Chapter 6

Inverse Domination Saturation

6.1 Introduction

The concepts of domination saturation and domination saturation number $ds(G)$ were introduced by B.D. Acharya [1]. It was noted that $\gamma(G) \leq ds(G) \leq \gamma(G) + 1$. A graph $G$ is said to be of class I with respect to domination saturation if $ds(G) = \gamma(G)$, where as $G$ is said to be of class II if $ds(G) = \gamma(G) + 1$ [2].

In this chapter, we introduce inverse domination saturation. As in domination saturation, we classify graphs into type I and type II inverse domination saturated graphs. Also we introduce inverse domination unsaturation and we classify type I and type II inverse domination unsaturated graphs.
In section 6.2, we give some graphs which motivate to define inverse domination saturation. We define and give some examples for inverse domination saturation, inverse domination unsaturation and inverse domination saturation number with respect to a vertex and that to a graph. Also we give some results related to inverse domination saturation and classify into type I and type II inverse domination saturated graphs. In section 6.3, we give some results related to inverse domination unsaturation and classify into type I and type II inverse domination unsaturated graphs.

6.2 Classification of Inverse Domination Satturation

In this section, we classify the inverse domination saturation into two classes.

Example 6.2.1. Consider a cycle $G = C_5$ with $V(C_5) = \{1, 2, 3, 4, 5\}$. Here $D = \{2, 5\}$ is a $\gamma-$ set of $G$. $D_1^{'} = \{1, 4\}$ and $D_2^{'} = \{1, 3\}$ are $\gamma^{'}$-sets of $G$ with respect to $D$. One can note that $D$ is an inverse dominating set with respect to a $\gamma$-set $D_1^{'}$ of $G$. Hence $D$ is also a $\gamma^{'}-$ set of $G$. Thus each vertex of $G = C_5$, is contained in a $\gamma^{'}$-set of $G$. 

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Example 6.2.2. Consider a path $G = P_5$ with $V(P_5) = \{1, 2, 3, 4, 5\}$. Here, $D = \{2, 5\}$ is a $\gamma$-set of $G$ and $D' = \{1, 4\}$ is a $\gamma'$-set of $G$ and vice versa. Note that there is no $\gamma'$-set of $P_5$, containing the vertex 3. However, $\{1, 3, 4\}$ is an inverse dominating set with respect to $D$, containing 3.

Example 6.2.3. Consider the graph $G = K_{1,n}$ with $V(K_{1,n}) = \{0, 1, 2, \ldots, n\}$. Let 0 be the vertex of degree $n$. Then $D = \{0\}$ is the $\gamma$-set of $G$ and $D' = \{1, 2, \ldots, n\}$ is the $\gamma'$-set of $G$. One can realize that $\{0\}$ is not contained in any inverse dominating set of $G$ and so in any $\gamma'$-set of $G$.

From the above, we consider the following cases:

(i) There are graphs for which each vertex is contained in some $\gamma'$-set.

(ii) There are graphs for which some vertices are contained in some inverse dominating set but not contained in any $\gamma'$-set of $G$.

(iii) There are graphs in which some vertices are neither contained in any $\gamma'$-set of $G$ nor in any inverse dominating set of $G$. 

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Hence we define inverse domination saturation of a vertex and inverse domination saturation of a graph.

**Definition 6.2.4.** Let $G$ be a connected graph and $u \in V(G)$. Then $u$ is said to be inverse domination saturated if there exists an inverse dominating set containing $u$, and is said to be inverse domination unsaturated if there exists no inverse dominating set containing $u$.

**Example 6.2.5.** Every vertex of $C_5$ is inverse domination saturated.

**Example 6.2.6.** Let $u$ be a vertex of a tree $T$, which is adjacent to at least two pendent vertices. Then $u$ lies in every $\gamma$-set of $T$. Hence there exists no inverse dominating set containing $u$. Therefore $u$ is inverse domination unsaturated.

**Definition 6.2.7.** Let $G$ be a connected graph and let $u \in V(G)$. Define $d'S(u)$ by

$$
\begin{cases}
\text{Min}\{ |S'|; \text{ where } S' \text{ is an inverse dominating set of } G \text{ containing } u \} \\
0 \text{ if no such set exists.}
\end{cases}
$$
Example 6.2.8. Consider $G = K_{1,n}$. Let $u$ be the vertex of degree $n$. Then $d^{'S}(u) = 0$ and $d^{'S}(v) = n$, for other vertices $v \in V(G) - u$.

