CHAPTER-4

Fixed Points in Symmetric Spaces
Introduction

In 1976, Cicchese [29] introduced the notion of contraction map in semi-metric spaces, coined by Menger [72], and obtained the first fixed point theorem for this class of spaces. Later, Hicks and Rhoades [48] obtained a common fixed point theorem of commuting and continuous maps in symmetric spaces. Afterwards, Aamri and Moutawakil [2] proved two common fixed point theorems of weakly compatible maps by assuming property (E.A) as well as a contractive condition in symmetric spaces. On the other hand, Branciari [20] introduced the notion of contractive condition of integral type and obtained a fixed point theorem of this class of maps. Further, a few papers dealt with fixed points of this class of maps were obtained (for instance, [7], [100], [118]). Meanwhile, Zhang [121] introduced a generalized contractive condition in which the integral operator is replaced by a monotone non decreasing function, and proved common fixed point theorems that extended results of [20], [100], [118]. In 2008, Cho et al. [27] gave some results on coincidence and fixed point theorems in symmetric space. Recently, Arandelović and Petković [10] extended the results of Aamri and Moutawakil [2] to maps which satisfied a more general contractive condition and further showed that many fixed point theorems using contractive conditions of integral type could be obtained as corollaries.

In 2008, Gopal et al. [45] made an attempt to generalize the Pant’s result [82] by introducing a new notion of absorbing pair of maps.

This chapter consists of four sections. The first is the introductory in which relevant definitions and results needed in the following sections are furnished.

In the second, we extend the results of Cho et al. [27, Theorems 3.2 and Theorem 3.5] and Arandelović and Petković [10, Theorem 6] by employing Bari and Vetro’s generalized contractive condition in symmetric spaces. The result of
this section has been presented in International Congress of Mathematicians (ICM), 2010, Hyderabad, INDIA.

In the third section, we extend the results of Gopal et al. [44, Theorems 3.1 and 3.6] to symmetric spaces.

Let $X$ be a nonempty set. A symmetric on $X$ is a map $d : X \times X \to [0, \infty)$ such that

(i) $d(x, y) = 0$ if and only if $x = y$, and

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

The pair $(X, d)$ is a symmetric space.

A metric on a set is an example of a symmetric but a symmetric is not necessarily a metric. A simple illustration of this fact is provided in the following.

Example 4.1.1. Let $X = \mathbb{R}$. Define $d : X \times X \to [0, \infty)$ by $d(x, y) = (x - y)^2$, for all $x, y \in X$. Then $d$ is a symmetric on $X$, but the triangle inequality fails to hold and therefore it is not a metric on $X$.

Let $d$ be a symmetric on a set $X$. For $x \in X$ and $r > 0$, we write $B(x, r) = \{ y \in X : d(x, y) < r \}$. A topology $\tau(d)$ on $X$ is defined as $U \in \tau(d)$ if and only if for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$. A subset $S$ of $X$ is a neighbourhood of $x \in X$ if and only if there exists $U \in \tau(d)$ such that $x \in U \subseteq S$.

The distinction between a symmetric and semi-metric is apparent as one can easily construct a symmetric $d$ such that $B(x, r)$ need not be a neighbourhood of $x$ in the topology $\tau(d)$ (refer to [52]). A symmetric $d$ is said to be a semi-metric if for each $x \in X$ and for each $r > 0$, $B(x, r)$ is a neighbourhood of $x$ in the topology $\tau(d)$.
A symmetric (semi-metric) space \((X,d)\) is a topological space whose topology \(\tau(d)\) on \(X\) is induced by symmetric (semi-metric) \(d\). For every sequence \(\{x_n\} \subset X\) and \(x \in X\), \(\lim_{n \to \infty} d(x_n, x) = 0\) if and only if \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\) in the topology \(\tau(d)\).

As symmetric spaces are not essentially Hausdorff in general, in order to obtain fixed point theorems in symmetric spaces, some additional axioms / properties are required.

The following two axioms were appeared in Wilson [120].

Let \((X,d)\) be a symmetric space.

(W.3) for any sequence \(\{x_n\}\) in \(X\) and \(x,y \in X\), \(\lim_{n \to \infty} d(x_n,x) = 0\) and \(\lim_{n \to \infty} d(x_n,y) = 0\) imply \(x = y\);

(W.4) for any sequences \(\{x_n\}, \{y_n\}\) in \(X\) and \(x \in X\), \(\lim_{n \to \infty} d(x_n,x) = 0\) and \(\lim_{n \to \infty} d(y_n,x) = 0\) imply \(\lim_{n \to \infty} d(y_n,x) = 0\).

In the case of semi-metric \(d\), if \(\tau(d)\) is Hausdorff, then (W.3) holds.

Galvin and Shore [42] introduced 1-continuous (in short, 1C).

A symmetric \(d\) on a set \(X\) is said to be 1-continuous if for any sequence \(\{x_n\}\) in \(X\) and for any \(x,y \in X\), \(\lim_{n \to \infty} d(x_n,x) = 0\) implies \(\lim_{n \to \infty} d(x_n,y) = d(x,y)\).


