graph $G$ satisfies $\gamma(G) = \left\lceil \frac{n}{2} \right\rceil$ if and only if $G \in \mathcal{G} = \bigcup_{i=1}^{6} \mathcal{G}_i$.

Theorem 2.3.18. [25] For any two arbitrary graphs $G$ and $H$ and for $k \geq 1$, $|V(G)| \leq \gamma(kGoH) \leq 2|V(G)|$.

Theorem 2.3.19. [25] Let $G$ be a connected graph. If $H$ has a dominating vertex, then $\gamma(kGoH) = 2|V(G)|$.

Corollary 2.3.20. [25] Let $D$ be the $\gamma(kGoH)$-set with $|D| = n = |V(G)|$ and $\gamma(H) \geq 2$. Then $k \leq d(G)$, where $d(G)$ is the domatic number of $G$.

Theorem 2.3.21. [20] For any tree $T$, $\gamma(T) = n - \Delta(G)$ if and only if $T$ is a wounded spider.