Example 6.2.9. Consider the graph $P_8$ with $V(P_8) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let $\{1, 8\}$ be the pendant vertices. Then $D = \{2, 5, 8\}$ is a $\gamma$-set and $D^{'} = \{1, 4, 7\}$ is a $\gamma^{' }$-set of $P_8$ and vice versa. Hence $d^{'S}(i) = 3$ for $i = 1, 2, 4, 5, 7, 8$. Note that there is no $\gamma^{' }$-set containing $3$ or $6$. But $\{1, 3, 6, 7\}$ is an inverse dominating set containing $3$ and $6$. Hence $d^{'S}(i) = 4$ for $i = 3, 6$.

Definition 6.2.10. Let $G$ be a connected graph. Then $G$ is said to be an inverse domination saturated, if each vertex $u \in V(G)$ is inverse domination saturated and is said to be an inverse domination unsaturated if at least one vertex $u \in V(G)$ is inverse domination unsaturated.

Example 6.2.11. (1) All cycles $C_n$ are inverse domination saturated.

(2) The corona $G = H \circ K_1$ is inverse domination saturated .

(3) $G = K_{1,n}$ is inverse domination unsaturated.
Definition 6.2.12. Let $G$ be an inverse domination saturated graph. Then the inverse domination saturation number denoted by $d' S(G)$ is defined by

$$d' S(G) = \text{Max}_{u \in V(G)} \{d' S(u)\}.$$  

Example 6.2.13. Consider the following 3-regular graph given in Figure 6.1.

![Figure 6.1](image)

Here $D_1 = \{1, 2, 10\}$, $D_2 = \{4, 5, 8\}$, $D_3 = \{1, 5, 9\}$ are $\gamma$-sets, and $D'_1 = \{3, 4, 7\}$, $D'_2 = \{2, 3, 6\}$, $D'_3 = \{3, 4, 7\}$ are the $\gamma'$-sets respectively and vice versa. Thus for each $i, 1 \leq i \leq 10$, there exists a $\gamma'$-set containing $i$. Thus $d' S(i) = 3$ for $1 \leq i \leq 10$ and so $d' S(G) = 3$. Also note that $\gamma'(G)=3$. 
Remark 6.2.14. A connected graph $G$ is inverse domination saturated if and only if there exists a $\gamma$-set not containing $v$, for each $v \in V(G)$. Also for a connected graph $G$, if $\gamma(G) = \gamma'(G)$, then $G$ is inverse domination saturated.

Remark 6.2.15. On the converse of Remark 6.2.14 need not be true. Consider the graph given in Figure 6.2. $D_1 = \{1, 4, 7, 11\}, D_2 = \{2, 5, 6, 11\}, D_3 = \{2, 7, 9, 12\}$ are $\gamma$-sets of $G$ and $D_{1,1}' = \{2, 5, 6, 9, 10\}, D_{1,2}' = \{2, 5, 8, 10, 12\}, D_{2,1}' = \{1, 4, 7, 10, 12\}, D_{2,2}' = \{1, 3, 7, 9, 10\}, D_{3,1}' = \{3, 4, 6, 8, 11\}, D_{3,2}' = \{1, 4, 5, 6, 11\}$ are the corresponding $\gamma'$-sets of $G$. Thus each vertex in $G$ is inverse domination saturated. Hence the graph $G$ is inverse domination saturated. But $\gamma(G) \neq \gamma'(G)$.

Figure 6.2
Lemma 6.2.16. Let $G$ be a connected graph with $\gamma(G) = \gamma'(G)$. Then $\gamma'(G) \leq d'S(G) \leq \gamma'(G) + 1$.

Proof. Let $G$ be a connected graph with $\gamma(G) = \gamma'(G)$. Then by Remark 6.2.14, $G$ is an inverse domination saturated graph.

Case 1 Suppose that $D_v'$ is a $\gamma'$-set containing $v$, for each $v \in V(G)$, then $d'S(v) = \gamma'(G)$ for each vertex $v \in V(G)$. Then $d'S(G) = \gamma'(G)$.