A symmetric space \((X,d)\) satisfies the property (H.E) if for any sequences \(\{x_n\}, \{y_n\}\) in \(X\) and \(x \in X\), \(\lim_{n \to \infty} d(x_n,x) = 0\) and \(\lim_{n \to \infty} d(y_n,x) = 0\) imply \(\lim_{n \to \infty} d(x_n,y_n) = 0\).

Pathak et al. [91] introduced the property (CE.2).
A symmetric space \((X,d)\) satisfies the property (CE.2) if for any sequences \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) in \(X\),
\[\lim_{n \to \infty} d(x_n, y_n) = 0\] implies
\[\limsup_{n \to \infty} d(z_n, x_n) = \limsup_{n \to \infty} d(z_n, y_n).
\]

It is noted that (1C) is called (CE.1) by Pathak et al. [91], and (C.C) by Cho et al. [27]. Also, (W.4) \(\Rightarrow\) (W.3) and (1C) \(\Rightarrow\) (W.3), but converse implications are not true in general, and further all other possible implications amongst (W.3), (W.4), (1C) and (H.E) are not generally true (refer to [27]). All these conditions can be used as a partial replacement of the triangle inequality.

A semi-metric space becomes metric when the so-called triangle axiom / inequality is added (i.e. when the triangle inequality holds). It is noted that (W.3), (W.4), (H.E) and (1C) are automatically satisfied if \(d\) is a metric.

We retrieve relevant definitions and results needed in the sequel.

**Definition 4.1.2** ([2]). Self-maps \(A\) and \(B\) of a symmetric space \((X,d)\) are said to be weakly compatible if they commute at their coincidence points, i.e. \(ABx = BAx\) whenever \(Ax = Bx\) for some \(x \in X\).

**Definition 4.1.3** ([2], [52]). Self-maps \(A\) and \(B\) of a symmetric space \((X,d)\) are said to satisfy property (E.A) if there exists a sequence \(\{x_n\}\) in \(X\) such that
\[\lim_{n \to \infty} d(Ax_n, t) = \lim_{n \to \infty} d(Bx_n, t) = 0\] for some \(t \in X\).

**Definition 4.1.4** ([71]). Two pairs of self-maps \(\{A,B\}\) and \(\{S,T\}\) defined on a symmetric space \((X,d)\) are said to satisfy the common property (E.A) if there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) such that
\[\lim_{n \to \infty} d(Ax_n, t) = \lim_{n \to \infty} d(Bx_n, t) = \lim_{n \to \infty} d(Sy_n, t) = \lim_{n \to \infty} d(Ty_n, t) = 0.\]
**Definition 4.1.5** ([27]). A subset $S$ of a symmetric space $(X,d)$ is said to be $d$-closed if for a sequence $\{x_n\}$ in $S$ and a point $x \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ implies $x \in S$.

For a symmetric space $(X,d)$, $d$-closeness implies $\tau(d)$-closeness, and if $d$ is a semi-metric, the converse is also true.

### 4.2. Fixed points under Bari and Vetro’s generalized contractive condition in symmetric spaces

In 2006, Imdad et al. [52] extended the results of Aamri and Moutawakil [1, Theorem 1], and Pant and Pant [83, Theorems 2.1 and 2.3] to symmetric spaces under tight conditions. In the same year, Aliouche [7] extended the main result of Aamri and Moutawakil [2, Theorem 2.2] under a contractive condition of integral type and further, extended the result of Aamri and Moutawakil [1, Theorem 2] to symmetric space. In 2008, Cho et al. [27] extended Pant and Pant’s result [83, Theorems 2.1 and 2.3] to symmetric space that satisfy additional conditions (H.E) and (1C) and further, extended the results of Imdad et al. [52, Theorems 2.1, 2.2 and 2.3] to four self-maps. Arandelović and Petković [10] extended the maps in the main result of Aamri and Moutawakil [2, Theorem 2.2] and Aliouche [7, Theorem 1] to those satisfying a versatile contractive condition which can deduce a contractive condition of integral type.

In 2008, Bari and Vetro [15] introduced a generalized contractive condition which is versatile of deducing a contractive condition of integral type in metric space.

Now, we employ this new Bari and Vetro’s generalized contractive condition in symmetric space and extend the results of Cho et al. [27, Theorems 3.2 and Theorem 3.5] and Arandelović and Petković [10, Theorems 5 and 6] to those satisfying such contractive condition. Moreover, we improve the same by assuming...
a more general condition, such as closeness of one of the four range spaces instead of one of the two (four) range spaces to be closed (complete) respectively.

Following Bari and Vetro [15], let $F$ be a family of all functions $F : [0, \infty) \to [0, \infty)$ such that $F$ is non decreasing, continuous and $F(0) = 0 < F(t)$ for every $t > 0$.

