CHAPTER - 2
NEAR MEAN GRAPH

2.1 Introduction

In 1966, Rosa [23] introduced $\alpha$ - valuation of a graph. Golomb subsequently introduced graceful labeling. In 1980, Graham and Sloance [5] introduced the harmonious labeling of a graph. Several graph labelings have been introduced in Gallian survey [4]. In [21], Ponraj introduced the concept of Mean Labeling and it was studied in [22, 24, 25, 26, 27, 29].

Near Graceful Labeling introduced by Frucht [3] and Near $\alpha$-labeling introduced by S.El – Zanati, M.Kenig and C.Vanden Eynden [33]. It motives us to define the concept of Near Mean Labeling as follows:

Let $G$ be a $(p, q)$ - graph. A vertex labeling of $G$ is an assignment $f: V(G) \rightarrow \{0, 1, 2, ..., q-1, q+1\}$. For a vertex labeling $f$, the induced edge labeling $f^*$ is defined by $f^*(uv) = \left\lceil \frac{f(u) + f(v)}{2} \right\rceil$, for any edge $uv$ in $G$ where $\lceil x \rceil$ denotes the least integer which is greater than or equal to $x$. A vertex labeling $f$ is called Near Mean Labeling of $G$ if its induced edge labeling is $f^*(E) = \{1, 2, ..., q\}$. If a graph $G$ has Near Mean Labeling, then we call $G$ as a Near Mean Graph (NMG).

For example, the graphs $K_4$ and $K_{1,4}$ are Near Mean Graphs as shown in Fig. 2.1.
Note that the graphs $K_4$ and $K_{1,4}$ are not Mean graphs. So, there are graphs which are not Mean Graphs but they are Near Mean Graphs.

In this chapter, we study some basic theorems on Near Mean Graphs and also find the total number of possible Mean labeling of a given graph. Also we investigate Near Mean Labeling of some graphs. Near meaness of family of trees [10] and join of graphs [11] are also studied.

### 2.2 Basic Results

**Theorem 2.2.1** : The total number of possible labelings of a NMG is $2(q-1)(q-1)!$.

**Proof** : The set of labeling for the edges $\{1, 2, \ldots, q\}$ are as follows:

<table>
<thead>
<tr>
<th>Edge Label</th>
<th>Choice of adjacent vertices labels</th>
<th>No. of edge labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>$(q+1, q-1), (q+1, q-2)$</td>
<td>2</td>
</tr>
<tr>
<td>$q-1$</td>
<td>$(q+1, q-3), (q+1, q-4), (q-1, q-2)$</td>
<td>3</td>
</tr>
<tr>
<td>$q-2$</td>
<td>$(q+1, q-5), (q+1, q-6), (q-1, q-3), (q-1, q-4), (q-2, q-3)$</td>
<td>5</td>
</tr>
</tbody>
</table>
Hence total number of possible Near Mean labeling

\[ = (2 \ 3 \ 5 \ 7 \ \ldots \ q-1) \ (q-1) \ (q-2, q-4, \ldots, 4, 2) \]

\[ = 2 \ (q-1) \ (2 \ 3 \ 4 \ \ldots \ q-1) = 2(q-1)(q-1)! \]

**Remarks 2.2.2 :** If G is NMG then 0 and q+1 must appear as vertex labels.

**Theorem 2.2.3 :** Let G be a Near Mean Graph and let f be a Mean Labeling of G.

Let t be the number of edges whose one vertex label is even and the other is odd, then

\[ \sum_{v \in V(G)} d(v)f(v) + t = q(q+1), \text{ where } d(v) \text{ is degree of vertex } v. \]

**Proof:**

\[
f'(xy) = \begin{cases} 
\frac{f(x) + f(y)}{2} & \text{if } f(x) + f(y) \text{ is even} \\
\frac{f(x) + f(y) + 1}{2} & \text{if } f(x) + f(y) \text{ is odd}
\end{cases}
\]

then,

\[ \sum_{v \in V(G)} d(v)f(v) = 2 \left[ \sum_{xy \in E(G)} f'(xy) - \frac{1}{2} \right] \]
\[ = 2 \left[ 1 + 2 + \ldots + q \right] - t \]
\[ = q \left[ q + 1 \right] - t \]

Hence, \( \sum_{v \in V(G)} d(v) f(v) + t = q(q+1) \)

**Example 2.2.4**: The above result can be verified for \( K_4 \) (See Fig. 2.2).

![Fig 2.2](image)

Defining \( f : V(K_4) \to \{0, 1, 2, 3, 4, 5, 7\} \) by

\[ f(v_1) = 0 \quad f(v_3) = 4 \]
\[ f(v_2) = 2 \quad f(v_4) = 7 \]
\[ f^*(v_1v_2) = 1 \quad f^*(v_1v_3) = 2 \]
\[ f^*(v_1v_4) = 4 \quad f^*(v_2v_3) = 3 \]
\[ f^*(v_2v_4) = 5 \quad f^*(v_3v_4) = 6 \]

Here \( t = 3 \).

\( K_4 \) is a Near Mean Graph.

Then \( d(v_1)f(v_1) + d(v_2)f(v_2) + d(v_3)f(v_3) + d(v_4)f(v_4) + t \)
\[ = 3 \times 0 + 3 \times 2 + 3 \times 4 + 3 \times 7 + 3 \]
\[ = 42 = 6 \times 7 = q(q+1) \]
Remark 2.2.5: Since $t \leq q$, from the above theorem, we have

$$\sum_{v \in V(G)} d(v)f(v) \geq q^2.$$ 

Theorem 2.2.6: Let $G$ be 2-regular Near Mean Graph. Let $f$ be Near Mean labeling of $G$ and $x \in \{0, 1, 2, \ldots, q-1, q+1\} - f[V(G)]$. Then $x \leq \frac{q+2}{2}$.

Proof: Since $G$ is 2-regular, $q = p$. Then there exists a positive integer $x$ such that $x \in \{0, 1, \ldots, q-1, q+1\} - f(V(G))$.

By the above Remark,

$$q^2 \leq \sum_{v \in V(G)} d(v)f(v)$$

$$\leq 2 \sum_{v \in V(G)} f(v)$$

$$\leq 2 \left[ \sum_{v \in V(G)} f(v) + x - x \right]$$

$$\leq 2 \left[ \sum_{v \in V(G)} f(v) + x \right] - 2x$$

$$\leq 2[1+2+\ldots+(q-1)+(q+1)] - 2x$$

$$\leq 2 \left( \frac{q(q+1)}{2} \right) + 2 - 2x$$

$$q^2 \leq 2 \left( \frac{q(q+1)}{2} \right) + 2 - 2x$$

$$2x \leq q+2$$

$$x \leq \frac{q+2}{2}$$
Remark 2.2.7 : If \( p > q+1 \), then the graph \( G(p, q) \) is not a Near Mean graph.

Remark 2.2.8 : Let \( G \) be a Near Mean Graph containing a cycle \( C_3 \). Then any three successive integers can not be labels for the vertices of \( C_3 \). That is, if \( u_i \), \((1 \leq i \leq 3)\) are the vertices of \( C_3 \) in \( G \), then \( x \), \( x+1 \) and \( x+2 \) can not be labels of \( u_1 \), \( u_2 \) and \( u_3 \) respectively.

