Chapter 5

Sum of Domination and Inverse Domination numbers

In this chapter we deal about results concerning sum of domination and inverse domination numbers.

5.1 Introduction

A Gallai-type theorem has the from \( \alpha(G) + \beta(G) = n \), where \( \alpha(G) \) and \( \beta(G) \) are parameters defined on \( G \), and \( n \) is the number of vertices in \( G \). Cockayne, Hedetniemi and Laskar [9] give Gallai-type theorems for graphs. Gayla Domke, Jean Dunbar and Lisa Markus [13] characterize the graphs for which \( \gamma(G) + \gamma'(G) = n \). In this chapter, we characterize the graphs which satisfy \( \gamma(G) + \gamma'(G) = n - 1 \).

In the section 5.2, we characterize the class of graphs with
minimum degree at least two for which the sum of their domination number and inverse domination number is $n - 1$. In the section 5.3, we construct classes of graphs with minimum degree one for which the sum of their domination number and inverse domination number is $n - 1$. Further we prove that the class of graphs constructed are the only graphs with minimum degree one, for which the sum of their domination number and inverse domination number is $n - 1$.

5.2 Graphs with minimum degree at least two and $\gamma(G) + \gamma'(G) = n - 1$

Here, we attempt to get characterization for graphs with $\gamma(G) + \gamma'(G) = n - 1$ and $\delta(G) \geq 2$. To attain this aim, we first present certain general results which are useful in the further discussion.

**Lemma 5.2.1.** Let $G$ be a connected graph on $n$ vertices with $\delta(G) \geq 2$. Then $\gamma(G) + \gamma'(G) = n - 1$ implies $\gamma(G) = \gamma'(G)$.

*Proof.* Assume $\gamma(G) + \gamma'(G) = n - 1$. Suppose $\gamma(G) < \gamma'(G)$. Let $D$ be a $\gamma$-set of $G$ and let $D'$ be a $\gamma'$-set of $G$
with respect to $D$. Let $\{w\} = V(G) - (D \cup D')$. Let $S \subseteq D$ be those vertices that are adjacent to more than one vertex in $D'$. Let $S' = N(S) \cap D'$.

We claim that there is at most one vertex in $S'$, that is adjacent to a vertex in $D - S$. Suppose there are at least two vertices $t', r'$ in $S'$, that are adjacent to vertices $t, r \in D - S$ respectively. There are two possibilities:

1. either both $t, r$ in $D - S$ are adjacent to $w$.
2. at least one of them is not adjacent to $w$.

**Case A** When both $t, r \in D - S$ are adjacent to $w$, then $D_1 = D \cup \{w\} - \{t, r\}$ is a dominating set of $G$, since $t', r'$ in $S'$ are the only vertices adjacent to $t, r \in D - S$, which are already dominated by some vertex in $S$. A contradiction to $D$ is a $\gamma$-set of $G$.

**Case B** When at least one of them say $t$ is not adjacent to $w$, since $t \in D - S$ and since $\delta(G) \geq 2$, $t$ is adjacent to a vertex $u$ in $D$. Therefore $D_1 = D - \{t\}$ is a dominating set of $G$. A contradiction to $D$ is a $\gamma$-set of $G$. Hence there is at most one vertex $t' \in S'$, that is adjacent to a vertex $t \in D - S$. 

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Let \( S' = S' - \{ t' \} \) if \( t' \) exists as described above. And \( S' = S' \) if no such \( t' \) exists.

Now we have the following four possibilities. They are:

1. \( w \) is adjacent to no vertex in \( S' \).
2. \( w \) is adjacent to exactly one vertex in \( S' \).
3. \( w \) is adjacent to exactly two vertices in \( S' \).
4. \( w \) is adjacent to more than two vertices in \( S' \).

**Case 1** When \( w \) is adjacent to no vertex in \( S' \), we have two possibilities. They are either there exists at least one vertex say \( u' \) in \( S' \), that is adjacent to a vertex in \( D' \) or each vertex in \( S' \) is adjacent to at least two vertices in \( S \).

**Sub case 1.1** Suppose there exists a vertex say \( u' \) in \( S' \), that is adjacent to a vertex in \( D' \), then \( D' = D' - \{ u' \} \) is an inverse dominating set with respect to \( D \). Therefore \( \gamma(G) + \gamma'(G) \leq n - 2. \)

**Sub case 1.2** Suppose that each vertex in \( S' \) is adjacent to at least two vertices in \( S \). By our construction, for every \( v' \in D' - S' \), there exists an unique vertex \( v \in D - S \) such that \( v \) is adjacent to \( v' \). Also \( |D| < |D'| \). Hence \( |S| < |S'| \). Therefore we can find two vertices \( x, y \) in \( S \) that are
adjacent to at least three vertices say $x', y', z'$. W. L. G. let $x', y' \in N(x) \cap S'$ be such that both $x', y'$ are adjacent to some vertices in $S - x$. Now $D_1 = D - \{x\} \cup \{y'\}$ is a $\gamma$-set of $G$. Let $D_1' = D' - \{x', y'\} \cup \{x\}$. Since $x'$ is adjacent to some vertex in $S$ and to the vertex $x \in D_1'$ and each vertex in $S$, that were dominated by $x' \in D'$ are dominated by some other vertex in $D_1'$, we see that $D_1'$ is an inverse dominating set with respect to $D_1$. Therefore $\gamma(G) + \gamma'(G) \leq n - 2$.