By $\Psi$, we denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi$ is non decreasing, right continuous and $\psi(t) < t$ for every $t > 0$.

If $A, B, S$ and $T$ are self-maps of a symmetric space $(X, d)$, in the sequel, we set

$$M(x, y) = \max \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Sx, By) + d(Ax, Ty)}{2}\} \quad \ldots (4.2.1)$$

**Theorem 4.2.1.** Let $(X, d)$ be a symmetric space that satisfies properties (W.4), (H.E), (1C) and (CE.2). Let $A, B, S$ and $T$ be self-maps of $X$ such that

(i) $AX \subset TX$ and $BX \subset SX$;

(ii) for any $x, y \in X$,

$$F(d(Ax, By)) \leq \psi(F(M(x, y))),$$

where $\psi \in \Psi$ and $F \in F$;

(iii) one of the pairs $\{A, S\}$ and $\{B, T\}$ satisfies the property (E.A.).

If one range of the maps $A, B, S$ and $T$ is a $d$-closed subspace of $X$, then each pair (viz. $\{A, S\}$ and $\{B, T\}$) has point of coincidence in $X$. Further, if each pair is weakly compatible, then $A, B, S$ and $T$ have a unique common fixed point.

**Proof.** Suppose that the pair $\{B, T\}$ satisfies property (E.A.). Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} d(Bx_n, t) = 0$ and $\lim_{n \to \infty} d(Tx_n, t) = 0$ for some
Since $BX \subset SX$, there exists a sequence $\{y_n\}$ in $X$ such that $Bx_n = Sy_n$ for all $n \in \mathbb{N}$ and hence $\lim_{n \to \infty} d(Sy_n, t) = 0$. By property (H.E), we obtain

$$\lim_{n \to \infty} d(Bx_n, Tx_n) = \lim_{n \to \infty} d(Sy_n, Tx_n) = 0 \quad \ldots \quad (4.2.2)$$

By (4.2.2) and property (CE.2), we have

$$\limsup_{n \to \infty} d(Ay_n, Tx_n) = \limsup_{n \to \infty} d(Ay_n, Bx_n).$$

Now, we claim that $\lim_{n \to \infty} d(Ay_n, t) = 0$. It is enough to show that $\limsup_{n \to \infty} d(Ay_n, Bx_n) = 0$. For this, suppose that $\limsup_{n \to \infty} d(Ay_n, Bx_n) = \varepsilon > 0$. Using (4.2.1), we obtain

$$M(y_n, x_n) = \max \{d(Sy_n, Tx_n), d(Ay_n, Sy_n), d(Bx_n, Tx_n),$$

$$d(Sy_n, Bx_n) + d(Ay_n, Tx_n)\}/2$$

and subsequently, from (iii) it follows that

$$F(d(Ay_n, Bx_n)) \leq \psi(F(M(y_n, x_n))).$$

Taking lim sup and as $F$ being continuous and $\psi$ right continuous, we obtain

$$F(\varepsilon) \leq \psi(F(\varepsilon))$$

$$< F(\varepsilon), \quad \text{a contradiction.}$$

Therefore, $\limsup_{n \to \infty} d(Ay_n, Bx_n) = 0$ which implies that $\lim_{n \to \infty} d(Ay_n, Bx_n) = 0$. By (W.4), it follows that $\lim_{n \to \infty} d(Ay_n, t) = 0$.

We now consider the $d$-closeness of range space in four cases.

Case I. Suppose that $SX$ is a $d$-closed subspace of $X$. Then there exists a point $u \in X$ such that $t = Su$. Consequently, we obtain $\lim_{n \to \infty} d(Ay_n, Su) = \lim_{n \to \infty} d(Bx_n, Su) = \lim_{n \to \infty} d(Sy_n, Su) = \lim_{n \to \infty} d(Tx_n, Su) = 0$. To show that $Au = Su$, suppose that $Au \neq Su$. Using (4.2.1), we have
\[ M(u, x_n) = \max \{ d(Su, Tx_n), d(Au, Su), d(Bx_n, Tx_n), \]
\[ \left[ d(Su, Bx_n) + d(Au, Tx_n) \right]/2 \]

and therefore, from (iii) it follows that

\[ F(d(Au, Bx_n)) \leq \psi(F(M(u, x_n))). \]

As \( n \to \infty \), \( F \) being continuous, \( \psi \) right continuous and using (1C), we have

\[ F(d(Au, Su)) \leq \psi(F(d(Au, Su))) \]
\[ < F(d(Au, Su)). \]

This leads to contradiction and therefore, \( Au = Su \).

