CHAPTER - 3
FUZZY UNIFORM FUZZY BOX PRODUCTS*

3.1 Introduction

The concept of fuzzy uniformity has been defined by many authors in more or less similar terms. Here we are interested in the fuzzy uniform structure $\mathcal{U}$ in the sense of Lowen [LO].

In the second section of the chapter we give the necessary preliminary ideas like fuzzy filter, fuzzy uniform space, compatible fuzzy uniform base, fuzzy uniform fuzzy topological space etc.

In the third section we introduce the concept fuzzy uniform fuzzy box product.

In the fourth section we investigate the completeness property of fuzzy uniformities in fuzzy box products. Also we introduce the notion of fuzzy topologically complete spaces and prove the main theorem that for a family of fuzzy topologically complete spaces, their fuzzy box product is also fuzzy topologically complete.

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3.2 Preliminaries

3.2.1 Definition [KA] A fuzzy filter on X is a family $\mathcal{U}$ of non empty fuzzy subsets of $X \times X$ which satisfies the following conditions.

i) If $U, V \in \mathcal{U}$ then $U \wedge V \in \mathcal{U}$

ii) If $U \in \mathcal{U}$ and $U \leq V$ then $V \in \mathcal{U}$

3.2.2 Definition [P;A] A family $\mathcal{V}$ of non empty fuzzy subsets of $X \times X$ is called a fuzzy filter base if it satisfies the condition:

if $U_1, U_2 \in \mathcal{V}$ then there exists $U_3 \in \mathcal{V}$ such that $U_3 \leq U_1 \wedge U_2$.

Let $U$ be a fuzzy subset of $X \times X$, $U$ is symmetric if $U = U^{-1}$.

If $X$ is a set, the diagonal of $X \times X$ is denoted by $D(X)$.

That is, $D(X) = \{(x, x) : x \in X\}$.

The following definitions are from [LO]3

3.2.3 Definition A fuzzy uniformity on a set $X$ is a fuzzy filter $\mathcal{U}$ on $X \times X$ which satisfies the following conditions.

(U1) For all $U \in \mathcal{U}$, $D(X) \subseteq U$

(U2) For all $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$ where $U^{-1}(x, y) = U(y, x)$

(U3) For all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \leq U$ where

$V \circ V(x, y) = \sup_{z \in X} \{ V(x, z) \wedge V(z, y) \}$ for all $(x, y) \in X \times X$

The pair $(X, \mathcal{U})$ is called a fuzzy uniform space.
3.2.4 **Note**

(i) (U1) is equivalent to saying that $U(x,x) = 1$ for all $U \in \mathcal{H}$

(ii) If $V \in I^{X \times X}$ and $n \in \mathbb{N}$, we denote by $V^n$ the fuzzy set $V^n = V \circ V^{n-1}$

inductively defined from $V^2 = V \circ V$. Clearly then (U3) is equivalent to saying that for all $n \in \mathbb{N}$ and for all $U \in \mathcal{H}$ there exists $V \in \mathcal{H}$ such that $V^n \leq U$.

3.2.5 **Definition** If $V$ is a fuzzy subset of $X \times X$ and $A$ is a fuzzy subset of $X$, then the section of $V$ over $A$ is the fuzzy subset of $X$, defined by

$$V<A>(x) = \sup_{y \in X} (A(y) \wedge V(y,x))$$

for all $x \in X$.

If $A = \{x\}$ we write $V<x>$ for $V<A>$ and call it the $x$-section of $V$.

3.2.6 **Definition** A fuzzy uniform base is a fuzzy filter base $\mathcal{H}$ on $X \times X$

which satisfies (U1),(U2) and (U3).