Remark 2.2.9 : The union of two or more trees is not a Near Mean Graph.

Proof : Let \( G_1 = (p_1, q_1) \), \( G_2 = (p_2, q_2) \) be two trees.

Then \( q_1 = p_1 - 1 \), \( q_2 = p_2 - 1 \),

Let \( G = G_1 \cup G_2 = G(p, q) \)

Then \( p = p_1 + p_2 \), \( q = q_1 + q_2 \)

\[ q = q_1 + q_2 = p_1 + p_2 - 2 = p - 2 \]

Hence, \( p = q + 2 \)

By Remark 2.2.7, \( G = G_1 \cup G_2 \) is not a Near Mean graph.

2.3 Near Mean of Standard Graphs

It is easy to verify that the following are Near Mean Graphs [9] :

Theorem 2.3.1 : The path \( P_n \) is a Near Mean Graph.

Proof: Let \( P_n \) be a path on \( n \) vertices.

Let \( V(P_n) = \{ u_1, u_2, \ldots, u_n \} \) and

\[ E(P_n) = \{ (u_i, u_{i+1}) / i = 1, 2, \ldots, n-1 \} \]

Define \( f : V(P_n) \rightarrow \{ 0, 1, 2, \ldots, n-2, n \} \) by
\[ f(u_i) = i - 1, \ 1 \leq i < n \]

\[ f(u_n) = n \]

It can be verified that the induced edge labeling is given by \( f^e(u_iu_{i+1}) = i, \ 1 \leq i < n \)

Hence, \( P_n \) is a Near Mean Graph.

For example, Near Mean Labeling of \( P_4 \) is given in Fig 2.3.

**Fig. 2.3**

**Theorem 2.3.2:** The graph \( P_n^+ \) is a Near Mean Graph.

**Proof:** Let \( P_n \) be the path of length \( n-1 \) having \( n \) vertices

Let \( P_n = (v_1v_2v_3 \ldots v_n) \)

Let \( V(P_n^+) = \{ (u_i, v_i) / 1 \leq i \leq n \} \) and

\[ E(P_n^+) = \{ [(v_iv_{i+1}) : 1 \leq i \leq n-1] \cup [(u_iv_i) : 1 \leq i \leq n] \}. \]

\( P_n^+ \) is obtained by joining each \( v_i \) to a vertex \( u_i \)

\( P_n^+ \) has 2n vertex and 2n-1 edges.

**Case i:** Suppose \( n \) is odd. Let \( n = 2m+1 \)

Define \( f : V(P_n^+) \rightarrow \{0, 1, 2, \ldots, 4m, 4m+2\} \)

\[ f(v_1) = 0 \]

\[ f(v_{2i+1}) = 4i \quad 1 \leq i \leq m \]

\[ f(v_{2i}) = 4i - 1 \quad 1 \leq i \leq m \]

\[ f(u_{2i+1}) = 1 + 4i \quad 0 \leq i < m \]

\[ f(u_n) = 4m+2 \]
\[ f(u_{2i}) = 2 + 4(i-1) \quad 1 \leq i \leq m \]

**Case ii**: Suppose \( n \) is even. Let \( i = 2m \)

Define \( f : V(P_n^+) \rightarrow \{0, 1, 2, \ldots, 4m-2, 4m\} \)

\[
\begin{align*}
  f(v_{2i-1}) &= 4(i-1) \quad 1 \leq i \leq m \\
  f(v_{2i}) &= 4i - 1 \quad 1 \leq i \leq m - 1 \\
  f(v_{2m}) &= 4m \\
  f(u_{2i-1}) &= 1 + 4(i-1) \quad 1 \leq i \leq m \\
  f(u_{2i}) &= 2 + 4(i-1) \quad 1 \leq i \leq m
\end{align*}
\]

The induced edge labels are

\[
\begin{align*}
  f^*(v_i v_{i+1}) &= 2i \quad 1 \leq i \leq n-1, \\
  f^*(u_i v_i) &= 2i-1 \quad 1 \leq i \leq n
\end{align*}
\]

Hence, \( P_n^+ \) is a Near Mean Graph.

For example, Near Mean Labeling of \( P_7^+ \) and \( P_6^+ \) are shown in Fig. 2.4 and Fig. 2.5 respectively.

![Fig. 2.4](image)

![Fig. 2.5](image)
Theorem 2.3.3 : The graph $P^2_n$ is a Near Mean Graph.

Proof:

Let $V(P^2_n) = \{u_i : 1 \leq i \leq n\}$ and 

$$E(P^2_n) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_i u_{i+2}) : 1 \leq i \leq n-2\}$$

Define $f : V \to \{0, 1, 2, \ldots, 2n-4, 2n-2\}$ by 

$$f(u_i) = 2i-2 \quad 1 \leq i \leq n$$

The induced edge labels are 

$$f^*(u_i u_{i+1}) = 2i-1 \quad 1 \leq i \leq n-1$$

$$f^*(u_i u_{i+2}) = 2i \quad 1 \leq i \leq n-2$$

It is clear that every edge in $P^2_n$ get distinct labels from $\{1, 2, \ldots, q\}$

Hence, $P^2_n$ is a Near Mean Graph.

For example, Near Mean labeling of $P^5_3$ is shown in Fig. 2.6.

![Fig 2.6](image)

Theorem 2.3.4 : $C_n$ is a Near Mean Graph.

Proof:

Let $V(C_n) = (u_1, u_2, u_3, \ldots, u_n, u_1)$ and 

$$E(C_n) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{u_i u_n\}$$

Case i : Let $n$ be even, say $n = 2m$

Define $f : V(C_n) \to \{0, 1, 2, \ldots, 2m-1, 2m+1\}$ by
\[ f(u_i) = i - 1 \quad 1 \leq i \leq m. \]
\[ f(u_{m+j}) = m+j \quad 1 \leq j < m. \]
\[ f(u_{2m}) = 2m+1 \]

Clearly, \( f \) is injective. The set of edge labels of \( C_n \) is \( \{1, 2, \ldots, q\} \).

**Case ii:** Let \( n \) be odd, say \( n = 2m+1 \)

Define \( f : V(C_n) \to \{0, 1, 2, \ldots, 2m, 2m+2\} \) by
\[ f(u_i) = i - 1 \quad 1 \leq i \leq m. \]
\[ f(u_{m+j}) = m+j \quad 1 \leq j \leq m. \]
\[ f(u_{2m+1}) = 2m+2 \]

Clearly, \( f \) is injective. The set of edge labels of \( C_n \) is \( \{1, 2, \ldots, q\} \).

Hence, \( C_n \) is a Near Mean Graph.

For example, Near Mean Labeling of \( C_7 \) and \( C_6 \) are shown in Fig. 2.7 and Fig 2.8 respectively.

**Theorem 2.3.5:** \( K_n \), \( n > 4 \) is not a Near Mean Graph.

**Proof:** Let \( f: V(G) \to \{0, 1, 2, \ldots, q-1, q+1\} \)

To get the edge label 1, we must have either 0 and 1 (or) 0 and 2 as vertex labels.
In either case, 0 must be label of some vertex. In the same way to get edge label $q$, we must have either $q-1$ and $q+1$ as vertex labels or $q-2$ and $q+1$ as vertex labels.

