CHAPTER-3

Fixed Points of Occasionally Weakly Compatible Maps and Weakly Biased Maps in Metric Spaces
**Introduction**

On studying common fixed point theorems of compatible maps (respectively, its types), most of them invariably require assumptions on completeness of space or continuity (at least one) of the maps involved, besides some contractive conditions, and aim at weakening one or more of these conditions. However, the study of fixed point theorems of a map satisfying a contractive condition that does not require continuity at each point in its domain, but the map involved is continuous at the fixed point, was initiated by Kannan [68] in 1968.

On the other hand, Pant ([79], [80]) initiated the study of common fixed point theorems of noncompatible maps. The utility and importance of the study of noncompatibility can be highlighted from the fact that this study can be extended to the class of non-expansive or Lipschitz type maps without assuming continuity of the maps involved or completeness of the underlying space. In 1998, the same author [81] then introduced the concept of reciprocal continuity which is weaker than continuity, and obtained a common fixed point theorem by using a minimal commutativity condition (i.e. pointwise $R$-weakly commuting and compatible maps) and reciprocal continuity in metric space, in which each map involved is discontinuous at the fixed point. In 2002, Kumar and Chugh [103] proved a common fixed theorem by employing the minimal commutativity condition and reciprocal continuity in metric space. Later on, Al-Thagafi and Shahzad [8] introduced the concept of occasionally weakly compatible, a very general notion of weakly compatible.

On the other hand, in 1995, Jungck and Pathak [62] coined a less restrictive notion of biased maps and weakly biased maps as a generalization of compatible maps, and also extended a theorem of Kang and Rhoades [67]. It is noted that compatible maps are biased maps, but the converse is not true ([62]). In 1998, Pathak et al. [87] weakened the condition of compatible maps of type (A) to
that of biased maps of type (A) and weakly biased maps of type (A), and showed the existence of solutions of nonlinear integral equations. It is worth mentioning that compatible maps of type (A) are biased maps of type (A), but the converse is not necessarily true (refer to [87]). Recently, M. R. Singh and Mahendra [111] characterized biased concepts (viz, weakly biased, weakly biased of type (A)) with that of weakly compatible.

This chapter consists of four sections. The first is the introductory in which relevant definitions and results needed in the following sections are furnished.

In the second, we substantially improve the main result of Esakkiappan [39, Theorem 3.2] by considering the completeness of one of the four range spaces in lieu of the completeness of the whole space, and by omitting reciprocal continuity.

In the third section, we deal with characterization of occasionally weakly compatible maps with weakly biased maps. In the fourth section, we establish common fixed point theorems of two pairs of weakly biased (respectively, of type (A)) maps by using property (E.A.), without appeal to continuity and completeness of space. Our theorems improve and extend Theorem 2.11 of [111] and Theorem 3.5 of [87]. The result of this section is to be appeared in International Journal of Pure and Applied Mathematics (IJPAM), vol. 86, no. 1, 2013.

We retrieve relevant definitions and results needed in the sequel.

Let $A$ and $S$ be two self-maps of a metric space $(X,d)$ and $C(A,S)$ the set of coincidence points of $A$ and $S$, i.e. $C(A,S) = \{ x : Ax = Sx \}$.

**Definition 3.1.1 ([59])**. $A$ and $S$ are said to be weakly compatible if they commute at their coincidence points, i.e. $SAT = AST$ for all $t \in C(A,S)$ (also refer to [64]).

The following concept (Definition 3.1.2) is a proper generalization of nontrivial weakly compatible maps which have a coincidence point.
**Definition 3.1.2 ([8]).** A and $S$ are said to be occasionally weakly compatible (shortly, owc) iff there is a point $t \in X$ such that $At = St$ at which $A$ and $S$ commute, i.e. $SAt = AS\tau$ for some $t \in C(A, S)$.

Clearly, weakly compatible maps are owc, but the converse is not true.

**Remark 3.1.3.** If $C(A, S) = \emptyset$, then the pair $\{A, S\}$ is trivially weakly compatible as well as occasionally weakly compatible maps.

**Definition 3.1.4 ([62]).** $A$ and $S$ are said to be weakly $S$-biased iff $At = St$ implies $d(SAt, St) \leq d(ASt, At)$.