Case 2 Suppose that there exists a vertex say $u \in V(G)$ such that there exists no $\gamma'$-set containing $u$. Let $T$ and $T'$ be a $\gamma$-set and $\gamma'$-set of $G$ respectively. Then $u \notin T \cup T'$. Therefore $T' \cup \{u\}$ is the smallest inverse dominating set containing $u$. Hence $d'S(G) = Max\{d'S(u)\}$ for $u \in V(G)$, implies that $d'S(G) = \gamma'(G) + 1$. Thus $d'S(G) = \gamma'(G)$ or $\gamma'(G) + 1$. Hence $\gamma'(G) \leq d'S(G) \leq \gamma'(G) + 1$. □

Proposition 6.2.17. Let $G(\neq K_n)$ be a connected graph with at least two dominating vertices. Then $d'S(G) = \gamma'(G) + 1 = 2$. 

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Proof. Since $G$ has at least two dominating vertices, say $u, v$, we have $\gamma(G) = \gamma'(G)=1$. Then $D = \{u\}$ is a $\gamma$-set of $G$ and $D' = \{v\}$ is a $\gamma'$-set of $G$. Since $G \neq K_n$, there exists at least one vertex say $w \in V(G)$, such that $d(w) < n - 1$. Hence $D' \cup \{w\}$ is the smallest inverse dominating set containing $w$. Therefore $d'S(w) = \gamma'(G) + 1$. Thus $d'S(G) = \gamma'(G) + 1=2$. \qed

Definition 6.2.18. An inverse domination saturated graph $G$ is said to be of type I inv. dom. saturated if $d'S(G) = \gamma'(G)$ and it is said to be of type II inv. dom. saturated otherwise.

Example 6.2.19. $K_n$ ($n \geq 2$) and $P_4$ are type I inv. dom saturated.

Example 6.2.20. Note that $\gamma'(P_5) = 2$ and $d'S(P_5) = 3$. Hence $P_5$ is type II inv. dom saturated.

Theorem 6.2.21. For all $n \geq 3$, $C_n$ is type I inv. dom saturated.
Proof. In view of Theorem 2.3.5, there exists a $\gamma'$-set containing each vertex of $C_n$. Hence $C_n$ is type I inv.dom saturated for all $n$. □

Lemma 6.2.22. For $n \geq 3$, $P_n$ is inv. dom saturated if and only if $n \not\equiv 0(\text{mod } 3)$. Also $P_n$ is of type I inv. dom. saturated if $n \equiv 1(\text{mod } 3)$ and is type II inv. dom. saturated if $n \equiv 2(\text{mod } 3)$.

Proof. Assume that $P_n$ is inv. dom saturated and let $V(P_n) = \{1, 2, 3, \ldots 3k\}$. Suppose $n \equiv 0(\text{mod } 3)$ and so $n = 3k$ for some $k$. Then $D = \{2, 5, 8, \ldots 3k - 1\}$ is a $\gamma$-set of $G$, and $D_1' = \{1, 4, 7, \ldots 3k - 2, 3k\}$, $D_2' = \{1, 3, 6, 9, \ldots 3k\}$ are $\gamma'$-sets of $P_{3k}$. Note that $\gamma(G) = k$ and $\gamma'(G) = k + 1$ and that there exists an inverse dominating set containing each element of $V - D$. But there exists no inverse dominating set containing any element of $D$, since $D$ is the only $\gamma$-set of $G$. Hence $P_{3k}$ is inv. dom. unsaturated. Therefore $n \not\equiv 0(\text{mod } 3)$.

Conversely, assume that $n \not\equiv 0(\text{mod } 3)$. Then either $n \equiv 1(\text{mod } 3)$ or $n \equiv 2(\text{mod } 3)$.  

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**Case (i)** When $n \equiv 1(\text{mod } 3)$, $n = 3k + 1$ for some $k \geq 1$. Then $D = \{1, 4, 7, \ldots, 3k - 2, 3k + 1\}$ is a $\gamma$-set of $G$, with $k + 1$ elements and $D_1' = \{2, 5, 8, \ldots, 3k - 1, 3k\}$, $D_2' = \{2, 3, 6, 9, \ldots, 3k\}$ are $\gamma'$-sets of $G$. Note that $D$ is an inverse dominating set of $G$ with respect to $D_1'$. Since $|D| = |D_1'| = k + 1$, $D$ is also a $\gamma'$-set of $G$. Hence for each $i \in V(G)$, there exists a $\gamma'$-set containing $i$. Thus $P_{3k+1}$ is type I inv. dom. saturated.