Let $u$ be a vertex whose vertex label is 0.

**Case i:** To get the edge label $q$, assign vertex labels $q-1$ and $q+1$ to the vertices $w$ and $x$ respectively.

**Subcase a:** Let $v$ be a vertex whose vertex label be 2. Then the edges $vw$ and $ux$ get the same label.

**Subcase b:** Let $v$ be a vertex whose vertex label be 1. Then the edges $vw$ and $ux$ get the same label, when $q$ is odd.

Similarly, when $q$ is even, the edges $uw$ and $vw$ get the same label and also the edges $ux$ and $vx$ get the same label.

**Case ii:** To get the edge label $q$, assign the vertex label $q-2$ and $q+1$ to the vertices $w$ and $x$ respectively.

**Subcase a:** Let $v$ be the vertex whose vertex label be 1.

As $n > 4$, to get the edge label 2, there should be a vertex (say) $z$ whose vertex label is either 3 or 4.

When the vertex label of $z$ is 3, the edges $ux$ and $wz$ have the same label and also the edges $uz$ and $vz$ get the same edge label.

When the vertex label of $z$ is 4, the edges $vx$ and $wz$ have the same label.

**Subcase b:** Let $v$ be a vertex whose vertex label 2.

As $n > 4$, to get edge label 2, there should be a vertex (say) $z$ whose vertex label is either 3 or 4.
When vertex label of z is 3, the edges ux and wz get the same label.

Suppose the vertex label of z is 4.

If q is even then the edges ux and wz have the same label.

If q is odd then the edges vw and ux have the same label.

Hence, $K_n (n \geq 5)$ is not a Near Mean Graph.

**Remarks 2.3.6 :** $K_2$, $K_3$ and $K_4$ are Near Mean graphs as shown in Fig. 2.9.

![Fig. 2.9](image)

**Theorem 2.3.7 :** $K_{1,n} (n > 4)$ is not a Near Mean Graph.

**Proof:** Let $V(K_{1,n}) = \{ u, v_i : 1 \leq i \leq n \}$ and

$$E(K_{1,n}) = \{ (u v_i) : 1 \leq i \leq n \}$$

To get the edge label 1, either 0, 1 (or) 0, 2 are assigned to $u$ and $v_i$ respectively for some $i$.

In either case, 0 must be a label of some vertex.

To get the edge label $q$, we need the following vertex labelings $q-1$ and $q+1$ (or) $q-2$ and $q+1$.

Suppose $f(u) = 0$, then in order to get an edge label $q$, $q$ must be either 1 or 2.
As n > 4, this is not possible.

So assume that \( f(v_1) = 0 \).

To get the edge label 1, \( f(u) \) is either 1 or 2.

**Case (i) :** Let \( f(u) = 1 \)

To get edge label q, q is either 2 or 3. As q > 4, it is not possible to get the edge label q.

**Case (ii) :** Let \( f(u) = 2 \)

As in case (i), to get the edge value q, the value of q is either 3 or 4. As q > 4, it is not possible to get the edge label q.

From both the cases it is not possible to get the edge value q, when q > 4.

Hence, \( K_{1,n} \ (n \geq 5) \) is not a Near Mean Graph.

**Remark 2.3.8 :** \( K_{1,n} \ , \ n \leq 4 \) is a Near Mean Graph as shown in Fig 2.10.
2.4 Near Meanness on Product Graphs

Near meanness of product graphs are studied in [8].

**Theorem 2.4.1**: The Book $K_{1,n} \times K_2$ is a Near Mean Graph.

**Proof**: Let $K_{1,n} \times K_2 = (V, E)$ such that

$V = \{ u, v, (u_i, v_i : 1 \leq i \leq n) \}$

$E = \{(uu_i) \cup (vv_i) : 1 \leq i \leq n \} \cup (uv) \cup [(u_i,v_i) : 1 \leq i \leq n \}$

**Case (i)**: When $n$ is even,

Let $f : V \rightarrow \{0, 1, 2, \ldots, 3n, 3n+2\}$ by

$f(u) = 0$

$f(v) = 3n + 2$

$f(u_i) = 4i - 3$ \hspace{1cm} $1 \leq i \leq \frac{n}{2}$

$f(u_{n+1-i}) = 4i - 1$ \hspace{1cm} $1 \leq i \leq \frac{n}{2}$

$f(v_i) = 2n - 2(i - 1)$ \hspace{1cm} $1 \leq i \leq \frac{n}{2}$

$f(v_{n+1-i}) = 3n - 2(i - 1)$ \hspace{1cm} $1 \leq i \leq \frac{n}{2}$

Clearly, $f$ is injective.

The induced edge labels are

$f^e(uu_i) = 2i - 1$ \hspace{1cm} $1 \leq i \leq \frac{n}{2}$

$f^e(uu_{n+1-i}) = 2i$ \hspace{1cm} $1 \leq i \leq \frac{n}{2}$
\[ f^*(u_i v_i) = n + i \quad 1 \leq i \leq \frac{n}{2} \]

\[ f^*(u v) = \frac{3n + 2}{2} \]

\[ f^*(u_{n+1-i} v_{n+1-i}) = \frac{3n + 2}{2} + i \quad 1 \leq i \leq \frac{n}{2} \]

\[ f^*(v v_i) = \frac{5n}{2} + 2 - i \quad 1 \leq i \leq \frac{n}{2} \]

\[ f^*(v v_{n+1-i}) = 3n + 2 - i \quad 1 \leq i \leq \frac{n}{2} \]

It can be easily seen that each edge gets different label from \{1, 2, ..., 3n+1\}

Hence, \( K_{1,n} \times K_2 \) (n is even) is a Near Mean Graph.

For example, \( K_{1,4} \times K_2 \) is a Near Mean Graph as shown in Fig. 2.11.

**Case (ii) :** When n is odd

Let \( f : V \rightarrow \{0, 1, 2, ..., 3n, 3n + 2\} \) by

\[ f(u) = 0 \]

\[ f(v) = 3n + 2 \quad \text{Let } x = \frac{n+1}{2} \]

\[ f(u_i) = \begin{cases} 2i - 1 & \text{if } 1 \leq i \leq x \\ 2i & \text{if } x + 1 \leq i \leq n \end{cases} \]

\[ f(v_i) = 3n - 4(i - 1) \quad 1 \leq i \leq x \]

\[ f(v_{x+i}) = 3n - 2 - 4(i - 1) \quad 1 \leq i \leq n - x \]

Clearly, \( f \) is injective.
The induced edge labels are

\[ f^*(uu_i) = i \quad 1 \leq i \leq n \]

\[ f^*(u_iv_i) = \frac{3n + 3}{2} - i \quad 1 \leq i \leq x \]

\[ f^*(uv) = \frac{3n + 3}{2} \]

\[ f^*(u_{n+1-i}v_{n+1-i}) = \frac{3n + 3}{2} + i \quad 1 \leq i \leq x - 1 \]

\[ f^*(vv_i) = 3n + 3 - 2i \quad 1 \leq i \leq x \]

\[ f^*(vv_{x+i}) = 3n + 2 - 2i \quad 1 \leq i \leq x - 1 \]

It can be easily seen that each edge gets different label from \{1, 2, ..., 3n+1\}

Hence, \( K_{1,n} \times K_2 \) (n is odd) is a Near Mean Graph.