The pair $\{A, S\}$ is weakly $A$-biased if $A$ and $S$ are interchanged in Definition 3.1.4. From [111], it is seen that weakly compatible maps are weakly biased, but implication is not reversible.

**Definition 3.1.5 ([87]).** $A$ and $S$ are said to be weakly $S$-biased of type $(A)$ if $At = St$ implies $d(SS\tau, At) \leq d(ASt, St)$.

To be weakly $A$-biased of type $(A)$, $A$ and $S$ have to satisfy the inequality $d(AAt, St) \leq d(SAt, At)$, in lieu of inequality in Definition 3.1.5.

In [111], it is clear that the notions of weakly biased and weakly biased of type $(A)$ are invariant. In [62] and [87], they claimed that biased (biased of type $(A)$) is weakly biased (weakly biased of type $(A)$). But, in [111], it has already shown that the said claim is true only when the pair of maps is continuous. The biased of type $(A)$ is distinct from biased (refer to Examples 2.11, 2.12 of [87]). Moreover, $A$-biased of type $(A)$ and $S$-biased of type $(A)$ do not imply each other, in general (refer to Examples 2.6, 2.7 of [87]).
3.2. Fixed points of occasionally weakly compatible maps in metric spaces

For the existence of common fixed point of $g$-contraction, Delbosco [34] considered the set $\mathcal{D}$ of all continuous functions $g:[0,\infty)^3 \to [0,\infty)$ satisfying the following properties:

(i) $g(1,1,1) = h < 1$,

(ii) if $u,v \geq 0$ are such that $u \leq g(u,v,u)$ or $u \leq g(v,u,v)$ or $u \leq g(v,v,u)$, then $u \leq hv$.

Later on, Constantin [31] considered the family $G$ of all real continuous functions $g:[0,\infty)^5 \to [0,\infty)$ satisfying the following properties:

(g$_1$) $g$ is non decreasing in the 4$^{th}$ and 5$^{th}$ variables.

(g$_2$) there are $h>0$ and $k>0$ such that $hk < 1$ and if $u,v \in [0,\infty)$ satisfy $u \leq g(v,v,u,u+v,0)$ or $u \leq g(v,u,v,u+v,0)$, then $u \leq hv$ and if $u,v \in [0,\infty)$ satisfy $u \leq g(v,v,u,0,u+v)$ or $u \leq g(v,u,v,0,u+v)$, then $u \leq kv$.

(g$_3$) if $u \in [0,\infty)$ is such that $u \leq g(u,0,0,u,u)$ or $u \leq g(0,u,0,u,u)$ or $u \leq g(0,0,u,u,u)$, then $u = 0$.

On the hand, Kumar and Chugh [103] generalized the results of Jungck [58], Fisher [40], Kang and Kim [66], Jachymski [53], Jungck et al. [60], Rhoades et al. [101] and Pant ([78], [81]) in 2002. Recently, Esakkiappan [39] improved the main theorem of Kumar and Chugh [103, Theorem 3.1] by employing occasionally weakly compatible and reciprocally continuous maps in complete metric space.

Now, we substantially improve the main result of Esakkiappan [39, Theorem 3.2] by omitting reciprocal continuity and considering the completeness of one of the four range spaces in lieu of the completeness of the whole space.
Let $A, B, S$ and $T$ be self-maps of a metric space $(X, d)$ satisfying the following conditions:

$$AX \subseteq TX \text{ and } BX \subseteq SX,$$  \hspace{1cm} (3.2.1)

and

$$d(Ax, By) \leq g \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\},$$ \hspace{1cm} (3.2.2)

for all $x, y \in X$, where $g \in G$.

Then for an arbitrary point $x_0 \in X$, by (3.2.1), we choose a point $x_1 \in X$ such that $Sx_1 = Bx_0 = y_1$ and for this point $x_1$, there exists a point $x_2 \in X$ such that $Tx_2 = Ax_1 = y_2$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in $X$ such that

$$Sx_{2n+1} = Bx_{2n} = y_{2n+1} \text{ and } Tx_{2n+2} = Ax_{2n+1} = y_{2n+2}, \quad n \in \mathbb{N}_0 \quad \text{ ... (3.2.3)}$$

Before starting our main theorem, we prove the following lemma and proposition.