**Case (ii)** When $n \equiv 2(\text{mod } 3)$, $n = 3k + 2$ for some $k \geq 1$. Then $D = \{1, 4, 7, \ldots, 3k + 1\}$ is a $\gamma$-set of $G$, with $k + 1$ elements and $D_1' = \{2, 5, 8, \ldots, 3k + 2\}$, is a $\gamma'$-set of $G$. Note that $D$ is an inverse dominating set of $G$ with respect to $D_1'$ and also that $|D| = |D_1'| = k + 1$. Therefore $D$ is also a $\gamma'$-set of $G$. Hence for each $i \in (D \cup D')$, there exists a $\gamma'$-set containing $i$. But for the vertices of form $3i$, there exists no $\gamma'$-set containing it. However $D_2' = \{2, 3, 6, 9, \ldots, 3k, 3k + 2\}$ is an inverse dominating set with respect to $D$ containing the vertices of form $3i$. Also note that $|D_2'| = k + 2 = \gamma'(G) + 1$. Therefore $P_{3k+2}$ is type II inv. dom. saturated. \hfill \Box
Theorem 6.2.23. Let $G$ be a connected graph with $\gamma(G).d(G) = n$, where $d(G)$ is the domatic number of $G$. Then $G$ is of type I inv.dom. saturated.

Proof. Let $D_1, D_2, D_3, \ldots D_d$ be a domatic partition of $V(G)$ into dominating sets. Therefore $|D_i| \geq \gamma(G)$, for each $i$. Hence $n = \sum_{i=1}^{d} |D_i| \geq |D_i|d(G) \geq \gamma(G)d(G) = n$. This implies that $|D_i| = \gamma(G)$, for all $i$. Hence each $D_i$ is a $\gamma$-set of $G$. Then each $D_i$ is an inverse dominating set with respect to $D_j$, for $i \neq j$, and hence a $\gamma'$-set of $G$. Thus each vertex in $G$ lies in a $\gamma'$-set of $G$. Hence $G$ is of type I inv. dom. saturated. \qed

Remark 6.2.24. The converse of the Theorem 6.2.23 is not true in general. For, consider the cycle $C_5$. Let $V(C_5) = \{1, 2, 3, 4, 5\}$. Then $D = \{1, 4\}$ is a $\gamma$-set and $D_1' = \{2, 5\}$ and $D_2' = \{3, 5\}$ are $\gamma'$ sets containing each vertex of $V(G) - D$. Also note that $D$ is an inverse dominating set of $G$ with respect to $D_1'$ and hence a $\gamma'$-set of $G$. Thus there exists a $\gamma'$-set containing each vertex of $C_5$. Hence $C_5$ is of type I inv. dom. saturated. But $\gamma(G).d(G) \neq n$ since $\gamma(G) = 2$ and $d(G) = 2$. 

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Theorem 6.2.25. Let \( G \) be a regular graph and domatically full. Then \( G \) is of type I inv. dom. saturated.

Proof. Let \( G \) be a \( k \)-regular graph which is domatically full. Then \( d(G) = \delta(G) + 1 = k + 1 \). Let \( D_1, D_2, \ldots, D_{\delta(G)+1} \) be a domatic partition of \( G \). Let \( u \in V(G) \) be any vertex. Then either \( u \in D_i \) or exactly one of its neighbors is in \( D_i \), for each \( i \), since each \( D_i \) is a dominating set. Also for \( i \neq j \), each vertex in \( D_i \) is adjacent to exactly one vertex in \( D_j \). For if, a vertex in \( D_i \) is adjacent to more than one vertex in \( D_j \), since \( d(u) = k \) and there are \( k \) other sets \( D_i \), \( u \) cannot be adjacent to any vertex in some set \( D_k \). Also note that \( |D_i| = |D_j| \) for all \( i, j \). For if, \( |D_i| < |D_j| \) for some \( i \neq j \), then there are at least two vertices in \( D_j \) that are adjacent to one vertex in \( D_i \), which gives a contradiction.

Next we claim that \( |D_i| = \gamma \), for all \( i \). Clearly \( |D_i| \geq \gamma \). Suppose \( |D_i| \geq \gamma + 1 \) for some \( i \), then \( n \geq k\gamma + (\gamma + 1) \). That is \( n \geq (k+1)\cdot \gamma(G) + 1 \), which implies \( \gamma(G) < \frac{n}{(k+1)} = \frac{n}{(\Delta(G)+1)} \), a contradiction to \( \gamma \geq \frac{n}{(\Delta(G)+1)} \). Hence \( |D_i| = \gamma \), for each \( i \).