For example, \( K_{1,7} \times K_2 \) is a Near Mean Graph as shown in Fig. 2.12.

![Fig. 2.11](image-url)
**Theorem 2.4.2:** The Grid graph $P_n \times P_n$ is a Near Mean Graph.

**Proof:** Let $V(P_n \times P_n) = \{ u_{ij} : 1 \leq i \leq n, 1 \leq j \leq n \}$ and $E(P_n \times P_n) = \{ [(u_{ij} u_{ij+1}) : 1 \leq i \leq n, 1 \leq j \leq n-1] \cup [(u_{ij} u_{i+1j}) : 1 \leq i \leq n-1, 1 \leq j \leq n] \}.

Define an injective function $f : V \rightarrow \{0, 1, 2, \ldots, q-1, q+1\}$ by

For $i = 1, 2, 3, \ldots, n-1$

$$f(u_{ij}) = (i - 1)(2n-1) + (j - 1) \quad 1 \leq j \leq n$$

$$f(u_{nj}) = (n - 1)(2n-1) + (j - 1) \quad 1 \leq j \leq n-1$$

$$f(u_{nn}) = 2n(n-1) + 1$$

The induced edge labels are

For $i = 1, 2, 3, \ldots, n$,

$$f^e(u_{ij}, u_{ij+1}) = (i - 1)(2n-1) + j \quad 1 \leq j \leq n - 1$$

For $j = 1, 2, 3, \ldots, n$,
\[ f'(u_{ij}, u_{(i+1)j}) = (i-1)(2n-1) + (j-1) + n \quad 1 \leq i \leq n - 1 \]

It can be easily verified that each edge gets different label from the set \{1, 2, \ldots, q\}.

Hence, \( P_n \times P_n \) is a Near Mean Graph.

For example, \( P_4 \times P_4 \) is a Near Mean Graph as shown in Fig. 2.13.

\[ \text{Fig. 2.13} \]

**Theorem 2.4.3:** Prism \( P_m \times C_3 \), \( m \geq 2 \) is a Near Mean Graph.

**Proof:** Let \( G = P_m \times C_3 \)

\[ V(G) = \{ u_i, v_i, w_i : 1 \leq i \leq m \} \] and

\[ E(G) = \{ [(u_iu_{i+1}) \cup (v_iv_{i+1}) \cup (w_iw_{i+1}) : 1 \leq i \leq m-1] \cup \\ \quad [(u_i v_i) \cup (v_i w_i) \cup (u_i w_i) : 1 \leq i \leq m] \} \]

Define \( f : V(G) \to \{0, 1, 2, \ldots, 6m - 4, 6m - 2\} \) by

\[ f(u_1) = 0 \]

\[ f(v_1) = 4 \]
\[ f(w_1) = 2 \]
\[ f(w_m) = 6m - 2 \quad \text{if } m \text{ is even} \]
\[ f(u_i) = \begin{cases} 
6i - 4, & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\
6i - 2, & i \equiv 1 \mod 2, \ 3 \leq i \leq m 
\end{cases} \]
\[ f(v_i) = \begin{cases} 
6i - 4, & i \equiv 1 \mod 2, \ 3 \leq i \leq m \\
6i - 6, & i \equiv 0 \mod 2, \ 2 \leq i \leq m 
\end{cases} \]
\[ f(w_i) = \begin{cases} 
6i - 3, & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\
6i - 6, & i \equiv 1 \mod 2, \ 3 \leq i \leq m 
\end{cases} \]

Clearly, \( f \) is injective.

The induced edge labels are
\[ f^* (u_1v_1) = 2 \]
\[ f^* (u_1u_2) = 4 \]
\[ f^* (v_1w_1) = 3 \]
\[ f^* (v_1v_2) = 5 \]
\[ f^* (u_1w_1) = 1 \]
\[ f^* (w_1w_2) = 6 \]
\[ f^* (u_iv_i) = \begin{cases} 
6i - 5, & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\
6i - 3, & i \equiv 1 \mod 2, \ 3 \leq i \leq m 
\end{cases} \]
\[ f^* (v_iw_i) = \begin{cases} 
6i - 4, & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\
6i - 5, & i \equiv 1 \mod 2, \ 3 \leq i \leq m 
\end{cases} \]
\[ f^* (u_iw_i) = \begin{cases} 
6i - 3, & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\
6i - 4, & i \equiv 1 \mod 2, \ 3 \leq i \leq m 
\end{cases} \]
\[ f^* (u_iu_{i+1}) = 6i \quad 2 \leq i \leq m - 1 \]
\[ f^* (v_iv_{i+1}) = 6i - 2 \quad 2 \leq i \leq m - 1 \]
\[ f'(w_i, w_{i+1}) = 6i - 1 \quad 2 \leq i \leq m - 1 \]

It is clear that each edge gets unique label from the set \( \{1, 2, 3, \ldots, 6m-3\} \)

Hence, Prism \( P_m \times C_3 \) is a Near Mean Graph.

For example, \( P_5 \times C_3 \) and \( P_6 \times C_3 \) are Near Mean Graph as shown in Fig. 2.14 and Fig 2.15 respectively.

**Theorem 2.4.4**: The graph \( L_n \odot K_1 = (P_2 \times P_n) \odot K_1 \) is a Near Mean Graph.

**Proof**: Let \( L_n \odot K_1 = (V, E) \) such that

\[
V = \{u_i, v_i : 1 \leq i \leq 2n\} \quad \text{and} \quad E = \{(u_i u_{i+1}) : 1 \leq i \leq 2n-1\} \cup (u_1 u_{2n}) \cup \{(u_i v_i) : 1 \leq i \leq 2n\} \cup \{(u_i u_{2n+i}) : 2 \leq i \leq n-1\} \}
\]
Define $f : V \to \{0, 1, 2, \ldots, 5n-3, 5n-1\}$ by

\[
\begin{align*}
f(u_i) &= i & 1 \leq i \leq n \\
f(u_{n+i}) &= 5n-2-i & 1 \leq i \leq n \\
f(v_1) &= 0 \\
f(v_n) &= n+1 \\
f(v_{n+1}) &= 5n-1 \\
f(v_{2n}) &= 4n-3 \\
f(v_{n+i}) &= n+2+3i & 1 \leq i \leq n-2 \\
f(v_{n+1+i}) &= n+1+3i & 1 \leq i \leq n-2
\end{align*}
\]

Clearly, $f$ is injective.

The induced edge labels are

\[
\begin{align*}
f^\ast(u_1v_1) &= 1 \\
f^\ast(u_nv_n) &= n+1 \\
f^\ast(u_{n+1}v_{n+1}) &= 5n-2 \\
f^\ast(v_{2n}u_{2n}) &= 4n-2 \\
f^\ast(u_iu_{i+1}) &= i+1 & 1 \leq i \leq n-1 \\
f^\ast(u_iu_{n+i+1}) &= 3n-1 \\
f^\ast(u_{2n}u_1) &= 2n \\
f^\ast(u_{n+i}u_{n+i+1}) &= 5n-2-i & 1 \leq i \leq n-1 \\
f^\ast(u_{n+i}v_{n+i+1}) &= 2n+1-i & 2 \leq i \leq n-1 \\
f^\ast(u_{n+1+i}v_{n+1+i}) &= 3n+(i-1) & 1 \leq i \leq n-2 \\
f^\ast(u_iu_{2n+1-i}) &= 2n+(i-1) & 2 \leq i \leq n-1
\end{align*}
\]

Clearly, edge labels are $\{1, 2, \ldots, q\}$.
Hence, \( L_n \odot K_1 \) is a Near Mean Graph.