Since \( AX \subset TX \), there exists a point \( v \in X \) such that \( Au = Tv \). So, \( Au = Su = Tv = t \). We assert that \( Bv = t \). For this, suppose that \( Bv \neq t \). Using (4.2.1), we have

\[ M(u, v) = \max \{ d(Su, Tv), d(Au, Su), d(Bv, Tv), \left[ d(Su, Bv) + d(Au, Tv) \right]/2 \}
\]
\[ = \max \{ d(t, t), d(t, t), d(Bv, t), \left[ d(t, Bv) + d(t, t) \right]/2 \}
\]
\[ = d(t, Bv) \]

and hence from (iii), it follows that

\[ F(d(t, Bv)) = F(d(Au, Bv)) \]
\[ \leq \psi(F(M(u, v))) \]
\[ = \psi(F(d(t, Bv))) \]
\[ < F(d(t, Bv)). \]

This contradiction leads to \( Bv = t \). Subsequently, we obtain \( Au = Su = t = Bv = Tv \).

Case II. The proof is similar to that of Case I when \( TX \) is assumed to be a \( d \)-closed subspace of \( X \).

Case III. If \( BX \) is a \( d \)-closed subspace of \( X \), then there exists a point \( v \in X \) such that \( Bv = t \). Since \( t \in BX \subset SX \), there exists a point \( u \in X \) such that \( t = Su \). By the same argument as in Case I, it follows that \( Au = Su = t = Bv = Tv \).
Case IV. The proof is similar to that of Case III when \(AX\) is assumed to be a \(d\)-closed subspace of \(X\).

Thus, the same result holds in all cases.

The weak compatibility of \(A\) and \(S\) implies that \(ASu = SAu\), whenever \(Au = Su\). So, \(At = St\). We claim that \(d(At, t) = 0\). If \(d(At, t) \neq 0\), then the condition (4.2.1) gives

\[
M(t,v) = \max \{ d(StTv), d(At,St), d(BvTv), [d(St,Bv) + d(At,Tv)]/2 \}
\]

\[
= \max \{ d(At,t), d(At,t), d(t,t), [d(St,t) + d(At,t)]/2 \}
\]

\[
= d(At,t).
\]

Therefore from (iii), it follows that

\[
F(d(At,t)) = F(d(At,Bv))
\]

\[
\leq \psi(F(M(t,v)))
\]

\[
= \psi(F(d(At,t)))
\]

\[
< F(d(At,t)), \text{ a contradiction.}
\]

Therefore \(d(At,t) = 0\) which implies \(At = St = t\). Similarly, one can prove that \(Bt = Tt = t\). Thus, \(t\) is a common fixed point of \(A, B, S\) and \(T\).

If \(z \in X\) is also a common fixed point of \(A, B, S\) and \(T\) with \(t \neq z\), then using (4.2.1), we have

\[
M(t,z) = \max \{ d(StTz), d(At,St), d(Bz,Tz), [d(St,Bz) + d(At,Tz)]/2 \}
\]

\[
= \max \{ d(t,z), d(t,t), d(z,z), [d(t,z) + d(t,z)]/2 \}
\]

\[
= d(t,z).
\]

Therefore from (iii), it follows that

\[
F(d(t,z)) = F(d(At,Bz))
\]

\[
\leq \psi(F(M(t,z)))
\]

\[
= \psi(F(d(t,z)))
\]

\[
< F(d(t,z))
\]
which is a contradiction and hence $t = z$. This completes the proof.

If we assume that one range of the maps is complete instead of being closed in Theorem 4.2.1, we have the following corollary.

**Corollary 4.2.2.** Let $(X, d)$ be a symmetric space that satisfies properties (W.4), (H.E), (1C) and (CE.2). Let $A$, $B$, $S$ and $T$ be self-maps of $X$ satisfying (i)-(iii) as in Theorem 4.2.1.

If one range of the maps $A$, $B$, $S$ and $T$ is a complete subspace of $X$, then each pair (viz. $\{A, S\}$ and $\{B, T\}$) has point of coincidence in $X$. Further, if each pair is weakly compatible, then $A$, $B$, $S$ and $T$ have a unique common fixed point.

The proof of the Corollary 4.2.2 parallels to that of Theorem 4.2.1.

**Remark 4.2.3.** Contractive condition (ii) of Corollary 4.2.2 is more general than those of the main theorems of [10] and [2]. So, Corollary 4.2.2 is the improved version of those theorems.

Here, we demonstrate how the generalized contractive condition is being versatile of deducing contractive condition of integral type.

By $\Lambda$, we denote the set of all non-negative, Lebesgue-integrable function $\lambda : [0, \infty) \to [0, \infty)$ such that

$$\int_0^\varepsilon \lambda(t) \, dt > 0 \text{, for every } \varepsilon > 0.$$

Let $\lambda \in \Lambda$. If in Theorem 4.2.1, $F$ is defined by $F(s) = \int_0^s \lambda(t) \, dt$, for any $s \geq 0$, then the following theorem is deduced.

**Theorem 4.2.4.** Let $(X, d)$ be a symmetric space that satisfies properties (W.4), (H.E), (1C) and (CE.2). Let $\psi \in \Psi$ and $\lambda \in \Lambda$, and let $A$, $B$, $S$ and $T$ be self-maps of $X$ such that

(i) $AX \subseteq TX$ and $BX \subseteq SX$;
(ii) \( \int_0^d(x,y) \lambda(t) dt \leq \psi\left( \int_0^{M(x,y)} \lambda(t) dt \right) \) for any \( x, y \in X \);

(iii) one of the pairs \( \{A,S\} \) and \( \{B,T\} \) satisfies the property (E.A.).