**Case 2** When $w$ is adjacent to exactly one vertex say $x'$ in $S_1'$, either there exists at least one vertex in $S_1' - \{x'\}$, that is adjacent to a vertex in $D'$ or each vertex in $S_1' - \{x'\}$ is adjacent to at least two vertices in $S$.

**Sub case 2.1** Suppose there exists a vertex say $u'$ in $S_1' - \{x'\}$, that is adjacent to a vertex in $D'$, then as in sub case 1.1, we get $\gamma(G) + \gamma'(G) \leq n - 2$.

**Sub case 2.2** Suppose each vertex in $S_1' - \{x'\}$ is adjacent to at least two vertices in $S$, then as in sub case 1.2 we get $\gamma(G) + \gamma'(G) \leq n - 2$.

**Case 3** When $w$ is adjacent to exactly two vertices say $x', y'$ in $S_1'$, either there exists at least one vertex in $S_1'$, that
is adjacent to a vertex in $D'$ or not.

**Sub case 3.1** Suppose there exists a vertex say $u'$ in $S_1'$, that is adjacent to a vertex in $D'$, then as in sub case 1.1, we get $\gamma(G) + \gamma'(G) \leq n - 2$.

**Sub case 3.2** When there is no vertex in $S_1'$, that is adjacent to a vertex in $D'$, then $D_1' = D' \cup \{w\} - \{y'\}$ is an inverse dominating set. Note that $y'$ is the vertex that is neither in the $\gamma$-set $D$ of $G$, nor in the $\gamma'$-set $D_1'$ of $G$ with $y'$ being adjacent to exactly one vertex $w$ in the $\gamma'$-set $D_1'$. Then by replacing $w$ by $y$ as in case 2, we get $\gamma(G) + \gamma'(G) \leq n - 2$.

**Case 4.** When $w$ is adjacent to more than two vertices in $S_1'$, either there exists at least one vertex in $S_1'$, that is adjacent to a vertex in $D'$ or not.

**Sub case 4.1** Suppose there exists a vertex say $u'$ in $S_1'$, that is adjacent to a vertex in $D'$, then as in sub case 1.1, we get $\gamma(G) + \gamma'(G) \leq n - 2$.

**Sub case 4.2** When there is no vertex in $S_1'$, that is adjacent to a vertex in $D'$, then let $x', y', z' \in S_1'$ be adjacent to $w$. Suppose $x', y', z'$ are adjacent to the same set.
of vertices \(\{x, y\}\) in \(S\), then \(D_1' = D' \cup \{w\} - \{y', z'\}\) is an inverse dominating set with respect to \(D\). Otherwise there exists two vertices \(x', y' \in S_1'\) (W.L.G), such that \(N(x') \cap S \neq N(y') \cap S\). In this case, \(D_1' = D' \cup \{w\} - \{x', y'\}\) is an inverse dominating set with respect to \(D\). In either cases, we get \(\gamma(G) + \gamma'(G) \leq n - 2\).

Thus in all the cases, we get \(\gamma(G) + \gamma'(G) \leq n - 2\), a contradiction. Hence \(\gamma(G) = \gamma'(G)\).

**Theorem 5.2.2.** Let \(G\) be a connected graph with \(\delta(G) \geq 2\). Then \(\gamma(G) + \gamma'(G) = n - 1\) if and only if \(\gamma(G) = \gamma'(G) = \left\lfloor \frac{n}{2} \right\rfloor\) and \(n\) is odd.

**Proof.** Assume \(\gamma(G) + \gamma'(G) = n - 1\).

Then by Lemma 5.2.1, we get \(\gamma(G) = \gamma'(G)\). Therefore \(\gamma(G) = \gamma'(G) = \frac{n - 1}{2}\) and hence \(n\) is odd.

Conversely, assume that \(\gamma(G) = \gamma'(G) = \left\lfloor \frac{n}{2} \right\rfloor\) and \(n\) is odd.

Since \(n\) is an odd number we get \(\gamma(G) + \gamma'(G) = n - 1\). \(\square\)

**Remark 5.2.3.** Theorem 5.2.2 and Lemma 3.5.1 characterize the graphs with minimum degree two and which satisfy \(\gamma(G) +\)
\[ \gamma'(G) = n - 1. \] Therefore we have obtained the following result.

**Theorem 5.2.4.** Let \( G \) be a connected graph with minimum degree two. Then \( \gamma(G) + \gamma'(G) = n - 1 \) if and only if \( G \in \mathbb{A} \cup \mathbb{B} \), where \( \mathbb{A} \) and \( \mathbb{B} \) represent the classes of graphs shown in Figure 3.1 and in Figure 3.2.

### 5.3 Graphs with minimum degree one and \( \gamma(G) + \gamma'(G) = n - 1 \)

So far, we have identified the graphs with \( \delta(G) \geq 2 \) which satisfy \( \gamma(G) + \gamma'(G) = n - 1 \). In order to identify the graphs with \( \delta(G) = 1 \) which satisfy \( \gamma(G) + \gamma'(G) = n - 1 \), we construct a set of classes of graphs as detailed below. Further we prove that these are the only graphs which exactly fulfill our requirement.