For example, \( L_5 \odot K_1 = (P_2 \times P_3) \odot K_1 \) is a Near Mean Graph as shown in Fig. 2.16.

**Theorem 2.4.5**: The Cuboid \( C_4 \times P_m \) \( (m > 2) \) is a Near Mean Graph.

**Proof**: Let \( G = C_4 \times P_m \) where

\[
V(G) = \{ u_{ij} : 1 \leq i \leq m, \ 1 \leq j \leq 4 \} \quad \text{and} \\
E(G) = \{ [u_{ij}u_{i,j+1}, 1 \leq i \leq m, 1 \leq j \leq 3] \cup [u_{i1}u_{i1}] : 1 \leq i \leq m \} \cup
\quad [u_{ij}u_{i+1j}, 1 \leq i \leq m-1, 1 \leq j \leq 4] \}
\]

Define \( f : V(G) \rightarrow \{0, 1, 2, \ldots, q-1, q+1\} \) by

\[
f(u_{11}) = 0 \\
f(u_{12}) = 2 \\
f(u_{13}) = 4 \\
f(u_{14}) = 3 \\
f(u_{ii}) = \begin{cases} f(u_{i,i}) + 15 & i \equiv 0 \mod 2, 2 \leq i \leq m \\
f(u_{i,i}) + 1 & i \equiv 1 \mod 2, 3 \leq i \leq m \\
\end{cases}
\]
\[ f(u_{i2}) = \begin{cases} f(u_{i-1}) + 5 & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\ f(u_{i-1}) + 11 & i \equiv 1 \mod 2, \ 3 \leq i \leq m \end{cases} \]

\[ f(u_{i3}) = \begin{cases} f(u_{i-1}) + 6 & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\ f(u_{i-1}) + 10 & i \equiv 1 \mod 2, \ 3 \leq i \leq m \end{cases} \]

\[ f(u_{i4}) = \begin{cases} f(u_{i-1}) + 6 & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\ f(u_{i-1}) + 10 & i \equiv 1 \mod 2, \ 3 \leq i \leq m \end{cases} \]

Clearly, \( f \) is injective.

The induced edge labels are

\[ f^*(u_{11}u_{12}) = 1 \]
\[ f^*(u_{12}u_{13}) = 3 \]
\[ f^*(u_{13}u_{14}) = 4 \]
\[ f^*(u_{14}u_{11}) = 2 \]

\[ f^*(u_{i1}u_{i2}) = \begin{cases} f(u_{i-11}u_{i-12}) + 10 & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\ f(u_{i-11}u_{i-12}) + 6 & i \equiv 1 \mod 2, \ 3 \leq i \leq m \end{cases} \]

\[ f^*(u_{i2}u_{i3}) = \begin{cases} f(u_{i-12}u_{i-13}) + 6 & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\ f(u_{i-12}u_{i-13}) + 10 & i \equiv 1 \mod 2, \ 3 \leq i \leq m \end{cases} \]

\[ f^*(u_{i3}u_{i4}) = \begin{cases} f(u_{i-13}u_{i-14}) + 6 & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\ f(u_{i-13}u_{i-14}) + 10 & i \equiv 1 \mod 2, \ 3 \leq i \leq m \end{cases} \]

\[ f^*(u_{i4}u_{i1}) = \begin{cases} f(u_{i-14}u_{i-11}) + 10 & i \equiv 0 \mod 2, \ 2 \leq i \leq m \\ f(u_{i-14}u_{i-11}) + 6 & i \equiv 1 \mod 2, \ 3 \leq i \leq m \end{cases} \]

\[ f^*(u_{i1}u_{i+11}) = 8i \quad 1 \leq i \leq m - 1 \]

\[ f^*(u_{i2}u_{i+12}) = 8i - 3 \quad 1 \leq i \leq m - 1 \]

\[ f^*(u_{i3}u_{i+13}) = 8i - 1 \quad 1 \leq i \leq m - 1 \]

\[ f^*(u_{i4}u_{i+14}) = 8i - 2 \quad 1 \leq i \leq m - 1 \]
It is clear that $f'(E(G)) = \{1, 2, \ldots, q\}$.

Hence, $C_4 \times P_m \ (m > 2)$ is a Near Mean Graph.

For example, $C_4 \times P_5$ is a Near Mean Graph as shown in Fig. 2.17.

2.5 Near Meanness on Special Types of Graphs

Near meanness of special types of graphs are studied in [12].

**Theorem 2.5.1**: The graph $P_n + 2K_1$ is a Near Mean Graph.

**Proof**: Let $P_n + 2K_1 = (V, E)$, where

$$V(P_n + 2K_1) = \{ u_i : 1 \leq i \leq n, v, w \}$$

and

$$E(P_n + 2K_1) = \{ (u_i, u_{i+1}) : 1 \leq i \leq n - 1 \} \cup \{(v, u_i) \cup (w, u_i) : 1 \leq i \leq n\}$$

**Case (i)**: when $n$ is even

Let $f : V(P_n + 2K_1) \rightarrow \{0, 1, 2, \ldots, 3n - 2, 3n\}$ by
\[ f(u) = 0 \]
\[ f(w) = 3n \]
\[ f(u_{2i-1}) = 6i - 4 \quad 1 \leq i \leq \frac{n}{2} \]
\[ f(u_{2i}) = 6i - 2 \quad 1 \leq i \leq \frac{n}{2} \]

Clearly, \( f \) is injective.

The induced edge labels are
\[ f^*(u_iu_{i+1}) = 3i \quad 1 \leq i \leq n-1 \]
\[ f^*(vu_{2i-1}) = 3i-2 \quad 1 \leq i \leq \frac{n}{2} \]
\[ f^*(vu_{2i}) = 3i-1 \quad 1 \leq i \leq \frac{n}{2} \]
\[ f^*(wu_{2i-1}) = \frac{3n}{2} + 3i-2 \quad 1 \leq i \leq \frac{n}{2} \]
\[ f^*(wu_{2i}) = \frac{3n}{2} + 3i-1 \quad 1 \leq i \leq \frac{n}{2} \]

It can be easily verified that each edge gets different label from \( \{1, 2, \ldots, 3n-1\} \)

**Case (ii) :** When \( n \) is odd

Let \( f : V(P_n + 2K_1) \rightarrow \{0, 1, 2, \ldots, 3n -2, 3n\} \) by
\[ f(u) = 0 \]
\[ f(w) = 3n \]
\[ f(u_i) = 3i - 2 \quad 1 \leq i \leq n \]

The induced edge labels are
\[ f^*(u_iu_{i+1}) = 3i \quad 1 \leq i \leq n-1 \]
\[ f'(vu_i) = \begin{cases} 
\frac{3i-1}{2}, & \text{if } i \equiv 1 \mod 2, 1 \leq i \leq n \\
\frac{3i-2}{2}, & \text{if } i \equiv 0 \mod 2, 1 \leq i \leq n 
\end{cases} \]

\[ f'(wu_i) = \begin{cases} 
\frac{3(n+i) - 2}{2}, & \text{if } i \equiv 1 \mod 2, 1 \leq i \leq n \\
\frac{3(n+i) - 1}{2}, & \text{if } i \equiv 0 \mod 2, 1 \leq i \leq n 
\end{cases} \]

It can be easily verified that each edge gets different label from \( \{1, 2, ..., 3n-1\} \).