If one range of the maps \( A, B, S \) and \( T \) is a \( d \)-closed subspace of \( X \), then each pair (viz. \( \{A,S\} \) and \( \{B,T\} \)) has point of coincidence in \( X \). Further, if each pair is weakly compatible, then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** One can follow on the lines of the proof of Theorem 4.2.1.

Corollary similar to Corollary 4.2.2 can be outlined in the context of Theorem 4.2.4.

**Corollary 4.2.5.** Let \( (X,d) \) be a symmetric space that satisfies properties (W.4), (H.E), (1C) and (CE.2). Let \( \psi \in \Psi \) and \( \lambda \in \Lambda \), and let \( A, B, S \) and \( T \) be self-maps satisfying (i)-(iii) as in Theorem 4.2.4.

If one range of the maps \( A, B, S \) and \( T \) is a complete subspace of \( X \), then each pair (viz. \( \{A,S\} \) and \( \{B,T\} \)) has point of coincidence in \( X \). Further, each pair is weakly compatible, then \( A, B, S \) and \( T \) have a unique common fixed point.

**Remark 4.2.6.** Corollary 4.2.5 is an improved version of Aliouche [7, Theorem 1].

We obtain the following variant of Theorem 4.2.1 by employing common property (E.A) instead of property (E.A).

**Theorem 4.2.7.** Let \( (X,d) \) be a symmetric space that satisfies properties (H.E) and (1C). Let \( A, B, S \) and \( T \) be self-maps of \( X \) satisfying (i) and (ii) as in Theorem 4.2.1. Suppose that the pairs \( \{A,S\} \) and \( \{B,T\} \) satisfy the common property (E.A).

If one range of the maps \( A, B, S \) and \( T \) is a \( d \)-closed subspace of \( X \), then each pair (viz. \( \{A,S\} \) and \( \{B,T\} \)) has point of coincidence in \( X \). Further, if each pair is weakly compatible, then \( A, B, S \) and \( T \) have a unique common fixed point.
Proof. Since the pairs \( \{A,S\} \) and \( \{B,T\} \) satisfy the common property (E.A.), there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that
\[
\lim_{n \to \infty} d(Ay_n, t) = \lim_{n \to \infty} d(Bx_n, t) = \lim_{n \to \infty} d(Sy_n, t) = \lim_{n \to \infty} d(Tx_n, t) = 0 \quad \text{for some } t \in X.
\]
By property (H.E), we obtain
\[
\lim_{n \to \infty} d(Ay_n, Bx_n) = \lim_{n \to \infty} d(Ay_n, Sy_n) = \lim_{n \to \infty} d(Ay_n, Tx_n) = \lim_{n \to \infty} d(Bx_n, Sy_n) = \lim_{n \to \infty} d(Bx_n, Tx_n) = 0.
\]

We now consider the \( d \)-closeness of range space in four cases.

Case I. Suppose that \( SX \) is a \( d \)-closed subspace of \( X \). Then there exists a point \( u \in X \) such that \( t = Su \). The remaining part follows on the lines of the proof of Theorem 4.2.1.

Remark 4.2.8. In the context of Theorem 4.2.7, the variant corollaries and theorem of Corollary 4.2.2, Theorem 4.2.4 and Corollary 4.2.5 remain true.

Remark 4.2.9. In the setting of Theorems 4.2.1 and 4.2.7, \( A, B, S \) and \( T \) have a unique common fixed point even if condition (ii) is replaced by
for any \( x, y \in X \),
\[
F(d(Ax, By)) \leq \psi(F(m(x, y)))
\]
where \( \psi \in \Psi \), \( F \in F \) and
\[
m(x, y) = \max \{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty) \}.
\]

4.3. Fixed points of absorbing maps

A pair of weakly compatible maps having at least one coincidence point often provides a common fixed point in presence of suitable contractive conditions. But some situations may arise in such a manner that a pair of maps does not commute at their coincidence points (refer to [45, Example 1.1]). In such situations, the notion of absorbing pair of maps can be employed to obtain the existence of common fixed point theorems.
Definition 4.3.1 ([45]). A pair of self-maps \( \{f, g\} \) defined on a symmetric space \((X, d)\) is called \(g\)-absorbing if there exists some real number \(R > 0\) such that \(d(gx, gfx) \leq R d(fx, gx)\) for all \(x \in X\).

Analogously, the pair \( \{f, g\} \) is called \(f\)-absorbing if there exists some real number \(R > 0\) such that \(d(fx, gfx) \leq R d(fx, gx)\) for all \(x \in X\).