**Construction \( C_1 \):** Let \( H \) be any connected graph and \( S = V(H) \). Let \( \mathbb{P}(H) \) denoted the set of connected graphs obtained from \( H \) by adding a set of new vertices and edges such that each vertex in \( V(H) \) is adjacent to one or more new pendant vertices. Define \( \mathbb{G}_1 = \bigcup \mathbb{P}(H) \) for any connected
Let $G \in \mathcal{G}_1$ and let $S_1$ be the set of all vertices $v \in V(G)$ such that $v$ is adjacent to exactly one pendant vertex in $G$. Clearly $S_1 \subseteq S$ and let $S_2 = S - S_1$. Let $L$ denoted the set of all pendant vertices. Let $L_1$ denoted the set $N(S_1) \cap L$ and let $L_2$ denoted the set $N(S_2) \cap L$. Here after whenever, we denoted $S, \ S_1, \ S_2, \ L, \ L_1$ and $L_2$ they mean only these six sets. The class of graphs $\mathcal{G}_1$ are given in Figure 5.1.

![Figure 5.1](image)

**Construction** $C_2$: Let $\mathcal{Q}(G)$ denoted the set of connected graphs obtained from $G \in \mathcal{G}_1$ by adding a set of vertices say $I$ and a set of edges joining each vertex of $I$ with two or more vertices in $S_2$. Define $\mathcal{G}_2 = \cup \mathcal{Q}(G)$ where, $G \in \mathcal{G}_1$. The class of graphs $\mathcal{G}_2$ are given in Figure 5.2.
Construction $C_3$: Let $S(K)$ denote the set of connected graphs obtained from $K \in \mathcal{G}_2$ by adding a new vertex $x$ and a set of edges joining $x$ to two or more vertices in $S$ with at least one vertex in $S_1$. Then define $\mathcal{R}_1 = \bigcup S(K)$ where $K \in \mathcal{G}_2$. The class of graphs $\mathcal{R}_1$ are given in Figure 5.3.
Construction $C_4$: Let $T(K)$ denoted the set of connected graphs obtained from $K \in G_2$ by adding two vertices $x$ and $y$, with an edge $xy$ and joining each of $x, y$ to vertices in $S$ such that when both $x$ and $y$ are adjacent to vertices in $S_1$, then either $|N(x) \cap S| = 1$ or $|N(y) \cap S| = 1$. Then define $R_2 = \cup T(K)$, where $K \in G_2$. The class of graphs $R_2$ are given in Figure 5.4.

![Figure 5.4](image)

Construction $C_5$: Let $Z(K)$ denoted the set of connected graphs obtained from $K \in G_2$ by adding a new path $P(w, x, y)$ and edges joining each of $w$ and $y$ to one or more vertices in $S$. Then define $R_3 = \cup Z(K)$, where $K \in G_2$. The class of graphs $R_3$ are given in Figure 5.5.
Construction $C_6$: Let $\mathbb{V}(K)$ denoted the set of connected graphs obtained from $K \in \mathcal{G}_2$ by adding a new path $P_4$ with end vertices $x$ and $w$ and internal vertices $y$ and $u$. Also add the edges joining each of $x$ and $w$ to one or more vertices in $S_2$ together with or with out edges joining one of $y, u$ to vertices in $S$. Then define $\mathbb{R}_4 = \mathbb{V}(K)$, where $K \in \mathcal{G}_2$. The class of graphs $\mathbb{R}_4$ are given in Figure 5.6.
Construction $C_7$: Let $\mathcal{W}(K)$ denoted the set of connected graphs which may be obtained from any graph $K \in \mathcal{G}_2$, by adding a cycle $C_4$ consisting of vertices $p, q, r, s$ and another new vertex $v$, an edge joining $v$ and $p$ and edges joining $v$ to one or more vertices in $S$. Then define $\mathcal{R}_5 = \bigcup \mathcal{W}(K)$, where $K \in \mathcal{G}_2$. The class of graphs $\mathcal{R}_5$ are given in Figure 5.7.
Construction $C_8$: Let $X(K)$ denote the set of connected graphs obtained from a graph $K \in G_2$, by adding a cycle $C_4$ with vertices $u, v, w, x$ and edges joining one vertex of $C_4$ to vertices in $S$ with at least one edge in $S_1$ or edges joining at most three vertices of $C_4$ to vertices in $S_2$. Then define $R_6 = \bigcup X(K)$, where $K \in G_2$. The class of graphs $R_6$ are given in Figure 5.8.
Construction $C_9$: Let $\mathcal{Y}(K, M)$ be the set of connected graphs which may be formed from $K \in \mathcal{G}_2$ and $M \in \mathcal{B}$, the collection of graphs given in figure 2.2 by joining each vertex of $F \subseteq V(M)$ to one or more vertices in $S$, such that no set with fewer than $\gamma(M)$ vertices of $M$ dominates $V(M) - F$. Then define $\mathcal{R}_7 = \bigcup \mathcal{Y}(K, M)$ where $K \in \mathcal{G}_2$ and $M \in \mathcal{B}$. The class of graphs $\mathcal{R}_7$ are given in Figure 5.9
Theorem 5.3.1. For graphs $G$ in $\mathbb{R}_i$, $i = 1$ to 7, $\gamma(G) + \gamma'(G) = n - 1$.

Proof. Let us prove the result by the way of giving the necessary $\gamma$ and $\gamma'$-sets in each of the classes separately.

