CHAPTER - 4
FUZZY $\alpha$-PARACOMPACTNESS IN
FUZZY BOX PRODUCTS

4.1 Introduction

The notion of shading family was introduced in the literature by T.E. Gantner and others in [G;S;W] during the investigation of compactness in fuzzy topological spaces. The shading families are a very natural generalization of coverings. An approach to fuzzy $\alpha$-paracompactness using the notion of shading families was introduced by S.R. Malghan and S.S. Benchalli in [M;B].

The second section of this chapter describes the necessary definitions and results of shading families.

In the third section, we introduce and study the notion of fuzzy $\alpha$-paracompactness in fuzzy box products. Here we give a characterization of fuzzy $\alpha$-paracompactness through fuzzy entourages.

In the last section we introduce fuzzy $\alpha$-paracompact fuzzy topologically complete spaces. Here we have the main theorem that for a family of fuzzy $\alpha$-paracompact spaces, their fuzzy box product is fuzzy topologically complete.

* Some results of this Chapter were communicated to the Journal of Fuzzy Mathematics.
4.2 Shading families

The following definitions and results are from [M;B]1

4.2.1 Definition Let \((X,T)\) be a fuzzy topological space and \(\alpha \in [0,1)\). A collection \(\mathcal{U}\) of fuzzy sets is called an \(\alpha\)-shading of \(X\) if for each \(x \in X\) there exists \(g \in \mathcal{U}\) with \(g(x) > \alpha\). A subcollection of an \(\alpha\)-shading of \(X\) which is also an \(\alpha\)-shading is called an \(\alpha\)-subshading of \(X\).

4.2.2 Definition Let \(X\) be a set. Let \(\mathcal{U}\) and \(\mathcal{V}\) be any two collections of fuzzy subsets of \(X\). Then \(\mathcal{U}\) is a refinement of \(\mathcal{V}\) (\(\mathcal{U} < \mathcal{V}\)) if for each \(g \in \mathcal{U}\) there is an \(h \in \mathcal{V}\) such that \(g \leq h\).

If \(\mathcal{U}, \mathcal{V}, \mathcal{W}\) are collections such that \(\mathcal{U} < \mathcal{V}\) and \(\mathcal{U} < \mathcal{W}\) then \(\mathcal{U}\) is called a common refinement of \(\mathcal{V}\) and \(\mathcal{W}\).

4.2.3 Definition A family \(\{a_s : s \in S\}\) of fuzzy sets in a fuzzy topological space \((X, T)\) is said to be locally finite if for each \(x \in X\) there exists a fuzzy open set \(g\) with \(g(x) = 1\) such that \(a_s \leq 1-g\) holds for all but at most finitely many \(s \in S\).

4.2.4 Definition A family \(\{a_s : s \in S\}\) of fuzzy sets in a fuzzy topological space \((X, T)\) is said to be \(\sigma\)-locally finite if it is the union of countably many locally finite sets.
4.2.5 **Theorem** Let \( \{ a_s \} \) and \( \{ b_t \} \) be two \( \alpha \)-shadings of a fuzzy topological space \((X, T)\), where \( \alpha \in [0,1) \). Then

i) \( \{ a_s \land b_t \} \) is an \( \alpha \) - shading of \( X \) which refines both \( \{ a_s \} \) and \( \{ b_t \} \).

Further if both \( \{ a_s \} \) and \( \{ b_t \} \) are locally finite so is \( \{ a_s \land b_t \} \).

ii) Any common refinement of \( \{ a_s \} \) and \( \{ b_t \} \) is also a refinement of \( \{ a_s \land b_t \} \).

4.2.6 **Theorem** Let \( \{ a_s : s \in S \} \) be a locally finite family of fuzzy sets in a fuzzy topological space \((X,T)\) then

i) \( \{ \overline{a_s} : s \in S \} \) is also locally finite.

ii) For each \( S' \subset S \), \( \bigvee \{ \overline{a_s} : s \in S' \} \) is a fuzzy closed set.

4.3 **A Characterization of fuzzy \( \alpha \)-paracompactness**

4.3.1 **Definition** A fuzzy topological space \((X, T)\) is said to be \( \alpha \)-paracompact if each \( \alpha \) - shading of \( X \) by fuzzy open sets has a locally finite \( \alpha \)-shading refinement by fuzzy open sets.

We quote the following theorem from [SU]

4.3.2 **Theorem** For a fuzzy regular space the following are equivalent

1) \( X \) is \( \alpha \)-paracompact.

2) Every \( \alpha \)-shading of \( X \) by fuzzy open sets has a \( \sigma \)- locally finite \( \alpha \) - shading refinement by fuzzy open sets.
3) Every $\alpha$-shading of $X$ by fuzzy open sets has a locally finite $\alpha$-shading refinement by fuzzy open sets.

4) Every $\alpha$-shading of $X$ by fuzzy open sets has a locally finite $\alpha$-shading refinement by fuzzy closed sets.

We prove the following theorem.

