Chapter 1

Introduction

1.1 Introduction

One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, covering, matching and inverse domination. Several types of domination parameters have been studied by imposing several conditions on dominating sets. Though substantial work has been carried out on topics of domination and related topics in graphs, there are only a few results concerning inverse domination in graphs. The purpose of this thesis is to study about inverse domination in graphs.

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. For graph theoretic terminology we refer to Harry [18]. A subset $S$ of $V$ is called a domi-
nating set if for every vertex \( v \in V - S \), there exists a vertex \( u \) in \( S \) such that \( u \) is adjacent to \( v \). The smallest cardinality of a minimal dominating set in \( G \) is called the \textit{domination number} of \( G \) and is denoted by \( \gamma(G) \) or simply \( \gamma \) when there is no possibility of confusion. Any dominating set with \( \gamma(G) \) vertices is called a \( \gamma \)-set of \( G \). An excellent treatment of fundamentals of domination in graphs is given in Haynes et.al. [20].

Let \( D \) be a \( \gamma \)-set of \( G \). A dominating set \( D' \) contained in \( V - D \) is called an \textit{inverse dominating set} of \( G \) with respect to \( D \). The smallest cardinality of a minimal dominating set in \( V - D \) is called the \textit{inverse domination number} of \( G \) and is denoted by \( \gamma'(G) \) or simply \( \gamma' \). Any inverse dominating set with respect to a dominating set with \( \gamma'(G) \) vertices is called a \( \gamma' \)-set of \( G \). By Ore’s Theorem [26], if a graph \( G \) has no isolated vertices, then the complement \( V - D \) of every minimal dominating set \( D \) contains a dominating set. Thus, every graph without isolated vertices contains an inverse dominating set with respect to a minimum dominating set, and has an inverse domination number. Due to this, we therefore restrict ourselves to graphs with no isolated vertices. The concept of inverse domination was introduced by Kulli in [24].
By the definition of an inverse domination number $\gamma'(G)$, we have for any graph $G$, with no isolated vertices, $\gamma(G) + \gamma'(G) \leq n$ and $\gamma'(G) \geq \gamma(G)$. Also Domke et.al [10] characterized the graphs which satisfy $\gamma(G) + \gamma'(G) = n$. In that paper they give a lower bound for the inverse domination number for trees. They also provide a constructive characterization of those trees which achieve this bound.

1.2 Contents Overview

In Chapter 1, we present a brief introduction and contents of the dissertation.

In Chapter 2, we collect the basic definitions and theorems in graphs.

In Chapter 3, we consider classes of graphs $G$ for which $\gamma(G) = \gamma'(G)$. Hereafter $G$ denotes a simple graph on $n$ vertices with no isolated vertices. We give some bounds for $\gamma'(G)$ through $n$ and $\gamma(G)$. We characterize the graphs with $\gamma(G) = \gamma'(G) = \frac{n}{2}$. Also we characterize the graphs with $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ and we give the inverse domination number for some classes of graphs.
In Chapter 4 we consider the inverse domination of grid graphs. The domination number of $k \times n$ grid graphs $P_k \times P_n$, for $1 \leq k \leq 10; \ n \geq 1$ have been previously established by Jacobson and Kinch [22, 23], Tony Yu Chang, W. Edwin Clark and E. O. Hare [7]. In this chapter, we obtain the inverse domination number and an inverse dominating set for the graphs $P_k \times P_n; \ 1 \leq k \leq 7; \ n \geq 1$ through smaller grids $P_k \times P_m, \ (m < n)$.

In Chapter 5, we characterize the class of graphs with minimum degree at least two for which the sum of their domination number and inverse domination number is $n - 1$. Further we construct classes of graphs with minimum degree one for which the sum of their domination number and inverse domination number is $n - 1$ and finally we characterize all graphs for which the sum of domination number and inverse domination number is $n - 1$.

In Chapter 6, we introduce inverse domination saturation. As in domination saturation, we classify graphs into type I and type II inverse domination saturated graphs. Further we introduce inverse domination unsaturation and we classify type I and type II inverse domination unsaturated graphs. We give
motivation to define inverse domination saturation. We define and give some examples for inverse domination saturation, inverse domination unsaturation and inverse domination saturation number with respect to a vertex and that to a graph. Also we give some results related to inverse domination saturation and classify into type I and type II inverse domination saturated graphs. Also some results related to inverse domination unsaturation are obtained and we classify into type I and type II inverse domination unsaturated graphs.