The pair \( \{f, g\} \) is called absorbing if it is both \(g\)-absorbing and \(f\)-absorbing. In particular, if we take \(g = I_X\), then \(f\) is trivially \(I_X\)-absorbing. Similarly, \(I_X\) is also \(f\)-absorbing in respect of \(f\).

Definition 4.3.2 ([45]). A pair of self-maps \( \{f, g\} \) defined on a symmetric space \((X, d)\) is called pointwise \(g\)-absorbing if for a given \(x \in X\), there exists some real number \(R > 0\) such that \(d(gx, gfx) \leq R d(fx, gx)\).

Analogously, the pair of self-maps \( \{f, g\} \) is called pointwise \(f\)-absorbing if for a given \(x \in X\), there exists some real number \(R > 0\) such that \(d(fx, gfx) \leq R d(fx, gx)\).

It has been shown in [45] that a pair \( \{f, g\} \) of compatible or \(R\)-weakly commuting maps need not be \(g\)-absorbing or \(f\)-absorbing. Also, the class of absorbing pairs of maps is neither a subclass of compatible pairs nor subclass of noncompatible pairs of maps as the absorbing pairs of maps need not commute at their coincidence points.

We furnish an example to show that absorbing pair of maps need not commute at their coincidence points.

Example 4.3.3. Let \(X = [0,1]\) equipped with the symmetric defined by \(d(x, y) = (x - y)^2\) for all \(x, y \in X\). Define self-maps \(f\) and \(g\) on \(X\) by

\[f(x) = \begin{cases} 
0 & \text{if } x \leq 1/2 \\
1 & \text{if } x > 1/2 
\end{cases}
\]

\[g(x) = \begin{cases} 
1 & \text{if } x \leq 1/2 \\
0 & \text{if } x > 1/2 
\end{cases}
\]
We observe that $f$ and $g$ are coincident at $x \neq 1$.

If we take $x \neq 1$, $d(gx, gfx) \leq Rd(fx, gx)$ becomes $d(1, g(1)) \leq Rd(1,1)$, i.e. $d(1,1) \leq Rd(1,1)$ which is true for any $R>1$. The pair $\{f, g\}$ is $g$-absorbing for any $R>1$ as $d(gx, gfx) \leq Rd(fx, gx)$ holds if $x \neq 1$.

At the coincidence point $x=0$, we see that $d(fg(0), gf(0)) = d(f(1), g(1)) = d(0,1) = 1$. But, $fg(0) \neq gf(0)$. Hence, the absorbing pair $\{f, g\}$ of maps need not commute at their coincidence point $x=0$.

**Definition 4.3.4 ([104]).** Let $f$ and $g$ be two self-maps defined on a symmetric space $(X, d)$. Then $f$ is said to be $g$-continuous if $gx_n \to gx \Rightarrow fx_n \to fx$ whenever $\{x_n\}$ is a sequence in $X$ and $x \in X$.

In 1999, Pant [82] proved a common fixed point theorem for Lipschitz type maps in metric space which is considered as first of its kind. It was extended in several ways by various researchers. Sastry and Murthy [104] extended Pant’s result [82] by employing $g$-continuous instead of utilizing some Lipschitz or contractive type conditions. In the other way, Pant [84] extended Pant’s result [82] to a more general inequality. Gopal et al. [44] presented another extension of Pant’s result [84] and a variant of the result of Sastry and Murthy [104] by employing absorbing maps in place of $R$-weakly commuting and weakly compatible maps respectively.

As an extension of the results of Gopal et al. [44, Theorems 3.1 and 3.6] to symmetric spaces, we prove the following theorems.

Before stating our theorems, let us prove the following proposition.
**Proposition 4.3.5.** Let $f$ and $g$ be two self-maps of a symmetric space $(X,d)$ satisfying the following Lipschitz type condition

$$d(fx, fy) \leq k d(gx, gy) + a \max\{d(fx, gx), d(fy, gy), d(gx, fy), d(gx, fy)\}$$

for all $x, y \in X$ where $k > 0$ and $a < 1$.

Then the pair $\{f, g\}$ is pointwise absorbing iff the pair is pointwise $g$-absorbing.

**Proof.** To prove if part, suppose that the pair $\{f, g\}$ is pointwise $g$-absorbing. Two cases arise:

Case I. If $fx \neq gx$ for some $x \in X$, then one can define $R = d(fx, f gx) / d(fx, gx)$ so that $d(fx, f gx) \leq R d(fx, gx)$. So, the pair $\{f, g\}$ is pointwise $f$-absorbing and hence the pair $\{f, g\}$ is pointwise absorbing.

Case II. If $fx = gx$ for some $x \in X$, then one can obtain $gx = g f x$ which in turn yields $fx = gx = g f x = g g x$. Now applying the given Lipschitz condition, we obtain

$$d(fx, f gx) \leq k d(gx, g gx) + a \max\{d(fx, gx), d(fgx, gg x), d(gx, f gx), d(fx, gg x)\}$$

$$= k \cdot 0 + a \max\{0, d(fgx, fx), d(fx, f gx), 0\}$$

$$= a \cdot d(fx, f gx)$$

$$< d(fx, f gx),$$

a contradiction.