Hence, \( P_n + 2K_1 \) is a Near Mean Graph.

For example, Near Mean Labelings of \( P_6 + 2K_1 \) and \( P_7 + 2K_1 \) are shown in Fig 2.18 and Fig 2.19.
Theorem 2.5.2 : Triangular snake $T_n$ is a Near Mean Graph.

Proof: Let $T_n = (V, E)$, where

$$V(T_n) = \{ [u_i : 1 \leq i \leq n-1], [v_j : 1 \leq j \leq n] \} \text{ and}$$

$$E(T_n) = \{ [(u_i v_i) \cup (v_i v_{i+1}) : 1 \leq i \leq n-1] \cup [(u_i v_{i+1}) : 1 \leq i \leq n-1] \}$$

Define $f : V(T_n) \rightarrow \{0, 1, 2, \ldots, 3n-4, 3n-2\}$ by

$$f(u_i) = 3(i-1) \quad 1 \leq i \leq n-1$$

$$f(v_i) = 3i-2 \quad 1 \leq i \leq n$$

Clearly, $f$ is injective.

The induced edge labels are

$$f^*(u_i v_i) = 3i-2 \quad 1 \leq i \leq n-1$$

$$f^*(u_i v_{i+1}) = 3i-1 \quad 1 \leq i \leq n-1$$

$$f^*(v_i v_{i+1}) = 3i \quad 1 \leq i \leq n-1$$

Clearly, $f^*(E(T_n)) = \{1, 2, \ldots, q\}$.

Hence, $T_n$ is a Near Mean Graph.

For example, Near Mean Labeling of $T_4$ is shown in Fig. 2.20.

Fig. 2.20

Theorem 2.5.3 : Quadrilateral snake $Q_n$ is a Near Mean Graph.

Proof: Let $Q_n = (V, E)$ where
\[ V(Q_n) = \{ [u_i : 1 \leq i \leq n], [v_i, w_i : 1 \leq i \leq n-1] \} \] and 
\[ E(Q_n) = \{ [(u_i v_i) \cup (u_i w_i) : 1 \leq i \leq n-1] \cup [(u_{i+1} v_i) \cup (u_{i+1} w_i) : 1 \leq i \leq n-1] \}. \]

Define \( f : V(Q_n) \to \{0, 1, 2, \ldots, 4n-5, 4n-3\} \) by

\[
\begin{align*}
    f(u_1) &= 0 \\
    f(u_i) &= 4i - 5 \quad 2 \leq i \leq n \\
    f(v_i) &= 4i - 2 \quad 1 \leq i \leq n - 1 \\
    f(w_i) &= 4i \quad 1 \leq i \leq n - 2 \\
    f(w_{n-1}) &= 4n - 3
\end{align*}
\]

Clearly, \( f \) is injective.

The induced edge labels are

\[
\begin{align*}
    f^*(u_i v_i) &= 4i - 3 \quad 1 \leq i \leq n - 1 \\
    f^*(u_{i+1} v_i) &= 4i - 1 \quad 1 \leq i \leq n - 1 \\
    f^*(u_i w_i) &= 4i - 2 \quad 1 \leq i \leq n - 1 \\
    f^*(u_{i+1} w_i) &= 4i \quad 1 \leq i \leq n - 1
\end{align*}
\]

It is easy to verify that \( f^*(Q_n) = \{1, 2, \ldots, q\} \).

Hence, \( Q_n \) is a Near Mean Graph.

For example, Near Mean Labeling of \( Q_5 \) is shown in Fig. 2.21.
**Theorem 2.5.4**: Parachute $P_{2,n-2}$ is a Near Mean Graph.

**Proof**: Let $P_{2,n-2} = (V, E)$ such that

$$V = \{ v, u_i : 1 \leq i \leq n \} \text{ and }$$

$$E = \{ [(u_{i}u_{i+1}) : 1 \leq i \leq n-1] \cup (u_{1}u_{n}) \cup (vv_{1}) \cup (vu_{n}) \}.$$ 

Define $f : V \rightarrow \{0, 1, 2, \ldots, n+1, n+3\}$ by

$$f(v) = 0$$

$$f(u_{1}) = 2$$

$$f(u_{n}) = 3$$

$$f(u_{2}) = n + 3$$

$$f(u_{i+2}) = n + 2 - i,$$

$$f(u_{2i}) = n + 2 - i, \begin{cases} 
 1 \leq i \leq \frac{n-1}{2}, & \text{if } n \text{ is odd} \\
 1 \leq i \leq \frac{n}{2} - 1, & \text{if } n \text{ is even}
\end{cases}$$

$$f(u_{n-i}) = 3 + i,$$

$$f(u_{n+3}) = 3 + i, \begin{cases} 
 0 \leq i \leq \frac{n-3}{2}, & \text{if } n \text{ is odd} \\
 1 \leq i \leq \frac{n}{2} - 1, & \text{if } n \text{ is even}
\end{cases}$$

Clearly, $f$ is injective.

The induced edge labels are

$$f^*(vu_{1}) = 1$$

$$f^*(vu_{n}) = 2$$

$$f^*(u_{1}u_{n}) = 3$$

$$f^*(u_{1}u_{2}) = \begin{cases} 
  \frac{n+5}{2}, & \text{if } n \text{ is odd} \\
  \frac{n+6}{2}, & \text{if } n \text{ is even}
\end{cases}$$
\[ f'(u_{i+1}u_{i+2}) = \begin{cases} n+3-i, & 1 \leq i \leq \frac{n-1}{2}, \text{ if } n \text{ is odd} \\ 1 \leq i \leq \frac{n}{2}-1, & \text{if } n \text{ is even} \end{cases} \]

\[ f'(u_{n+1-i}u_{n-i}) = \begin{cases} 3 + i, & 1 \leq i \leq \frac{n-3}{2}, \text{ if } n \text{ is odd} \\ 1 \leq i \leq \frac{n}{2}-1, & \text{if } n \text{ is even} \end{cases} \]

It is clear that edges get distinguished label from \{1, 2, \ldots, q\}.

Hence, the Parachute is a Near Mean Graph.

For example, Near Mean Labeling of \((P_{2,5} : n \text{ - odd})\) and \((P_{2,6} : n \text{ - even})\) are shown in Fig. 2.22 and Fig. 2.23 respectively.