**Lemma 3.2.1.** Let $\{A, S\}$ and $\{B, T\}$ be two pairs of self-maps of a metric space $(X, d)$ satisfying (3.2.1). Suppose that the sequence $\{y_n\}$ in $X$ is defined by (3.2.3) and there exists $h, k \in (0, 1)$ such that for all $n \in \mathbb{N}_0$,

$$d(y_{2n+3}, y_{2n+2}) \leq k d(y_{2n+2}, y_{2n+1}) \quad \text{ ... (3.2.4)}$$

and

$$d(y_{2n+4}, y_{2n+3}) \leq hd(y_{2n+3}, y_{2n+2}). \quad \text{ ... (3.2.5)}$$

Then either

1. $C(A, S) \neq \phi$, $C(B, T) \neq \phi$ and the sequence $\{y_n\}$ converges to a point in $X$, or
2. $\{y_n\}$ is a Cauchy sequence in $X$, holds.

**Proof.** Suppose that there exists $n \in \mathbb{N}_0$ such that $y_{2n+1} = y_{2n+2}$. Then from the definition of $\{y_n\}$, $Ax_{2n+1} = Tx_{2n+2} = Bx_{2n} = Sx_{2n+1}$ and so, the maps $A$ and $S$ have a coincidence point $x_{2n+1}$. Therefore, $C(A, S) \neq \phi$. 

Further, from (3.2.4), we obtain \(d(y_{2n+3}, y_{2n+2}) \leq k d(y_{2n+2}, y_{2n+1}) = 0\) which implies that \(y_{2n+3} = y_{2n+2}\), i.e. \(Bx_{2n+2} = Tx_{2n+2}\). Therefore, the maps \(B\) and \(T\) have a coincidence point \(x_{2n+2}\) and so, \(C(B,T) \neq \emptyset\).

Moreover, from (3.2.5), we obtain \(d(y_{2n+4}, y_{2n+3}) \leq h d(y_{2n+3}, y_{2n+2}) = 0\) which implies that \(y_{2n+4} = y_{2n+3}\), i.e. \(Ax_{2n+3} = Sx_{2n+3}\). Therefore, the maps \(A\) and \(S\) have a coincidence point \(x_{2n+3}\) and so, \(C(A,S) \neq \emptyset\).

In addition, repeating use of (3.2.4) and (3.2.5) yields \(y_{2n+1} = y_m\) for each \(m > 2n+1\) and hence, the sequence \(\{y_n\}\) converges to a point in \(X\). The same conclusion holds if \(y_{2n+2} = y_{2n+3}\) for some \(n \in \mathbb{N}_0\).

To prove (2), we assume that \(y_{2n+1} \neq y_{2n+2}\) for all \(n \in \mathbb{N}_0\). Then in view of (3.2.4) and (3.2.5), we deduce
\[
d(y_{2n+2}, y_{2n+1}) \leq h d(y_{2n+1}, y_{2n}) \leq h k d(y_{2n}, y_{2n+1}) \quad \leq h^n k^n d(y_2, y_1)
\]
and
\[
d(y_{2n+3}, y_{2n+2}) \leq k d(y_{2n+2}, y_{2n+1}) \leq h k d(y_{2n+1}, y_{2n}) \quad \leq h^n k^{n+1} d(y_2, y_1).
\]

For any \(m > n\), where \(m, n \in \mathbb{N}\), it follows that
\[
d(y_{2n+1}, y_{2m+1}) \leq \sum_{i=2n+1}^{2m} d(y_i, y_{i+1}) \leq \left\{ \sum_{i=n}^{m-1} (hk)^i + k \sum_{i=n}^{m-1} (hk)^i \right\} d(y_2, y_1)
\]
\[
= \left[ (hk)^n \frac{1 - (hk)^{m-n}}{1 - hk} + k (hk)^n \frac{1 - (hk)^{m-n}}{1 - hk} \right] d(y_2, y_1)
\]
\[
\leq \frac{(hk)^n (1+k)}{1-hk} d(y_2, y_1) \quad \text{and}
\]
\[
d(y_{2n+2}, y_{2m+1}) \leq \sum_{i=2n+2}^{2m} d(y_i, y_{i+1}) \\
\leq \left\{ k \sum_{i=n}^{m-1} (hk)^i + \sum_{i=n+1}^{m} (hk)^i \right\} d(y_2, y_1) \\
= \left[ k(hk)^n \frac{1-(hk)^{m-n}}{1-hk} + (hk)^{n+1} \frac{1-(hk)^{m-n-1}}{1-hk} \right] d(y_2, y_1) \\
\leq \frac{(hk)^n(1+h)}{1-hk} d(y_2, y_1).
\]