Case 1 Let $G \in \mathbb{R}_1$. By construction, let $x \in V(G)$ be the new vertex that is adjacent to a vertex $u$ in $S_1$ and let $w$ be the pendant vertex adjacent to $u$. Clearly $D = S$ is a dominating set of $G$. Therefore $\gamma(G) \leq |S|$. Also since $\gamma \geq$ number of supports $= |S|$, we get $\gamma(G) = |S|$. Then $D = S \cup \{w\} - \{u\}$ is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{x\})$.
is an inverse dominating set of $G$ with respect to $D$. Therefore
\[ \gamma(G) + \gamma'(G) \leq (|D| + |D'|) = n - 1. \]
Note that each vertex in $S_2$ lies in every $\gamma$-set of $G$ and hence the vertices that are adjacent to only the vertices of $S_2$ lie in every $\gamma'$-set of $G$. Also observe that each vertex in $(S_1 \cup L_1)$ lies either in a $\gamma$-set of $G$ or in a $\gamma'$-set of $G$. Hence $\gamma(G) + \gamma'(G) \geq n - 1$. Therefore $\gamma(G) + \gamma'(G) = n - 1$.

**Case 2** Let $G \in \mathbb{R}_2$. Then as in case 1, we see that $D = S$ is a $\gamma$-set and $D' = V(G) - (D \cup \{y\})$ is is an inverse dominating set of $G$ with respect to $D$. Therefore
\[ \gamma(G) + \gamma'(G) \leq (|D| + |D'|) = n - 1. \]
Note that each vertex in $S_2$ lies in every $\gamma$-set of $G$ and hence the vertices that are adjacent to only the vertices of $S_2$ together with the vertex $\{x\}$or $\{y\}$ lie in every $\gamma'$-set of $G$. Also observe that each vertex in $(S_1 \cup L_1)$ lies either in a $\gamma$-set of $G$ or in a $\gamma'$-set of $G$. Hence $\gamma(G) + \gamma'(G) \geq n - 1$. Therefore $\gamma(G) + \gamma'(G) = n - 1$.

**Case 3** Let $G \in \mathbb{R}_3$. Let $D$ be a $\gamma$-set and $D'$ be a $\gamma'$-set of $G$. Then as discussed earlier, we see that $(S \cup L) \subseteq (D \cup D')$. Since at least one of the vertices of $P_3$, say $x$ is not adjacent to any of the vertices of $S$, $x$ or one of the neighbors
say \( w \in D \) and the other neighbor say \( y \) or \( x \) itself must lie in \( D' \). Thus the only vertex that is neither in the \( \gamma \)-set nor in the \( \gamma' \) set of \( G \) is either \( x \) or \( y \). Thus \( D = S \cup \{ w \} \), is a \( \gamma \)-set and \( D' = V(G) - (D \cup \{ x \}) \) is a \( \gamma' \)-set of \( G \). Hence \( \gamma(G) + \gamma'(G) = n - 1 \).

**Case 4** Let \( G \in \mathbb{R}_4 \). Let \( D \) be a \( \gamma \)-set and \( D' \) be a \( \gamma' \)-set of \( G \). Then as discussed earlier, we see that \( (S \cup L) \cup I \subseteq (D \cup D') \). Since at least one of the vertices of \( P_4 \) say \( y \) is not adjacent to any of the vertices of \( S \), \( y \in D \) and since none of the vertices of the path \( P_4 \) are adjacent to any of the vertices of \( S_1 \), at least two other vertices say \( x, w \) must lie in \( D' \). Thus the only vertex that is neither in the \( \gamma \)-set nor in the \( \gamma' \) set of \( G \) is \( u \). Thus \( D = S \cup \{ y \} \), where \( y \) is not adjacent to vertices in \( S \), is a \( \gamma \)-set of \( G \) and \( D' = V(G) - (D \cup \{ u \}) \) is a \( \gamma' \)-set of \( G \). Hence \( \gamma(G) + \gamma'(G) = n - 1 \).

**Case 5** Let \( G \in \mathbb{R}_5 \). Let \( D \) be a \( \gamma \)-set and \( D' \) be a \( \gamma' \)-set of \( G \). Then as discussed earlier, we see that \( (S \cup L) \cup I \subseteq (D \cup D') \). Since none of the vertices of \( C_4 \) is adjacent to any of the vertices of \( S \), two of the vertices of \( C_4 \) say \( r, s \) must lie in a \( \gamma \)-set and two other vertices say \( p, q \) must lie in the \( \gamma' \) set of \( G \).
Since $S_2 \subseteq D$, $v$ is dominated by a vertex $v_1 \in D$ and is dominated by $p \in D'$. Thus $D = S \cup \{q, s\}$ is a $\gamma$-set and $D' = V(G) - (D \cup \{v\})$ is a $\gamma'$-set of $G$. Thus the only vertex that is neither in the $\gamma$-set nor in the $\gamma'$ set of $G$ is $v$. Hence $\gamma(G) + \gamma'(G) = n - 1$.

**Case 6** Let $G \in \mathbb{R}_6$. Let $D$ be a $\gamma$-set and $D'$ be a $\gamma'$-set of $G$. Then as discussed earlier, we see that $((S \cup L) \cup I) \subseteq (D \cup D')$. Since at least one of the vertices of $C_4$ say $w$ is not adjacent to any of the vertices of $S$, $w \in D$ and since no vertex of the cycle $C_4$ is adjacent to any of the vertices of $S_1$, at least two other vertices say $x, w$ must lie in $D'$. Thus $D = S \cup \{w\}$, is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{u\})$ is a $\gamma'$-set of $G$. Hence $\gamma(G) + \gamma'(G) = n - 1$. Thus the only vertex that is neither in the $\gamma$-set nor in the $\gamma'$ set of $G$ is $v$. Hence $\gamma(G) + \gamma'(G) = n - 1$.