Thus \( \sum |D_i| = n \) implies that \( \gamma(G)d(G) = n \), where \( d(G) \) is the domatic number of \( G \). Then by Theorem 6.2.23, \( G \) is of type I inv. dom. saturated. \( \square \)
Remark 6.2.26. The converse need not be true in general. Consider the graph $G = C_5$. Clearly $G$ is a regular graph which is type I inv. dom. saturated. But here, $d(G) = 2 \neq \delta(G) + 1$. Therefore $G$ is not domatically full.

6.3 Classification of Inverse Domination Unsaturation

In this section, we classify the inverse domination unsaturation as type I inverse domination unsaturation and type II inverse domination unsaturation.

Example 6.3.1. Consider a twin star graph $G$ in Figure 6.3 with $V(G) = \{1, 2, \ldots, 8\}$.

Here $D = \{4, 5\}$ is the $\gamma$–set of $G$ and $D' = \{1, 2, 3, 6, 7, 8\}$ is the $\gamma'$–set of $G$. One can note that the vertices of $D$ is neither contained in a $\gamma'$–set of $G$ nor in an
inverse dominating set of $G$. Hence $G$ is inverse domination unsaturated.

**Example 6.3.2.** Consider the 3-regular graph given in Figure 6.4.

![Figure 6.4](image)

Here $D = \{1, 4, 9, 14\}$ is a $\gamma$-set of $G$, $D_1' = \{2, 5, 7, 8, 13, 15\}$ and $D_2' = \{3, 6, 7, 10, 12, 16\}$ are $\gamma'$-sets of $G$. Also note that $T = \{1, 5, 8, 15\}$ is a $\gamma$-set of $G$, $T_1' = \{2, 4, 7, 9, 13, 14\}$ and $T_2' = \{3, 6, 7, 11, 12, 16\}$ are $\gamma'$-sets of $G$. Thus $d'S(i) = 6$, for $i = 2$ to 16. But $d'S(1) = 0$. Hence $G$ is inverse domination unsaturated.

From these examples we infer that the class of inv. dom. unsaturated graphs are of two types. This motivates to define
type I inv. dom. unsaturated graphs and type II inv.dom. unsaturated graphs.

**Definition 6.3.3.** An inv. dom.unsaturated graph is said to be of type I inv. dom. unsaturated, if there exists no inverse dominating set containing any vertex of a $\gamma$-set of $G$, and is said to be of type II inv. dom. unsaturated if there exists an inverse dominating set containing at least one vertex of every $\gamma$-set of $G$.

**Example 6.3.4.** $K_{1,n}$ is type I inv. dom. unsaturated.

**Example 6.3.5.** $P_6$ is of type I inv. dom. unsaturated. For, $D = \{2, 5,\}$ is a $\gamma$-set of $P_6$ and $D_1' = \{1, 4, 6\}$ and $D_2' = \{1, 3, 6\}$ are the $\gamma'$-sets of $P_6$. Hence there is no inverse dominating set containing the vertices of the $\gamma$-set.

**Example 6.3.6.** The 3- regular graph cited above in Figure 6.4 is type II inv. dom. unsaturated. For, as seen in example 6.3.2, the vertex 1 is the only vertex in the $\gamma$-set of $G$, that is not contained in any inverse dominating set of $G$. 

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**Remark 6.3.7.** In view of Lemma 6.2.22, $P_n$ is type I inv. dom. unsaturated if and only if $n \equiv 0 \pmod{3}$.

**Theorem 6.3.8.** A wounded spider $G$ is of type II inv. dom. unsaturated if and only if $\gamma(G) \neq \Delta(G)$.

**Proof.** Let $G$ be a wounded spider which is inv. dom. unsaturated. Since $G$ is a wounded spider, by Theorem 3.2.11 we have $\gamma'(G) = \Delta(G)$. Suppose $\gamma(G) = \Delta(G)$, then we get $\gamma(G) = \gamma'(G)$. By Remark 6.2.14, $G$ is inv. dom. saturated, which is a contradiction. Hence $\gamma(G) \neq \Delta(G)$.