3.2.7 **Definition** Let $\mathcal{H}$ be a fuzzy uniform base. Then the fuzzy topology $T_{\mathcal{H}}$ induced by the fuzzy uniform base $\mathcal{H}$ is called fuzzy uniform fuzzy topology, is given by

$$T_{\mathcal{H}} = \{G \in I^X / \text{If } x \in X \text{ is such that } G(x) = 1 \text{ then there exists } U \in \mathcal{H} \text{ such that } U<x> \leq G\}$$

where $U<x>(y) = U(x,y)$ for all $y \in X$. 
3.2.8 Definition If \((X, T)\) is a fuzzy topological space and \(\mathcal{V}\) is a fuzzy uniform base such that \(T_{\mathcal{V}} = T\) then we say that \((X, T)\) fuzzy uniformizable or that the fuzzy uniform base \(\mathcal{V}\) on \(X\) is compatible with \((X, T)\). Then \((X, T_{\mathcal{V}})\) is called a fuzzy uniform fuzzy topological space.

3.2.9 Definition The neighbourhoods of the diagonal in the fuzzy product topology on \(X \times X\) with respect to the fuzzy uniform topology on \(X\) are called fuzzy entourages.

3.3 Fuzzy uniformities in fuzzy box products

3.3.1 Definition. Let \(\mathcal{V}_i\) be a compatible fuzzy uniform base on \(X_i\) for all \(i \in I\). Let \(U_i\) be a fuzzy subset of \(X_i \times X_i\) for all \(i \in I\).

That is, \(U_1 \times U_2 \times U_3 \ldots \leq (X_1 \times X_1) \times (X_2 \times X_2) \times \ldots \)

\[\leq (X_1 \times X_2 \times \ldots) \times (X_1 \times X_2 \times \ldots)\]

\[= \prod X_i \times \prod X_i\]

\[= (\prod X_i)^2\]

Let \(x = (x_i)_{i \in I}\) and \(y = (y_i)_{i \in I}\) be elements of \((\prod X_i)^2\).

Then

\[\Box_{i \in I} U_i (x, y) = \inf_{i \in I} U_i (x_i, y_i)\]

for each \(x = (x_i)_{i \in I}\) and \(y = (y_i)_{i \in I}\) in \((\prod X_i)^2\).
And for each $j \in I$,

$$(\mathbf{f}; u_i)(j) = u_j$$

So $\mathbf{f} \mathbf{U}_i = \sup_{i \in I} \{ \mathbf{f}; u_i \}$

Here $\mathbf{f} \mathbf{U}_i$ is the fuzzy uniform fuzzy box product of $\{ \mathbf{U}_i : i \in I \}$

### 3.3.2 Theorem

Let $\mathbf{U}_i$ be a compatible fuzzy uniform base on $X_i$, for each $i \in I$. Then $\mathbf{U} = \mathbf{f} \mathbf{U}_i$ is a compatible fuzzy uniform base on $X = \mathbf{f} X_i$.

**Proof:**

Given that $\mathbf{U}_i$ is a compatible fuzzy uniform base on $X_i$, for each $i \in I$. This means that $\mathbf{U}_i$ is a fuzzy filter base on $X_i \times X_i$ which satisfies the conditions (U1),(U2) and (U3) of definition 3.2.3 and $T_{\mathbf{U}_i} = T_i$ for all $i \in I$. Now we can verify that $\mathbf{U} = \mathbf{f} \mathbf{U}_i$ is a compatible fuzzy uniform base on $X = \mathbf{f} X_i$.

(U1) Let $U = \mathbf{f} u_i \in \mathbf{U} = \mathbf{f} \mathbf{U}_i$ where $U_i \in \mathbf{U}_i$ for all $i \in I$.

$$U(x,x) = \mathbf{f} u_i(x,x)$$

$$= \inf_{i \in I} U_i(x_i,x_i)$$

$$= 1 \quad \text{for all } U \in \mathbf{U}$$
Let $U = \bigcap_{i \in I} U_i$ and $\mathcal{U} = \bigcap_{i \in I} \mathcal{U}_i$ where $U_i \in \mathcal{U}_i$ for all $i \in I$.