**Case 7** Let $G \in \mathbb{R}_7$. Let $D_1$ be the $\gamma$-set and $D_1'$ be the $\gamma'$-set of $M \in \mathbb{B}$. Then $|D_1' \cup D_1'| = |V(M)| - 1$. Let $t$ be the vertex which is neither in the $\gamma$-set nor in the $\gamma'$-set of $M$. Then by our construction, $D = S \cup D_1$ is a $\gamma$-set and $D' = V(G) - (D \cup \{t\})$ is a $\gamma'$-set of $G$. Hence
\( \gamma(G) + \gamma'(G) = n - 1 \). Thus \( \gamma(G) + \gamma'(G) = n - 1 \), for all \( G \) in \( \cup \mathbb{R}_i \), \( i = 1 \) to 7. \( \square \)

**Theorem 5.3.2.** Let \( G \) be a connected graph with \( \delta(G) \geq 1 \). Let \( L \subseteq V(G) \) be the set of all pendant vertices in \( G \) and \( S = N(L) \). Let \( V - S \) be independent. Then \( \gamma(G) + \gamma'(G) = n - 1 \) if and only if \( G \in \mathbb{R}_1 \).

**Proof.** Let \( G \in \mathbb{R}_1 \). Then by Theorem 5.3.1, \( \gamma(G) + \gamma'(G) = n - 1 \).

Conversely, assume that \( \gamma(G) + \gamma'(G) = n - 1 \). Let \( M_1 = V - (S \cup L) \). If \( M_1 = \phi \), then we have \( V = S \cup L \). Hence \( \gamma(G) + \gamma'(G) = n \), a contradiction. Therefore \( M_1 \) is non-empty.

By Theorem 2.3.10, we have at least one vertex \( x \in M_1 \) such that at least one vertex in \( N(x) \) is adjacent to only one pendant vertex.

We claim that there exists exactly one vertex \( u \) in \( M_1 \) such that at least one vertex in \( N(u) \) is adjacent to only one pendant vertex. Suppose there are two vertices \( x \) and \( y \) in \( M_1 \) such that \( N(x) \) and \( N(y) \) each contain at least one vertex which
is adjacent to only one pendant vertex. Let \( x_1 \in N(x) \) and \( y_1 \in N(y) \) be the vertices in \( S \), that are adjacent to only one pendant vertex say \( x_2 \) and \( y_2 \) respectively. Since \( V - S \) is independent, \( S \) is a \( \gamma \)-set of \( G \). Therefore \( D = S \cup \{x_2, y_2\} - \{x_1, y_1\} \) is a \( \gamma \)-set of \( G \) and \( D' = V(G) - (D \cup \{x, y\}) \) is an inverse dominating set of \( G \) with respect to \( D \). Now neither \( x \) nor \( y \) are in \( D \cup D' \). Therefore \( \gamma(G) + \gamma'(G) \leq |D + D'| \leq n - 2 \), a contradiction.

Therefore there exists an unique \( x \) in \( M_1 \), such that at least one vertex in \( N(x) \) is adjacent to only one pendant vertex. Hence by construction \( C_3, G \in \mathbb{R}_1 \). \( \square \)

In the previous part we have constructed a class of graphs with minimum degree one which satisfy \( \gamma(G) + \gamma'(G) = n - 1 \). Now we prove that they are only graphs with such property.

**Notation:** Let \( G \) be a connected graph with \( \delta(G) = 1 \). Let \( L \subseteq V \) be the set of all vertices of degree one in \( G \). Let \( S = N(L) \). Let \( S_1 \) be the set of all vertices in \( S \), which are adjacent to only one pendant vertex and let \( S_2 = S - S_1 \). Let \( I \) denoted the set of independent vertices in \( G \) but not in \( S \cup L \), that are adjacent to vertices in \( S_2 \).
Theorem 5.3.3. Let $G$ be a connected graph with $\delta(G) = 1$. Then $\gamma(G) + \gamma'(G) = n - 1$ if and only if $G \in \bigcup_{i=1}^{7} R_i$.

Proof. One part of the theorem follows from Theorem 5.3.1. Conversely assume $\gamma(G) + \gamma'(G) = n - 1$. Let $S, L$ and $I$ denote the sets as mentioned above.

Case 1 When $V(G) - S$ is independent, by Theorem 5.3.2, we get $G \in \mathbb{R}_1$.

Case 2 When $V(G) - S$ is not independent, let $K = \langle I \cup (S \cup L) \rangle$. By construction $C_2, K \in \mathbb{G}_2$. Let $N = \langle V(G) - V(K) \rangle$. Note that $\delta(N) \geq 1$.

Sub case 2.1 When $\delta(N) = 1$, let $X$ be the set of pendant vertices in $N$. No vertex of $N$ is a pendant vertex, as a vertex of $G$. Hence each vertex of $X$, as a vertex of $G$ is adjacent to at least one vertex in $S$. Now there are two possibilities. They are either $N - X = \phi$ or $N - X \neq \phi$.