**4.3.3 Theorem** For a fuzzy regular space $X$, $X$ is $\alpha$-paracompact if and only if (*) every $\alpha$-shading $\mathcal{U}$ of $X$ by fuzzy open sets is refined by a fuzzy entourage $D$.

**Remark:** We say that $D$ refines $\mathcal{U}$ for some $D \subseteq X \times X$ if

$$\mathcal{S} = \{D \times x : x \in X\}$$

refines $\mathcal{U}$. In particular, this gives a refinement by fuzzy entourages.

**Proof of the above theorem**

We first prove that (4) in theorem (4.3.2) implies (*).

Let $\mathcal{U}$ be an $\alpha$-shading of $X$ by fuzzy open sets. So for each $x \in X$ there exists $U_\beta \in \mathcal{U}$ such that $U_\beta(x) > \alpha$.

Let $\mathcal{V} = \{V_\beta : \beta \in \wedge\}$ be a locally finite $\alpha$-shading refinement by closed sets. For each $\beta \in \wedge$ and $V_\beta < U_\beta$, 
Now $W_\beta$ is a fuzzy open neighbourhood of the diagonal in $X \times X$.

Let $V = \inf \{ W_\beta : \beta \in \Lambda \}$

So $V <x> \leq W_\beta <x>$ for each $x \in X$.

Therefore $\{ V <x> : x \in X \}$ is a refinement of $\mathcal{H}$.

Next we prove that $V$ is a fuzzy neighbourhood of the diagonal.

For each point of the diagonal we choose a fuzzy open set $g$ of $x$ with

$g(x) = 1$ and $V_\beta \leq 1 - g$ holds for all but atmost finitely many $\beta \in \Lambda$.

If $g \land V_\beta = 0$ then $g \leq 1 - V_\beta$

That is $g \times g \leq W_\beta$

But $V = \inf \{ W_\beta : \beta \in \Lambda \}$.

This means that $V$ is a fuzzy neighbourhood of the diagonal.

Before proving (*) implies (1) of theorem 4.3.2, we prove a lemma

**4.3.4 Lemma** Let $X$ be a fuzzy topological space such that each

$\alpha$ - shading of $X$ by fuzzy open sets is refined by a fuzzy entourage and let

$\mathcal{A} = \{ a_s : s \in S \}$ be a locally finite family of fuzzy subsets of $X$. Then

there is a neighbourhood $V$ of the diagonal in $X \times X$ such that the family of

all sets $V <a_s>$ for $s \in S$ is locally finite.
Proof

Let \( \mathcal{U} \) be an \( \alpha \) – shading of \( X \) by fuzzy open sets. That is, for each \( x \in X \) there exists a fuzzy open set \( U_\beta \in \mathcal{U} \) such that \( U_\beta(x) > \alpha \).

Since \( \{a_s : s \in S\} \) is locally finite, for each \( x \in X \) there exists a fuzzy open set \( g \) with \( g(x) = 1 \) and \( a_s \leq 1 \) - \( g \) holds for all but atmost finitely many \( s \in S \).

Let \( U \) be neighbourhood of the diagonal such that \( \{U^{<x>} : x \in X\} \) refines \( \mathcal{U} \). Then there exists a symmetric neighbourhood \( V \) of the diagonal such that \( V \circ V \leq U \), where \( V = V^{-1} \).

If \( V \circ V^{<x>} \land a_s = 0 \) then \( V^{<x>} \land V^{<a_s>} = 0 \)

For,

If \( (y_\alpha)_{\alpha > 0} \in V^{<x>} \land V^{<a_s>} \) then \( y_\alpha \in V^{<x>} \) and \( y_\alpha \in V^{<a_s>} \) where \( \alpha > 0 \).

That is, \( V(x, y) = \alpha \) and \( V^{<a_s>} (y) = \alpha \) where \( \alpha > 0 \).

Now \( V^{<a_s>} (y) = \sup_{y \in X} (a_s(z) \land V(z, y)) = \alpha \)

Therefore given \( \varepsilon > 0 \), there exists \( z \in X \) such that

\[ a_s(z) \land V(z, y) > \alpha - \varepsilon \]

That is, \( a_s(z) > \alpha - \varepsilon \) and \( V(z, y) > \alpha - \varepsilon \).
\[ V \circ V(x, z) = \sup_{y \in X} \{ V(x, y) \wedge V(y, z) \} > \alpha - \varepsilon \]

\[ \therefore V \circ V(x, z) \wedge a_s(z) > \alpha - \varepsilon \]

Which is a contradiction.

Therefore the family of all sets \( V_{a_s} \) for \( s \in S \) is locally finite.

Hence the lemma.

We prove (*) implies (1) of theorem 4.3.2.

Let \( \mathcal{U} \) be an \( \alpha \) – shading of \( X \) by fuzzy open sets.

Therefore for each \( x \in X \) there exists \( U_\beta \in \mathcal{U} \) such that \( U_\beta(x) > \alpha \).