$$U(x,y) = \bigcap_{i \in I} U_i (x, y) \in \mathcal{U} \quad \text{for all } U \in \mathcal{U}.$$ 

$$= \inf_{i \in I} U_i (x_i, y_i)$$

$$= \inf_{i \in I} U_i^{-1} (y_i, x_i)$$

$$= (\inf_{i \in I} U_i (y_i, x_i))^{-1}$$

$$= [\bigcap_{i \in I} U_i (y, x)]^{-1}$$

$$= [U(y, x)]^{-1}$$

$$= U^{-1}(x, y) \in \mathcal{U}.$$ 

That is, for all $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$.

Let $U = \bigcap_{i \in I} U_i \in \mathcal{U}$ and $V = \bigcap_{i \in I} V_i \in \mathcal{U}$ where $U_i, V_i \in \mathcal{U}_i$ for all $i \in I$.

Consider

$$(V \circ V) (x, y) = \sup_{z \in X} \{V(x, z) \land V(z, y)\}$$

$$= \sup_{z \in X} \{\bigcap_{i \in I} V_i(x, z) \land \bigcap_{i \in I} V_i(z, y)\}$$

$$= \sup_{z \in X} \{\inf_{i \in I} V_i(x_i, z_i) \land \inf_{i \in I} V_i(z_i, y_i)\}$$

$$\leq \inf_{i \in I} \{\sup_{i \in I} (V_i(x_i, z_i) \land V_i(z_i, y_i))\}$$
\[ \inf_{i \in I} U_i (x_i, y_i) \leq U(x, y) \leq \bigwedge_i U_i (x, y) = U(x, y) \]

That is, \( \forall \forall U \leq U \) for all \( U \in \mathcal{U} \)

Therefore \( U = \bigwedge_i U_i \) satisfies the conditions (U1),(U2) and (U3).

Hence \( \mathcal{U} = \bigwedge_i \mathcal{U}_i \) is a fuzzy uniform base on \( X = \bigwedge_i X_i \)

Next we prove that \( \mathcal{U} \) is compatible.

That is, to prove that \( T_{\mathcal{U}} = T \).

Since \( \mathcal{U}_i \) is compatible, we have \( T_{\mathcal{U}_i} = T_i \) for all \( i \in I \).

where \( T_{\mathcal{U}_i} = \{ G_i \in I^{X_i} / \text{If } x_i \in X_i \text{ is such that } G_i(x_i) = 1 \text{ then} \}

there exists \( U_i \in \mathcal{U}_i \) s.t. \( U_i \leq G_i \} \) for all \( i \in I \).

We have \( X = \bigwedge_i X_i \) & \( U = \bigwedge_i U_i \)

Now \( U(x,y) = U(x,y) \) for all \( y \in X \)

That is, \( U(x,y) = \bigwedge_i U_i (x,y) \)

\[ = \inf_{i \in I} U_i (x_i, y_i) \leq G_i (y_i) \text{ for all } i \in I. \]

\[ \leq G (y) \]

Therefore \( U(x,y) \leq G \) and

\[ T_{\mathcal{U}} = \{ G \in I^X / \text{If } x \in X \text{ is such that } G(x) = 1 \text{ then there exists } U \in \mathcal{U} \}

such that \( U(x,y) \leq G \)
Thus \( T_{\mathcal{U}} = T \) holds.

Therefore \( \mathcal{U} = \bigcap_{i} \mathcal{U}_i \) is compatible and it is a fuzzy uniform base on \( X = \bigcap_{i} X_i \).

Hence the theorem.

### 3.4 Fuzzy Topologically Complete Spaces

The following concepts are available in literature.

**3.4.1 Definition** Let \( \mathcal{U} \) be a compatible fuzzy uniform base on \((X,T)\). A fuzzy filter \( \mathcal{F} \) in a fuzzy uniform space \((X, \mathcal{U})\) is said to be Cauchy if \( U \in \mathcal{U} \Rightarrow \text{there exists } x \in X \text{ with } U(x) \in \mathcal{F} \).