Sub case 2.1.1 If $N - X = \phi$, then $N = X$, and each vertex of $N$ is a pendant vertex. Hence each component of $N$ is a $K_2$. Let the number of components in $N$ be $m$. Note that any vertex of $G$ is either in $S$ or adjacent to vertices in $S$. 

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Therefore $D = S$ is a $\gamma$-set of $G$. Also $D' = (L \cup l) \cup \{\text{one vertex in each of the } m \text{ components of } N\}$ is an inverse dominating set of $G$ with respect to $D$. Therefore $\gamma(G) + \gamma'(G) \leq n - m = n - 1$, by the assumption. This gives that $m = 1$. Therefore $N = K_2$. Assume that $V(K_2) = \{x, y\}$. Suppose $x$ and $y$ are adjacent to vertices only in $S_2$. Since $N = \{x, y\}$, by construction of $C_4, G = T(K), K \in G_2$ and so $G \in \mathbb{R}_2$. Similarly if only one of the vertices $\{x, y\}$ is adjacent to vertices in $S_1$, then too by construction $C_4, G \in \mathbb{R}_2$. If each of $x$ and $y$ are adjacent to vertices in $S_1$, then we claim that either $|N(x) \cap S| = 1$ or $|N(y) \cap S| = 1$. If not, let $x_1, x_2 \in N(x) \cap S$ where $x_1 \in S_1$ and let $y_1, y_2 \in (N(y) \cap S)$, where $y_1 \in S_1$. Let $N(x_1) = x_1', N(y_1) = y_1'$ in $L$. Since $S$ is a $\gamma$-set, $D = S \cup \{x_1', y_1'\} - \{x_1, y_1\}$ is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{x, y\})$ is an inverse dominating set with respect to $D$, so that $\gamma(G) + \gamma'(G) \leq n - 2$, a contradiction. Thus when both the ends $x$ and $y$ of $K_2$ of are adjacent to vertices in $S_1$, we have either $|N(x) \cap S| = 1$ or $|N(y) \cap S| = 1$. By construction $C_4, G = T(K), K \in G_2$ and so $G \in \mathbb{R}_2$. 

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Sub case 2.1.2 When $N - X \neq \phi$, there are three possibilities as discussed below. They are either $N - X$ has only isolates or no isolates or both isolates and non-isolates.

Sub case 2.1.2.1 If $N - X$ has only isolates, then each isolate $u \in N - X$ is adjacent to at least two vertices in $X$. Let $|N - X| = k$. Let $V(N - X) = \{u_1, u_2, \ldots, u_k\}$. Let $u_i$ be adjacent with $v_{i_1}, v_{i_2}, \ldots, v_{i_{k_i}}$. Then $D = S \cup \{v_{11}, v_{21}, \ldots, v_{k_1}\}$, is a $\gamma$-set of $G$, and $D' = (L \cup I) \cup \{u_1, u_2, \ldots, u_k\}$ is an inverse dominating set of $G$ with respect to $D$. Hence $\gamma(G) + \gamma'(G) \leq n - k$. Since we have $\gamma(G) + \gamma'(G) = n - 1$, we get that $k = 1$. Hence there is only one isolate in $N - X$. Let $u$ be the isolate in $N - X$. Next we claim that $u$ is adjacent to exactly two vertices in $X$. Suppose $u$ is adjacent to three or more vertices in $X$, say $\{u_1, u_2, \ldots, u_m\}$, $m \geq 3$. Then $D = S \cup \{u_1\}$ is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{u_2, u_3, \ldots, u_m\})$ is an inverse dominating set of $G$ with respect to $D$. Therefore $\gamma(G) + \gamma'(G) \leq n - 2$, a contradiction. Hence $m \leq 2$. Since $d_G(u) \geq 2$, $m = 2$. Therefore $u$ is adjacent to exactly two vertices, say $v$ and $w$ and so $N$ is the path $P_3$ with vertices $v, u$ and $w$. Since $v, w$ are the vertices in $X$, they are adjacent to vertices in $S$. 
We claim that $u$ is not adjacent with vertices in $S$.

Suppose $u$ is adjacent with vertices in $S$, then $D = S$ is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{v, w\})$ is an inverse dominating set of $G$ with respect to $D$. Therefore $\gamma(G) + \gamma'(G) \leq n - 2$, a contradiction. Hence $u$ is not adjacent with vertices in $S$. Hence by construction $C_5$, $G = Z(K)$ and so $G \in \mathbb{R}_3$.

**Sub case 2.1.2.2** If $N - X$ has no isolates, then let $N_1 = \langle N - X \rangle$. Note that each vertex in $X$ is adjacent to at least one vertex in $S$ and to exactly one vertex in $N_1$. So $D = S \cup D_1$ is a $\gamma$-set of $G$ where $D_1$ is a $\gamma$-set of $N_1$ and $D' = V(G) - (D \cup \{u\})$, with $u \in X$ is an inverse dominating set of $G$.

Let $D_1'$ be a $\gamma'$-set of $N_1$. We claim that $|N_1 \cap D'| \leq |D_1'|$. If not, $|N_1 \cap D'| > |D_1'|$. That is there are some vertices in $N_1$ which are not in $D_1'$ but in $D'$ to dominate the vertices of $X$. But by the choice of $D'$, all the vertices of $X$ except $u$ ($\in X$) are in $D'$. Thus $D'$ is not a $\gamma'$-set of $G$. Hence $\gamma(G) + \gamma'(G) < n - 1$. A contradiction. Hence the claim.