**Theorem 2.5.5:** \( P_{a, b} \ (b : \text{odd}) \) is a Near Mean Graph.

**Proof:** Let \( P_{a, b} = (V, E) \), where

\[ V(P_{a, b}) = \{v_{ij} : 1 \leq i \leq b, 0 \leq j \leq a \} \text{ with } v_{i0} = u, \ v_{ia} = v \and \]

\[ E(P_{a, b}) = \{ (v_{ij}v_{i(j+1)}) : 1 \leq i \leq b, 0 \leq j \leq a-1 \} \]

Define \( f : V(P_{a, b}) \rightarrow \{0, 1, 2, \ldots, ab - 1, ab + 1\} \) by
f(u) = 0
f(v) = ab + 1
f(v_{i1}) = 2i - 1 \quad 1 \leq i \leq b

f\left( \frac{v_{b+1}}{2},i \right) = b + 1

f\left( \frac{v_{b+1}}{2} - i,2 \right) = (b + 1) + 4i \quad 1 \leq i \leq \frac{b-1}{2}

f(v_{b+1-i,2}) = (b - 1) + 4i \quad 1 \leq i \leq \frac{b-1}{2}

f(v_{ij}) = f(v_{i,j-2}) + 2b \quad 1 \leq i \leq b, 3 \leq j \leq a - 1

clearly, f is injective.

The induced edge labels are

\[ f^e(uv_{i1}) = i \quad 1 \leq i \leq b \]

\[ f^e\left( \frac{v_{b+1}}{2} - i,1 \right) = bj + i + 1 \quad 0 \leq i < \frac{b+1}{2}, 1 \leq j \leq a - 2 \]

\[ f^e\left( v_{b+1-i,j} v_{b+1-i,j+1} \right) = bj + \frac{b+1}{2} + i \quad 1 \leq i \leq \frac{b-1}{2}, 1 \leq j \leq a - 2 \]

\[ f^e(v_{i,a-1}) = (a-1) b + i \quad 1 \leq i \leq b \]

Clearly, f is injective. The set of edge labels of \( P_{a,b} \) (b : odd) is \{1, 2, 3, ..., q\}

Hence \( P_{a,b} \) (b : odd) is a Near Mean Graph.

For example, Near Mean Labeling of \( P_{4,5} \) is shown in Fig 2.24.
Theorem 2.5.6: $p_a^b$ (b : odd) is a Near Mean Graph.

Proof: Let $p_a^b = (V, E)$, where

$$V(p_a^b) = \{u_i : 1 \leq i \leq a, \bigcup_{i=1}^{a-1} \bigcup_{k=1}^{b}(u_{i,k} : 1 \leq k \leq i) \} \text{ and}$$

$$E(p_a^b) = \{(u_{ij1}, u_{ijkl}) : 1 \leq i \leq a-1, 1 \leq j \leq b \}
\cup \{(u_{ijkl}, u_{ij(k+1)} : 1 \leq k \leq i-1, 1 \leq j \leq b, 2 \leq i \leq a-1) \}$$
\cup \{(u_{ij1}, u_{ij} : 1 \leq i \leq a-1, 1 \leq j \leq b) \}.$$

Define $f : V(p_a^b) \rightarrow \{0, 1, 2, ..., q - 1, q + 1 \}$ by

- $f(u_1) = 0$
- $f(u_i) = f(u_{i-1}) + bi \quad 2 \leq i \leq a-1$
- $f(u_a) = q + 1$
- $f(u_{i1}) = 2j - 1 \quad 1 \leq j \leq b$
- $f(u_{ij1}) = f(u_i) + 2j - 1 \quad 2 \leq i \leq a - 1, 1 \leq j \leq b$
Clearly, \( f \) is injective.

The induced edge labels are

\[
\begin{align*}
\hat{f}^\ast(u_i u_{ij1}) &= f(u_i) + j & 1 \leq i \leq a - 1, \ 1 \leq j \leq b \\
\hat{f}^\ast(u_{iji} u_{i+1}) &= f(u_{iji}) + b + (j-1) & i \equiv 1 \mod 2, 1 \leq i \leq a - 1, 1 \leq j \leq b \\
\hat{f}^\ast(u_{i(b+1-j)1} u_{i(b+1-j)2}) &= \hat{f}^\ast(u_{i11} u_{i12}) + j & 2 \leq i \leq a - 1, 1 \leq j \leq \frac{b-1}{2} \\
\hat{f}^\ast(u_{i(b+1-j)1} u_{i(b+1-j)2}) &= \hat{f}^\ast(u_{i11} u_{i12}) + j & 2 \leq i \leq a - 1, 1 \leq j \leq \frac{b-1}{2}
\end{align*}
\]
\[
\begin{align*}
\hat{f}(u_{ijk} u_{ijk+1}) & = \hat{f}(u_{ijk-1} u_{ijk}) + b \quad 3 \leq i \leq a-1, 1 \leq j \leq b, 2 \leq k \leq i-1
\end{align*}
\]

The set of edge labels of \( P^b_a \) (b : odd) is \( \{1, 2, 3, \ldots, q\} \)

Hence, \( P^b_a \) (b : odd) is a Near Mean Graph.

For example, Near Mean Labeling of \( P^5_4 \) is shown in Fig. 2.25.

\[\text{Fig. 2.25}\]

2.6 Near Meanness on Armed and Double Armed Crown of Cycles

**Theorem 2.6.1**: The graph \( C_4 \bowtie P_m \) is a Near Mean Graph.

**Proof**: Let \( C_4 \bowtie P_m = (V, E) \) such that

\[
V = \{ u, v, w, x, [u_i, v_i, w_i, x_i : 1 \leq i \leq m] \} \text{ and } \\
E = \{ (uv) \cup (vw) \cup (wx) \cup (xu) \cup \\
[(u_i u_{i+1}) \cup (v_i v_{i+1}) \cup (w_i w_{i+1}) \cup (x_i x_{i+1}) : 1 \leq i \leq m-1] \} \\
\text{with } u = u_m, v = v_m, w = w_m, x = x_m
\]

Define \( f : V \to \{0, 1, 2, \ldots, 4m-1, 4m+1\} \) by

**Case (i)**: When \( m \) is odd

\[
\begin{align*}
f(u_i) & = i - 1 \quad 1 \leq i \leq m \\
f(v_{2i+1}) & = 2m - 2i \quad 0 \leq i \leq \frac{m-1}{2}
\end{align*}
\]
\[ f(v_{2i}) = 2m - 2i \quad 1 \leq i \leq \frac{m-1}{2} \]

\[ f(w_i) = 2m + i \quad 1 \leq i \leq m \]

\[ f(x_1) = 4m + 1 \]

\[ f(x_{m+1-i}) = 3m + i \quad 1 \leq i \leq m - 1 \]

It implies, \( f(u) = f(u_m) \), \( f(v) = f(v_m) \), \( f(w) = f(w_m) \), and \( f(x) = f(x_m) \)

Clearly, \( f \) is injective.

The induced edge labels are

\[ f^*(u_i u_{i+1}) = i \quad 1 \leq i \leq m - 1 \]

\[ f^*(v_i v_{i+1}) = 2m - i \quad 1 \leq i \leq m - 1 \]

\[ f^*(w_i w_{i+1}) = 2m + 1 + i \quad 1 \leq i \leq m - 1 \]

\[ f^*(x_i x_{i+1}) = 4m - (i-1) \quad 1 \leq i \leq m - 1 \]

\[ f^*(uv) = m \]
\[ f^*(vw) = 2m + 1 \]
\[ f^*(wx) = 3m + 1 \]
\[ f^*(ux) = 2m \]

**Case (ii)** : When \( m \) is even

\[ f(u_i) = i - 1 \quad 1 \leq i \leq m - 1 \]

\[ f(u_m) = m \]

\[ f(v_{2i+1}) = 2m - 2i \quad 0 \leq i < \frac{m}{2} \]
\[ f(v_{2i}) = 2m - 1 - 2i \quad 1 \leq i \leq \frac{m}{2} \]
\[ f(w_i) = 2m + i \quad 1 \leq i \leq m \]
\[ f(x_{m+i}) = 3m + i \quad 1 \leq i \leq m - 1 \]
\[ f(x_1) = 4m + 1 \]

Clearly, \( f \) is injective.