So, $fx = fgx$ and therefore, the pair $\{f, g\}$ is pointwise $f$-absorbing. Hence the pair $\{f, g\}$ is pointwise absorbing.

The proof of the converse part is obvious by definition.

**Theorem 4.3.6.** Let $(X,d)$ be a symmetric space satisfying properties (H.E) and (1C). Let $f$ and $g$ be two self-maps of $X$ such that

(i) the pair $\{f, g\}$ satisfies the property (E.A);

(ii) for all $x, y \in X$, 


\[ d(fx, fy) \leq k d(gx, gy) + a \max \{d(fx, gx), d(fy, gy), d(gx, fy), d(fx, gy)\} \]

where \( k > 0 \) and \( a < 1 \);

(iii) \( gX \) is a closed subset of \( X \).

Then the pair \( \{f, g\} \) has a common fixed point provided the pair is pointwise \( g \)-absorbing.

**Proof.** In view of (i), there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} d(f x_n, t) = 0 \) and \( \lim_{n \to \infty} d(g x_n, t) = 0 \) for some \( t \in X \). By property (H.E), we get

\[ \lim_{n \to \infty} d(f x_n, g x_n) = 0 . \]

By (iii), there exists a point \( u \in X \) such that \( t = gu \). Using (ii), we obtain

\[ d(fu, fx_n) \leq k d(gu, gx_n) + a \max \{d(fu, gu), d(fx_n, gx_n), d(gu, fx_n), d(fu, gx_n)\} \]

\[ = k d(t, gx_n) + a \max \{d(fu, t), d(fx_n, gx_n), d(t, fx_n), d(fu, gx_n)\} . \]

Taking limits as \( n \to \infty \) and applying (1C), we obtain

\[ d(fu, t) \leq k \cdot 0 + a \max \{d(fu, t), 0, 0, d(fu, t)\} \]

\[ = a d(fu, t) . \]

This leads to contradiction and so, \( fu = t \) which yields that \( fu = gu = t \).

Employing pointwise \( g \)-absorbing property of the pair \( \{f, g\} \), one can obtain \( gu = gf u \) which in turn yields \( gt = t \). In view of Proposition 4.3.5, the pair \( \{f, g\} \) is pointwise absorbing and so, \( fu = fg u \) which in turn yields \( fl = t \). Thus, \( fl = gt = t \) which means that \( t \) is a common fixed point of \( f \) and \( g \).

The following theorem is a counterpart of Theorem 4.3.6.

**Theorem 4.3.7.** Theorem 4.3.6 remains true if the condition (iii) is replaced by \( fX \subseteq gX \), where \( fX \) denotes the closure of the range space \( fX \), besides retaining the rest of the hypotheses.
**Proof.** In view of (i), there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} d(fx_n, t) = 0 \) and \( \lim_{n \to \infty} d(gx_n, t) = 0 \) for some \( t \in X \). By property (H.E), we get \( \lim_{n \to \infty} d(fx_n, gx_n) = 0 \). By the closure of the range space \( fX, t \in \overline{fX} \). As \( \overline{fX} \subseteq gX \), we get \( t \in gX \). There exists a point \( u \in X \) such that \( t = gu \). One can follow on the lines of the proof furnished in Theorem 4.3.6.

The following example highlights the validity of Theorem 4.3.6 and Theorem 4.3.7.

**Example 4.3.8.** Consider \( X = [0,1] \) equipped with the symmetric \( d(x, y) = (x - y)^2 \). Define \( f, g : X \to X \) as follows:

\[
fx = \begin{cases} 
0, & 0 \leq x < 1/2 \\
1/2, & 1/2 \leq x \leq 1
\end{cases} \quad \text{and} \quad gx = \begin{cases} 
0, & 0 \leq x < 1/2 \\
1/3 + x/3, & 1/2 \leq x \leq 1
\end{cases}
\]

Clearly \( gX = \{0\} \cup [1/2, 2/3] \) is a closed subset of \( X \), and \( \overline{fX} = \{0, 1/2\} \subseteq gX \). For the sequence \( \{1/2 + 1/n\} \subseteq [0,1] \), we obtain \( \lim_{n \to \infty} d(f(1/2 + 1/n), 1/2) = 0 \) and \( \lim_{n \to \infty} d(g(1/2 + 1/n), 1/2) = 0 \), and so, the pair \( \{f, g\} \) enjoys the property (E.A).