Therefore $|N_1 \cap D| \cup |N_1 \cap D'| = |N_1 \cap (D \cup D')| \leq |D_1 \cup D_1'| \leq |V(N_1)|$. Since $\gamma(G) + \gamma'(G) = n - 1$ and $I \cup (S \cup L) \subseteq D \cup D'$, and since $u \in X$ is the only vertex which is neither in the $\gamma$-set of $G$ nor in the $\gamma'$-set of $G$, we have
\[ |N_1 \cap (D \cup D')| = |V(N_1)|. \] Therefore \[ |D_1 \cup D_1'| = |V(N_1)|. \] That is \[ \gamma(N_1) + \gamma'(N_1) = |V(N_1)|. \]

**Sub case 2.1.2.2.1** When \( \delta(N_1) = 1. \) We claim that \[ |V(N_1)| = n_1 = 2. \] Since \( \delta(N_1) = 1, \) trivially \( n_1 \geq 2. \) If \( n_1 \geq 3, \) then by Theorem 2.3.10, there are at least two pendant vertices in \( N_1 \) and each pendant vertex in \( N_1 \) is adjacent with one vertex in \( X. \) Let \( \{x_1, x_2\} \) be two vertices in \( X \) that are adjacent with the pendant vertices of \( N_1. \) Let \( H \) be the sub-graph obtained from \( N_1 \) by deleting the pendant vertices of \( N_1. \) Let \( H_1 \) be the \( \gamma \)-set of \( N_1, \) with no pendant vertices of \( N_1. \) Then \( D = S \cup H_1 \) is a \( \gamma \)-set of \( G \) and \( D' = V(G) - (D \cup \{x_1, x_2\}) \) is an inverse dominating set of \( G \) with respect to \( D. \) Therefore \( \gamma(G) + \gamma'(G) \leq n - 2, \) a contradiction. Therefore \( n_1 = 2. \) Since \( \delta(N_1) = 1, \) \( N_1 = K_2 \) and so \( N = P_4. \)

Let \((x_1, x_2, x_3, x_4)\) be the path \( P_4. \) We claim that all the vertices of \( P_4 \) are not adjacent to vertices in \( S. \)

Suppose all the 4 vertices of \( P_4 \) are adjacent to vertices in \( S, \) then \( D = S \) is a \( \gamma \)-set of \( G \) and \( D' = V(G) - (D \cup \{x_1, x_4\}) \) is an inverse dominating set of \( G \) with respect to \( D. \) Therefore \( \gamma(G) + \gamma'(G) \leq n - 2, \) a contradiction. Therefore all the
vertices of \( P_4 \) are not adjacent to vertices in \( S \).

We claim that both \( x_1 \) and \( x_4 \) are adjacent to vertices in \( S_2 \) only.

Since \( x_1 \) and \( x_4 \) are pendant vertices in \( N \), they are adjacent to vertices in \( S \). Suppose at least one of \( x_1 \) and \( x_4 \), say \( x_1 \) is adjacent to vertices in \( S_1 \). Let \( x_1 \) be adjacent with \( y_1 \) in \( S_1 \) where \( z_1 \) is the pendant vertex adjacent to \( y_1 \). Then \( D = (S \cup \{z_1, x_2\}) - \{y_1\} \) is a \( \gamma \)-set of \( G \) and \( D' = V(G) - (D \cup \{x_1, x_4\}) \) is an inverse dominating set of \( G \) with respect to \( D \). Therefore \( \gamma(G) + \gamma'(G) \leq n - 2 \), a contradiction. Therefore both \( x_1 \) and \( x_4 \) are adjacent to vertices in \( S_2 \) only. Hence by the above claims and if one of \( \{x_2, x_3\} \) is or is not adjacent with vertices in \( S \), we get \( G = V(K) \) by construction \( C_6 \). Thus \( G \in \mathbb{R}_4 \).

**Sub case 2.1.2.2.2** When \( \delta(N_1) \geq 2 \). Since \( \gamma(N_1) \) + \( \gamma'(N_1) = |V(N_1)| \), by the Theorem 2.3.9, we observe that \( N_1 = C_4 \). We claim that \( |X| = 1 \).

If not, \( |X| \geq 2 \). Let \( \{x_1, x_2\} \) be two vertices in \( X \), which are adjacent to vertices of \( C_4 \). Let \( \{a, b, c, d\} \) be the vertices of \( C_4 \). Without loss of generality, let \( ax_1 \) and \( bx_2 \) be the edges in \( N \). Then \( D = S \cup \{c, d\} \) is a \( \gamma \)-set of \( G \) and \( D' = \).
$V(G) - (D \cup \{x_1, x_2\})$, is an inverse dominating set of $G$ with respect to $D$. Hence $\gamma(G) + \gamma'(G) \leq n - 2$, a contradiction. Therefore $|X| = 1$. Hence assume $ax$ is an edge in $N$. Since $x$ is a pendant vertex in $N$, $x$ is adjacent to vertices in $S$.

We claim that no vertex of $C_4$ is adjacent with vertices in $S$.