**3.4.2 Definition** A fuzzy filter is convergent if it contains a fuzzy neighbourhood base at some point.

**3.4.3 Definition** A fuzzy uniform space \((X, \mathcal{U})\) is said to be complete if every Cauchy fuzzy filter converges.

**3.4.4 Definition** A fuzzy topological space \((X,T)\) is said to be fuzzy topologically complete if there exists a fuzzy uniformity \( \mathcal{U} \) for \( X \) such that \((X, \mathcal{U})\) is complete and \( T_{\mathcal{U}} = T \).

**3.4.5 Theorem** If \( X_i \) is fuzzy topologically complete for each \( i \in I \), then \( \bigcap_{i} X_i \) is fuzzy topologically complete.
Proof:

Since $X_i$ is fuzzy topologically complete, it possess a compatible complete fuzzy uniform base $U_i$, for each $i \in I$. But we proved (in theorem 3.3.2) that $U = \bigoplus_i U_i$ is a compatible fuzzy uniform base on $X = \bigoplus_i X_i$. So it is enough to prove that $U = \bigoplus_i U_i$ is complete.

Suppose $\mathcal{F}$ is a $\bigoplus_i U_i$ Cauchy fuzzy filter on $\bigoplus_i X_i$.

Define

$$\mathcal{F}_i = \{ F \subseteq X_i : \Pi_i^{-1}(F) \in \mathcal{F} \}$$

That is, $\Pi_i^{-1}(F)(x) = F(\Pi_i(x)) = F(x_i) \in \mathcal{F}_i$, for all $i \in I$.

Now $\mathcal{F}_i$ is a $U_i$ - Cauchy fuzzy filter on $X_i$ for each $i \in I$.

For,

$$U_i \in U_i \Rightarrow \text{there exists } x_i \in X_i \text{ with } U_i \cdot x_i i \in \mathcal{F}_i$$

where $U_i \cdot x_i (y_i) = U_i(x_i, y_i)$ for each $i \in I$.

Since $U_i$ is complete, every Cauchy fuzzy filter converges.

Let $x \in \prod_i X_i$

Assume that $\mathcal{F}_i$ converges to $x_i$ for all $i \in I$. By definition, for $U = \bigoplus_i U_i \in \bigoplus_i U_i$ there is a symmetric $V = \bigoplus_i V_i \in \bigoplus_i U_i$ such that $V \cdot V \cdot V \leq U$
That is, \((\text{VoVoV})(p, q) \leq U(p, q)\)

where \((\text{VoVoV})(p, q) = \sup_{r,s \in V} \{ V(p, r) \land V(r, s) \land V(s, q) \}\)

Since \(\mathcal{F}\) is a \(\bigcap_{i} \mathcal{U}_i\) Cauchy fuzzy filter there exists \(y \in \Pi_i X_i\) such that \(V<y> \in \mathcal{F}\), where \(V<y>(x) = V(y, x)\) for all \(x \in X\)

Therefore \(V(y, x) = \bigcap_{i} V_i(y, x)\)

\[= \inf_{\in I} V_i(y_i, x_i)\]

where \(x = (x_i)i \in I\) & \(y = (y_i)i \in I\) in \((\Pi_i X_i)^2\)

\(\in \mathcal{F}\).

Also, for \(U = \bigcap_{i} U_i \in \bigcap_{i} \mathcal{U}_i\), there exists \(x \in \Pi_i X_i\) such that \(U<x> \in \mathcal{F}\), where \(U(x, y) = \inf_{\in I} U_i(x_i, y_i)\)

Thus we get, \(V<y> \leq U<x>\)

Therefore \(\mathcal{F}\) converges to \(x\).

Thus \(\bigcap_{i} \mathcal{U}_i\) is complete.

Hence the theorem.