Conversely let $G$ be a wounded spider with $\gamma(G) \neq \Delta(G)$. Since $\gamma'(G) = \Delta(G)$ for a wounded spider and $\gamma(G) \neq \Delta(G)$, we see that at most $\Delta - 2$ edges of $K_{1, \Delta(G)}$ are subdivided. Let $V(G) = \{v, 1, 2, \ldots, \Delta, v_1, v_2, \ldots, v_k\}$, where $1, 2, \ldots, \Delta$ are the vertices incident with $v$ and each $v_i$ is adjacent with $i$ for $i = 1, 2, 3, \ldots, k \leq \Delta - 2$. Then we see that $D = \{v, v_1, v_2, \ldots, v_k\}$ is a $\gamma$-set and $D' = \{1, 2, \ldots, \Delta\}$ is a $\gamma'$-set of $G$. Also $T = \{v, 1, 2, \ldots, k\}$ is a $\gamma$-set and $T' = \{v_1, v_2, \ldots, v_k, k + 1, \ldots, \Delta\}$ is a $\gamma'$-set of $G$. However there is no inverse dominating set containing $v$. Hence $G$ is of type II inv. dom. unsaturated. \qed
**Theorem 6.3.9.** Let $G$ be a connected graph. Then $G$ is of type I inv. dom. unsaturated if and only if $G$ has an unique $\gamma$-set.

**Proof.** Let $G$ be a connected graph such that $G$ is of type I inv. dom. unsaturated. Then there exists no inverse dominating set containing any vertex of at least one $\gamma$-set say $D$ of $G$. Suppose $G$ has more than one $\gamma$-set, then we have the following cases.

**Case (i)** Suppose that there exists at least two disjoint $\gamma$-sets say $D_1$ and $D_2$. Since $D_j \subseteq V(G) - D_i$, for $i, j = 1, 2$ and $i \neq j$, each vertex of $D_i$, for $i = 1, 2$ and lies in the inverse dominating set $V(G) - D_j$. Also each vertex of $V(G) - D_i$, for $i = 1, 2$ lies in the inverse dominating set $D_j \cup \{x\}$. Hence $G$ is inv. dom. saturated, a contradiction.

**Case (ii)** Let $D_1$ and $D_2$ be any two $\gamma$-sets with $D_1 \cap D_2 \neq \phi$. Then each vertex of $D_1 - D_2$ lies in the inverse dominating set $V(G) - D_2$ and each vertex of $D_2 - D_1$ lies in the inverse dominating set $V(G) - D_1$. Thus at least one vertex in each $D_i$, for $i = 1, 2$ is inverse domination saturated. Hence $G$ is not of type I inv. dom. unsaturated, a contradiction. Therefore $G$ has an unique $\gamma$-set.
Conversely assume that $G$ has an unique $\gamma$-set, say $D$. Then trivially there exists no inverse dominating set containing any vertex of $D$. Hence $G$ is type I inv. dom. unsaturated. \qed

**Theorem 6.3.10.** Let $G$ be a connected graph which is type I inv. dom. unsaturated. Then there exists an inverse dominating set $D'$ in which at least one vertex has no private neighbor other than itself with respect to $D'$.

**Proof.** Let $G$ be a connected graph which is type I inv. dom. unsaturated. Then by Theorem 6.3.9, $G$ has an unique $\gamma$-set say $D$. Then by Lemma 2.3.12, every vertex in $D$ has at least two private neighbors other than itself. Let $D' = V(G) - D$. Then any vertex $x \in D$ is not a private neighbor of any vertex in $D'$. Hence at least one vertex in $D'$ has no private neighbors other than itself with respect to $D'$. \qed

**Corollary 6.3.11.** Let $G$ be a connected graph which is type I inv. dom. unsaturated. Then there exists a $\gamma$-set $D$ in which no vertex is a private neighbor of any vertex with respect to an inverse dominating set $D'$. 

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Proposition 6.3.12. Let $G$ be a connected graph which is type I inv. dom. unsaturated. Then there exists a $\gamma$-set $D$, such that $\gamma(G - x) \geq \gamma(G)$, for all $x \in D$.

Proof. Let $G$ be a connected graph which is type I inv. dom. unsaturated. Then by Theorem 6.3.9, $G$ has an unique $\gamma$-set say, $D$. Now by Lemma 2.3.15, $\gamma(G - x) \geq \gamma(G)$, for all $x \in D$. \qed