Similarly, \(d(y_{2n+1}, y_{2m+2}) \leq \frac{(hk)^n(1+k)}{1-hk} d(y_2, y_1)\) and
\[
d(y_{2n+2}, y_{2m+2}) \leq \frac{(hk)^n(1+h)}{1-hk} d(y_2, y_1).
\]

Taking limits as \(n \rightarrow \infty\), we obtain \(d(y_n, y_{n+m}) \rightarrow 0\). Therefore, the sequence \(\{y_n\}\) is a Cauchy sequence in \(X\).

**Proposition 3.2.2.** Let \(\{A, S\}\) and \(\{B, T\}\) be two pairs of self-maps of a metric space \((X, d)\) satisfying (3.2.1), (3.2.2) and (3.2.3). If one of the range spaces \(AX, BX, SX\) and \(TX\) is a complete subspace of \(X\), then \(C(A, S) \neq \emptyset\) and \(C(B, T) \neq \emptyset\).

**Proof.** Suppose that \(\{y_n\}\) is a sequence in \(X\) and is defined by (3.2.3). We assert that the sequence \(\{y_n\}\) satisfies conditions (3.2.4) and (3.2.5). By (3.2.2), we obtain
\[
d(y_{2n+3}, y_{2n+2}) = d(Bx_{2n+2}, Ax_{2n+1}) \\
\leq g \{ d(Sx_{2n+1}, Tx_{2n+2}), d(Ax_{2n+1}, Sx_{2n+1}), d(Bx_{2n+2}, Tx_{2n+2}) \}, \\
d(Ax_{2n+1}, Tx_{2n+2}), d(Bx_{2n+2}, Sx_{2n+1}) \} \\
= g \{ d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+3}, y_{2n+2}), \\
d(y_{2n+2}, y_{2n+2}), d(y_{2n+3}, y_{2n+1}) \} \\
\]
\[
\begin{align*}
&\leq g\{d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+3}, y_{2n+2}), 0, \\
&\quad d(y_{2n+3}, y_{2n+2}) + d(y_{2n+2}, y_{2n+1})\}.
\end{align*}
\]

By property \((g_2)\), \(d(y_{2n+3}, y_{2n+2}) \leq k \cdot d(y_{2n+2}, y_{2n+1})\).

By \((3.2.2)\), we obtain
\[
\begin{align*}
d(y_{2n+4}, y_{2n+3}) &= d(Ax_{2n+3}, Bx_{2n+2}) \\
&\leq g\{d(Sx_{2n+3}, Tx_{2n+2}), d(Ax_{2n+3}, Sx_{2n+3}), d(Bx_{2n+2}, Tx_{2n+2}), \\
&\quad d(Ax_{2n+3}, Tx_{2n+2}), d(Bx_{2n+2}, Sx_{2n+3})\} \\
&= g\{d(y_{2n+3}, y_{2n+2}), d(y_{2n+4}, y_{2n+3}), d(y_{2n+3}, y_{2n+2}), \\
&\quad d(y_{2n+4}, y_{2n+2}) + d(y_{2n+3}, y_{2n+2}), 0\}.
\end{align*}
\]

By property \((g_2)\), \(d(y_{2n+4}, y_{2n+3}) \leq h \cdot d(y_{2n+3}, y_{2n+2})\). So, the conditions \((3.2.4)\) and \((3.2.5)\) of Lemma 3.2.1, are satisfied.

If \(y_n = y_{n+1}\) for some \(n \in \mathbb{N}\), then from (1) of Lemma 3.2.1, \(C(A,S) \neq \emptyset\) and \(C(B,T) \neq \emptyset\), and the sequence \(\{y_n\}\) converges to a point in \(X\). In fact, without loss of generality, we may assume that \(y_n \neq y_{n+1}\) for any \(n \in \mathbb{N}\). Thus, from (2) of Lemma 3.2.1, the sequence \(\{y_n\}\) is a Cauchy sequence in \(X\).