By (*) there exists a fuzzy neighbourhood \( V \) of the diagonal which refines \( \mathcal{U} \).

That is \( \{ V_{a_s} : x \in X \} \) refines \( \mathcal{U} \).

That is \( V_{a_s} \leq U_\beta \) where \( U_\beta \in \mathcal{U} \).

Let \( \{ a_s : s \in S \} \) be a locally finite family of fuzzy subsets of \( X \).

Then by above lemma there exists a neighbourhood \( V \) of the diagonal in \( X \times X \) such that \( \{ V_{a_s} : s \in S \} \) is locally finite.

where \( V_{a_s}(y) = \sup_{x \in X} (a_s(x) \wedge V(x, y)) \) for all \( y \in X \)

So for each \( s \in S \), choose a fuzzy open set \( U_\beta \in \mathcal{U} \) such that \( a_s \leq U_\beta \).

Let \( W_\beta = U_\beta \wedge V_{a_s} \).
Therefore $W_{\beta}$ is a locally finite $\alpha$ – shading refinement of $\mathcal{U}$.

Hence $X$ is fuzzy $\alpha$-paracompact.

4.4. Fuzzy $\alpha$-paracompact fuzzy topologically complete spaces

We first prove a lemma.

4.4.1 Lemma  If $X$ is a fuzzy $\alpha$-paracompact space, then the fuzzy filter of entourages of $X$ is a complete fuzzy uniformity compatible with $X$.

Proof

Let $\mathcal{N}$ be the fuzzy filter of entourages of $X$. We prove that $\mathcal{N}$ is a fuzzy uniformity.

Let $D \in \mathcal{N}$. For each $x \in X$ choose an $\alpha$ – shading $U_x$ of $x$ by fuzzy open sets with $U_x(x) > \alpha$ and $U_x \times U_x \subseteq D$. By theorem 4.3.3, every $\alpha$ – shading of $X$ by fuzzy open sets is refined by a fuzzy entourage.

That is, there exists $E \in \mathcal{N}$ which refines $\mathcal{U} = \{U_x, x \in X\}$.

Let $D = E \wedge E^{-1}$

So we have.

i) $D(x, x) = 1$

ii) $D \in \mathcal{N} \Rightarrow D^{-1} \in \mathcal{N}$

iii) Let $(x, y) \in E$ and $(y, z) \in E$
Consider \( E \circ E (x, z) = \sup_{y \in X} \{ E(x, y) \land E(y, z) \} \leq D(x, z) \).

That is, for \( D \in \mathcal{N} \), there exists \( E \in \mathcal{N} \) such that \( E \circ E \leq D \).

Now \( D \in \mathcal{N} \) is a fuzzy open subset of \( X \times X \) and \( D(x) \) is a fuzzy open subset of \( X \) for every point \( x \in X \).

Again, if given a fuzzy open set \( G \) of \( y \) in \( X \) with \( G(y) = 1 \) for all \( y \in X \), then there exists \( F \in \mathcal{N} \) such that \( F(y) \leq G \),

Where \( F = (G \times G) \cup ((X \setminus \{ y \}) \times (X \setminus \{ y \})) \).

That is \( F(y) (x) \leq G(x) \) for all \( x \in X \).

Therefore \( T_{\mathcal{N}} = \{ G \in \mathcal{I}^X / \text{If } y \in X \text{ is such that } G(y) = 1 \text{ then there exists } F \in \mathcal{N} \text{ such that } F(y) \leq G \} \).

Thus \( \mathcal{N} \) is compatible with \( X \).

Claim \( \mathcal{N} \) is complete.

We have to prove that every \( \mathcal{N} \)-Cauchy fuzzy filter is convergent.

It is enough to prove that a non-convergent fuzzy filter is not \( \mathcal{N} \)-Cauchy.

Suppose \( \mathcal{F} \) is a non-convergent fuzzy filter on \( X \). Then for each \( y \in X \) there exists an \( \alpha \)-shading \( U_y \) of \( y \) with \( U_y (y) > \alpha \) and \( U_y \in \mathcal{F} \).

But by theorem 4.3.3, every \( \alpha \)-shading of \( X \) by fuzzy open sets is refined by fuzzy entourages.
That is there exists \( D \in \mathcal{N} \) which refines \( \mathcal{U} = \{ U_x : x \in X \} \).

But this is not possible by our above argument. Therefore \( \mathcal{F} \) is not \( \mathcal{N} \)-Cauchy.

Thus \( \mathcal{N} \) is complete. Hence the theorem.

**4.4.2 Corollary** Each fuzzy \( \alpha \)-paracompact space is fuzzy topologically complete.

**4.4.3 Theorem** Suppose that \( \{ X_i : i \in I \} \) be a family of fuzzy \( \alpha \)-paracompact spaces. Then \( \bigcup_{i \in I} X_i \) is fuzzy topologically complete.

**Proof**

Proof follows from lemma 4.4.1 and theorem 3.4.5.