If not, at least one vertex say $a$ (or $b$) is adjacent to vertices in $S$. Then $D = S \cup \{c\}$ (or $D = S \cup \{d\}$ accordingly) is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{x, d\})$, (or $D' = V(G) - (D \cup \{x, b\})$ accordingly) is an inverse dominating set of $G$ with respect to $D$. Hence $\gamma(G) + \gamma'(G) \leq n - 2$, a contradiction. Therefore no vertex of $C_4$ is adjacent with vertices in $S$.

Thus only $x$ is adjacent with vertices in $S$. Hence by construction $C_7$, $G = W(K)$ and $G \in \mathbb{R}_5$.

Sub case 2.1.2.2.3 From the above two cases we see that the components of $N - X$ shall be either with only isolates or with no isolates, since in each case we have one vertex which is neither in the $\gamma$-set of $G$ nor in the $\gamma'$-set of $G$. 
Sub case 2.2 When $\delta(N) \geq 2$. Let $D$ and $D_1$ be the $\gamma$-sets of $G$ and $N$ respectively. Also let $D'$ and $D_1'$ be the $\gamma'$-sets of $G$ and $N$ respectively. Since either the vertices of $S$ or the vertices of $L$ lie in any dominating set of $G$, we need no vertex of $N$ to dominate vertices in $S$. Therefore $|N \cap (D \cup D')| \leq |D_1 \cup D_1'| \leq |V(N)|$. Thus either $|N \cap (D \cup D')| \leq |D_1 \cup D_1'|$ or $|N \cap (D \cup D')| = |D_1 \cup D_1'|$ are two possibilities.

Sub case 2.2.1 Suppose $|N \cap (D \cup D')| < |D_1 \cup D_1'|$, then $|N \cap (D \cup D')| < |D_1 \cup D_1'| \leq |V(N)|$. Since $\gamma(G) + \gamma'(G) = n - 1$ and $S \cup L \subseteq D \cup D'$, we have $|N \cap (D \cup D')| = |V(N)| - 1$. Therefore $|D_1 \cup D_1'| = |V(N)|$. That is $|D_1| + |D_1'| = |V(N)|$. Thus we have $\gamma(N) + \gamma'(N) = |V(N)|$ and since $\delta(N) \geq 2$, by Theorem 2.3.9, we have $N = C_4$.

Sub case 2.2.1.1 Only one vertex of $C_4$ is adjacent to vertices in $S_1$, if adjacent to vertices in $S_1$.

If not, there are at least two vertices of $C_4$ that are adjacent to vertices in $S_1$ or one vertex of $C_4$ adjacent to vertices in $S_1$ and at least one or more of the other vertices is adjacent to vertices in $S_2$. 
Let $e$, $f$, $g$ and $h$ be the vertices of $C_4$. Let $e$, $f$ be the vertices adjacent to $x_1$ and $y_1$ in $S_1$ respectively. Let $x_2$ and $y_2$ be the neighbors of $x_1$ and $y_1$ in $L$ respectively.

Then $D = (S \cup \{y_2, g\}) - \{y_1\}$ is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{e, f\})$ is an inverse dominating set of $G$ with respect to $D$. Hence $\gamma(G) + \gamma'(G) \leq n - 2$, a contradiction. Similarly we get a contradiction in the other case too. Hence, if the vertices of $C_4$ are adjacent to vertices in $S_1$, only one vertex of $C_4$ is adjacent to vertices vertices in $S_1$. Therefore by construction $C_8, G \in \mathbb{R}_6$.

Sub case 2.2.1.2 At most three vertices of $C_4$ are adjacent to vertices in $S_2$. If not, all the vertices of $C_4$ are adjacent to vertices in $S_2$. Then $D = S$ is a $\gamma$-set of $G$ and $D' = V(G) - (D \cup \{e, f\})$ is an inverse dominating set of $G$ with respect to $D$. Hence $\gamma(G) + \gamma'(G) \leq n - 2$, a contradiction. Hence if the vertices of $C_4$ are adjacent to vertices in $S_2$, at most three vertices of $C_4$ are adjacent to vertices in $S_2$. Therefore by construction $C_8$, we get $G \in \mathbb{R}_6$.

Sub case 2.2.2 Suppose $|N \cap (D \cup D')| = |D_1 \cup D_1'|$, then $|N \cap (D \cup D')| = |D_1 \cup D_1'| \leq |V(N)|$. But $|N \cap (D \cup D')| = |V(N)| - 1$. Therefore $|D_1 \cup D_1'| = |V(N)| - 1$. That is
\[ |D_1| + |D_1'| = |V(N)| - 1. \] Also \( \delta(N) \geq 2. \) Therefore by Theorem 5.2.4, \( N \in (\mathbb{A} \cup \mathbb{B}). \) However \( N \) may be further restricted. It can be verified that if \( N \in \mathbb{A}, \) then for any \( U \neq \phi, V(N) - U \) may be dominated by less than \( \gamma(N) \) vertices, so that \( \gamma(G) + \gamma'(G) \leq n - 2. \) Therefore for \( N \in \mathbb{B}, \) there exists at least one suitable non empty subset \( U \) of \( V(N), \) such that \( V(N) - U \) is dominated by exactly \( \gamma(N) \) vertices. Thus only for these sets \( U, \) each vertex in \( U \) is adjacent with one or more vertices in \( S. \) Thus by construction \( C_9, \) we get \( G \in \mathbb{R}_7. \) Thus \( G \in \bigcup_{i=1}^{7} \mathbb{R}_i. \) \( \square \)