The induced edge labels are
\[ f^*(u_iu_{i+1}) = i \quad 1 \leq i \leq m - 1 \]
\[ f^*(v_iv_{i+1}) = 2m - i \quad 1 \leq i \leq m - 1 \]
\[ f^*(w_iw_{i+1}) = 2m + i \quad 1 \leq i \leq m - 1 \]
\[ f^*(x_ix_{i+1}) = 4m - (i-1) \quad 1 \leq i \leq m - 1 \]
\[ f^*(uv) = m \]
\[ f^*(vw) = 2m \]
\[ f^*(vx) = 3m + 1 \]
\[ f^*(ux) = 2m + 1 \]

It is clear that in both the cases, \( f^*(E(C_4 \odot P_m)) = \{1, 2, \ldots, q\} \).

Hence, \( C_4 \odot P_m \) is a Near Mean Graph.

For example, Near Mean Labeling of \((C_4 \odot P_7 : m \text{ - odd})\) and \((C_4 \odot P_6 : m \text{ - even})\) are shown in Fig 2.25 and Fig. 2.26 respectively.
Theorem 2.6.2: The graph $C_4 \odot 2P_m$ is a Near Mean Graph.

Proof: Let $C_4 \odot 2P_m = (V, E)$, where

$$V = \{[u_{ij}: 1 \leq i \leq 4, 1 \leq j \leq 2m-1], v_i: 1 \leq i \leq 4 \} \text{ with } u_{im} = v_i: 1 \leq i \leq 4$$

and

$$E = \{ ([u_{ij}u_{i+1}]: 1 \leq i \leq 4, 1 \leq j \leq 2m-2] \cup [(v_i v_{i+1}): 1 \leq i \leq 3] \cup (v_1v_4) \}$$

Define $f: V \rightarrow \{0, 1, 2, \ldots, 8m-5, 8m-3\}$ by

- $f(u_{ij}) = j - 1$ \hspace{1cm} $1 \leq j \leq 2m-1$
- $f(u_{ij}) = 2m - 2 + j$ \hspace{1cm} $1 \leq j \leq 2m-1$
- $f(u_{ij}) = \begin{cases} 4m - 2 + i \quad 1 \leq i \leq m - 1 \\ 5m - 1 \quad i = m \end{cases}$
- $f(u_{2m-1}) = q + 1 = 8m - 3$
- $f(u_{4i}) = 6m - 3 + i$ \hspace{1cm} $1 \leq i \leq 2m-2$

The induced edge labels are

- $f^*(u_{ij}u_{i+1}) = j$ \hspace{1cm} $1 \leq j \leq 2m-2$
- $f^*(v_1v_2) = 2m-1$
- $f^*(u_{2j}u_{2j+1}) = 2m - 1 + j$ \hspace{1cm} $1 \leq j \leq 2m-2$
- $f^*(v_2v_3) = 4m-1$
- $f^*(u_{3j}u_{3j+1}) = 4m - 1 + j$ \hspace{1cm} $1 \leq j \leq m-1$
- $f^*(v_3v_4) = 6m-2$
- $f^*(v_1v_4) = 4m-2$
- $f^*(u_{4j}u_{4j-1}) = 8m - 3 - i$ \hspace{1cm} $1 \leq j \leq 2m-2$

Case (i): When $m$ is odd

$$f(u_{3,2m-1}) = \begin{cases} 5m \quad \text{if } m \geq 5 \\ 4m+1 \quad \text{if } m = 3 \end{cases}$$
\[ f(u_{3,2m-2}) = 5m \text{ if } m = 3 \]
\[ f(u_{3,m+i}) = 6m - 2 - i \quad 1 \leq i \leq \frac{m-1}{2} \]
\[ f \left( u_{3,\frac{3m+1}{2}} \right) = 5m - 2 \]
\[ f \left( u_{3,\frac{3m+1+i}{2}} \right) = 5m - 1 + i \quad 1 \leq i \leq \frac{m-3}{2} \]

Clearly, \( f \) is injective.

The induced edge labels are

\[ f^* (u_{3,m} u_{3,m+1}) = \frac{11m - 3}{2} \]
\[ f^* (u_{3,m+1} u_{3,m+2}) = 5m - 1 \text{ if } m = 3 \]
\[ f^* (u_{3,m+i} u_{3,m+1+i}) = 6m - 2 - i \quad 1 \leq i \leq \frac{m-3}{2} \]
\[ f^* \left( u_{3,\frac{3m-1}{2}} u_{3,\frac{3m+1}{2}} \right) = 5m \text{ if } m > 3 \]
\[ f^* \left( u_{3,\frac{3m+1}{2}} u_{3,\frac{3m+3}{2}} \right) = 5m - 1 \]
\[ f^* \left( u_{3,\frac{3m+1+i}{2}} u_{3,\frac{3m+3+i}{2}} \right) = 5m + i \quad 1 \leq i \leq \frac{m-5}{2} \]

**Case (ii) :** When \( m \) is even

\[ f(u_{3,m+i}) = 6m - 2 - i \quad 1 \leq i \leq \frac{m}{2} \]
\[ f \left( u_{3,\frac{3m}{2}+1} \right) = 5m - 2 \]
Clearly, \( f \) is injective.

The induced edge labels are

\[
f^e(u_{3,m} u_{3,m+1}) = \frac{11m - 4}{2}
\]

\[
f^e(u_{3,m+i} u_{3,m+i+1}) = 6m - 2 - i \quad 1 \leq i \leq \frac{m-2}{2}
\]

\[
f^e(u_{3,2m-2} u_{3,2m-1}) = 5m - 1 \quad \text{if } m = 4
\]

\[
f^e\left(\begin{bmatrix} 3m \\ -1 \end{bmatrix}, \begin{bmatrix} 3m+1 \\ -1 \end{bmatrix}\right) = 5m \quad \text{if } m > 4
\]

\[
f^e\left(\begin{bmatrix} 3m+2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3m+4 \\ -1 \end{bmatrix}\right) = 5m - 1 \quad \text{if } m > 4
\]

\[
f^e\left(\begin{bmatrix} 3m+4 \\ -1 \end{bmatrix}, \begin{bmatrix} 3m+4+1 \\ -1 \end{bmatrix}\right) = 5m + i \quad 1 \leq i \leq \frac{m-6}{2}
\]

Clearly, the edge labels of \( C_4 \odot 2P_m \) form the set \( \{1, 2, \ldots, q\} \).

Hence, \( C_4 \odot 2P_m \) is a Near Mean Graph.

For example, Near Mean Labeling of \( (C_4 \odot 2P_5 : m \text{ - odd}) \) and \( (C_4 \odot 2P_6 : m \text{ - even}) \) are shown in Fig. 2.27 and Fig. 2.28 respectively.
Fig. 2.27

Fig. 2.28