We consider completeness of range space in four cases.

Case I. Suppose that \(SX\) is a complete subspace of \(X\). Then the subsequence \(\{Sx_{2n+1}\} = \{y_{2n+1}\}\) of \(\{y_n\}\) is a Cauchy sequence in \(SX\). It follows that completeness of \(SX\) implies existence of \(u \in SX\) such that \(y_{2n+1} \to u\) as \(n \to \infty\). Subsequently, there exists \(v \in X\) such that \(Sv = u\).

Now, we prove that \(Av = u\). By \((3.2.2)\), we obtain
\[ d(Av, Bx_{2n}) \leq g\{ d(Sv, Tx_{2n}), d(Av, Sv), d(Bx_{2n}, Tx_{2n}) \} \]
\[ = g\{ d(u, y_{2n}), d(Av, u), d(y_{2n+1}, y_{2n}), d(Av, y_{2n}), d(y_{2n+1}, u) \}. \]

Taking limits as \( n \to \infty \), we obtain \( d(Av, u) \leq g\{ d(u, u), d(Av, u), d(u, u), d(Av, u) \} \)
\[ = g\{ 0, d(Av, u), 0, d(Av, u) \} \]
\[ \leq g\{ 0, d(Av, u), 0, d(Av, u) \}. \]

By property \((g_3)\), we obtain \( d(Av, u) = 0 \) which yields that \( Av = u \), and so, \( Av = Sv = u \).

As \( u \in AX \subseteq TX \), there exists \( w \in X \) such that \( u = Tw \). Now we assert that \( Bw = u \). From (3.2.2), we obtain \( d(u, Bw) = d(Av, Bw) \)
\[ = g\{ d(Sv, Tw), d(Av, Sv), d(Bw, Tw), d(Av, Tw), d(Bw, Sv) \} \]
\[ = g\{ d(u, u), d(u, u), d(Bw, u), d(u, u), d(Bw, u) \} \]
\[ \leq g\{ 0, 0, d(Bw, u), 0, d(Bw, u) \} \]
\[ = g\{ 0, 0, d(Bw, u), d(Bw, u), d(Bw, u) \}. \]

By property \((g_3)\), we have \( d(Bw, u) = 0 \) which yields that \( Bw = u \), and therefore, \( Bw = Tw = u \).

Case II. Suppose that \( TX \) is a complete subspace of \( X \). The proof is similar to that of Case I.

Case III. Suppose that \( BX \) is a complete subspace of \( X \). Then the subsequence \( \{Bx_{2n}\} = \{ y_{2n+1} \} \) of \( \{ y_n \} \) is a Cauchy sequence in \( BX \). It follows that completeness of \( BX \) implies existence of \( u \in BX \) such that \( y_{2n+1} \to u \) as \( n \to \infty \).

As \( u \in BX \subseteq SX \), there exists \( v \in X \) such that \( Sv = u \). By the same argument as in Case I, it follows that \( Av = Sv = u \) and \( Bw = Tw = u \).
Case IV. Suppose that $AX$ is a complete subspace of $X$. The proof is similar to that of Case III.

In all cases, $A v = S v = u$ and $B w = T w = u$. Therefore, $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$.

Now, we state our theorem.

**Theorem 3.2.3.** In addition to the hypotheses of Proposition 3.2.2 on $A, B, S$ and $T$, if the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible, then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** By Proposition 3.2.2, $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$. Since the pair $\{A, S\}$ is occasionally weakly compatible, there exists $t \in C(A, S)$ such that $A t = S t$ and $A S t = S A t$ which implies that $A A t = A S t = S A t = S S t$. By occasionally weakly compatibility of the pair $\{B, T\}$, there exists $z \in C(B, T)$ such that $B z = T z$ and $B T z = T B z$ which implies that $B B z = B T z = T B z = T T z$.

We assert that $A t = B z$. From (3.2.2), we obtain

$$d(A t, B z) \leq g \{ d(S t, T z), d(A t, S t), d(B z, T z), d(A t, T z), d(B z, S t)\}$$

$$= g \{ d(A t, B z), 0, 0, d(A t, B z), d(A t, B z)\}.$$  

By property $(g_3)$, $d(A t, B z) = 0$ which yields that $A t = B z$. So, $A t = B z = S t = T z$.