In particular, we take \( k = 2/3 \) and \( a = 1/2 \). Four cases arise:

**Case I.** If \( 0 \leq x < 1/2 \) and \( 0 \leq y < 1/2 \), then we obtain

\[
d(fx, fy) = d(gx, gy) = d(fx, gx) = d(fy, gy) = d(gx, fy) = d(fx, gy) = 0.
\]

**Case II.** If \( 0 \leq x < 1/2 \) and \( 1/2 \leq y \leq 1 \), then we obtain

\[
d(fx, fy) = d(gx, fy) = d(0, 1/2) = 1/4,
\]

\[
d(gx, gy) = d(fx, gy) = d(0, 1/3 + y/3) = (1 + y)^2/9,
\]

\[
d(fx, gx) = d(0, 0) = 0 \quad \text{and} \quad d(fy, gy) = d(1/2, 1/3 + y/3) = (1 - 2y)^2/36.
\]

**Case III.** If \( 1/2 \leq x \leq 1 \) and \( 0 \leq y < 1/2 \), then we obtain

\[
d(fx, fy) = d(fx, gy) = d(1/2, 0) = 1/4,
\]

\[
d(gx, fx) = d(gx, fy) = d(1/2, 1/3 + y/3) = 1/36,
\]

\[
d(gy, fx) = d(gy, fy) = d(1/2, 0) = 1/4.
\]
\[ d(gx, gy) = d(gx, fy) = d(1/3 + x/3, 0) = (1 + x)^2 / 9, \]
\[ d(fy, gy) = d(0, 0) = 0 \quad \text{and} \quad d(fx, gx) = d(1/2, 1/3 + x/3) = (1 - 2x)^2 / 36. \]

Case IV. If \( 1/2 \leq x \leq 1 \) and \( 1/2 \leq y \leq 1 \), then we obtain
\[ d(fx, fy) = d(1/2, 1/2) = 0, \]
\[ d(gx, gy) = d(1/3 + x/3, 1/3 + y/3) = (x - y)^2 / 9, \]
\[ d(fx, gx) = d(1/2, 1/3 + x/3) = (1 - 2x)^2 / 36, \]
\[ d(gx, fy) = d(1/3 + x/3, 1/2) = (2x - 1)^2 / 36 \quad \text{and} \]
\[ d(fy, gy) = d(fx, gy) = d(1/2, 1/3 + y/3) = (1 - 2y)^2 / 36. \]

In all cases, we obtain
\[ d(fx, fy) \leq \frac{2}{3} d(gx, gy) + \frac{1}{2} \max \{ d(fx, gx), d(fy, gy), d(gx, gy), d(fx, gy) \}. \]

We see that \( fx = gx \iff x = 0, 1/2 \). Since \( d(g(0), gf(0)) = 0 = d(f(0), g(0)) \), we observe that \( d(gx, gfx) \leq Rd(fx, gx) \) for any \( R > 0 \). Further, as \( d(g(1/2), gf(1/2)) = 0 = d(f(1/2), g(1/2)) \), one can find \( d(gx, gfx) \leq Rd(fx, gx) \) for any \( R > 0 \). So, the pair \( \{ f, g \} \) is pointwise \( g \)-absorbing at \( x = 0, 1/2 \).

Moreover, we see that \( f(0) = g(0) = 0 \) and \( f(1/2) = g(1/2) = 1/2 \). Thus, the pair \( \{ f, g \} \) satisfies all the conditions of Theorem 4.3.6 and Theorem 4.3.7 with \( k = 2/3 \) and \( a = 1/2 \). Hence, the pair \( \{ f, g \} \) has two common fixed points 0 and 1/2.

In the next theorem, we employ \( g \)-continuous in place of Lipschitz condition.

**Theorem 4.3.9.** Let \((X, d)\) be a symmetric space satisfying property (W3). Let \( f \) and \( g \) be two self-maps of \( X \) such that

(i) the pair \( \{ f, g \} \) satisfies the property (E.A);

(ii) \( f \) is \( g \)-continuous;
(iii) either $fX \subseteq gX$ or $gX$ is a closed subset of $X$, where $\overline{fX}$ denotes the closure of the range space $fX$.

Then the pair $\{A,S\}$ has a common fixed point provided the pair is pointwise absorbing.

**Proof.** In view of (i), there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} d(fx_n, t) = 0$ and $\lim_{n \to \infty} d(gx_n, t) = 0$ for some $t \in X$.

Two cases arise:

Case I. Suppose that $\overline{fX} \subseteq gX$. By the closure of $fX$, $t \in \overline{fX}$. As $\overline{fX} \subseteq gX$, $t \in gX$.

Case II. Suppose that $gX$ is closed. Then $t \in gX$.

In both the cases, $t \in gX$ and so, there exists a point $u \in X$ such that $t = gu$ which in turn yields that $\lim_{n \to \infty} d(fx_n, gu) = 0$ and $\lim_{n \to \infty} d(gx_n, gu) = 0$.

Now using $g$-continuous of $f$, $\lim_{n \to \infty} d(fx_n, fu) = 0$. By condition (W3), one can get $fu = gu = t$.

Employing pointwise absorbing property of the pair $\{A,S\}$, one can find $gu = gfu$ and $fu = fgu$ which in turn yield $ft = gt = t$. Hence $t$ is a common fixed point of $f$ and $g$.

*****