We need to prove that $A A t = A t$. By (3.2.2), we obtain

$$d(A A t, A t) = d(A A t, B z)$$

$$\leq g \{ d(S A t, T z), d(A A t, S A t), d(B z, T z), d(A A t, T z), d(B z, S A t)\}$$

$$= g \{ d(A A t, A t), 0, 0, d(A A t, A t), d(A A t, A t)\}.$$  

By property $(g_3)$, $d(A A t, A t) = 0$ which yields that $A A t = A t$. So, $A A t = S A t = A t$ which means that $A t$ is a common fixed point of the pair $\{A, S\}$.

We claim that $B B z = A t$. By (3.2.2), we obtain
\[d(At, BBz) \leq g \{ d(St, TBz), d(At, St), d(BBz, TBz), d(At, TBz), d(BBz, St)\}\]
\[= g \{ d(At, BBz), 0, 0, d(At, BBz), d(At, BBz)\}.
\]
By property \((g_3)\), \(d(At, BBz) = 0\) and so, \(BBz = At\). Therefore, \(BBz = TBz = At\) which implies that \(BAt = TAjt = At\). Thus, \(AAAt = SAAt = BAAt = TAAt = At\) which means that \(At\) is a common fixed point of \(A, B, S\) and \(T\).

For uniqueness, suppose that \(AAAt = SAAt = BAAt = TAAt = At'\). We assert that \(At = At'\). By (3.2.2), we have
\[d(At, At') = d(AAAt, BAAt')\]
\[\leq g \{ d(SAt, TAAt'), d(AAAt, SAAt), d(BAAt', TAAt'), d(AAAt, TAAt'), d(BAAt', SAAt)\}\]
\[= g \{ d(At, At'), 0, 0, d(At, At'), d(At, At')\}.
\]
By property \((g_3)\), \(d(At, At') = 0\) which yields that \(At = At'\). Hence, \(At\) is the unique common fixed point of \(A, B, S\) and \(T\). This completes the proof.

**Remark 3.2.4.** Two special cases of condition (3.2.2) are given below:

(i) By setting \(g(t_1, t_2, t_3, t_4, t_5) = h_i \max \{ t_1, t_2, t_3, t_4, t_5 \}\) for all \(t_1, t_2, t_3, t_4, t_5 \in [0, \infty)\), where \(0 \leq h_i < 1\), the condition (3.2.2) reduces to
\[d(Ax, By) \leq h_i \max \{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\}\]
\[\cdots (3.2.6)\]
for all \(x, y \in X\).

(ii) When \(g(t_1, t_2, t_3, t_4, t_5) = h_i \max \{ t_1, t_2, t_3, [t_4 + t_5]/2 \}\) for all \(t_1, t_2, t_3, t_4, t_5 \in [0, \infty)\), where \(0 \leq h_i < 1\), the condition (3.2.2) reduces to
\[d(Ax, By) \leq h_i \max \{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}\]
\[\cdots (3.2.7)\]
for all \(x, y \in X\).
Remark 3.2.5. The conclusion of Theorem 3.2.3 remains true even if the condition (3.2.2) is replaced by any one of the conditions (3.2.6) and (3.2.7), besides retaining the rest of the hypotheses.

Now, we furnish the following example in support of Theorem 3.2.3.

Example 3.2.6. Let $X = [1/3, 1)$ with the usual metric $d$. Define maps $A, B, S, T : X \to X$ as

$$Ax = \begin{cases} 1/2 & , 1/3 \leq x < 2/3 \\ 2/3 & , 2/3 \leq x < 1 \end{cases}, \quad Bx = \begin{cases} 3/4 & , 1/3 \leq x < 2/3 \\ 2/3 & , 2/3 \leq x < 1 \end{cases},$$

$$Sx = \begin{cases} 2/5 & , 1/3 \leq x < 2/3 \\ 1/3 + x/2 & , 2/3 \leq x < 1 \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1/2 & , 1/3 \leq x < 2/3 \\ 1-x/2 & , 2/3 \leq x < 1 \end{cases}.$$

Clearly $AX = \{ 1/2, 2/3 \} \subset [1/2, 2/3] = TX$ and $BX = \{ 2/3, 3/4 \} \subset \{ 2/3, 5/6 \} \cup \{ 2/5 \} = SX$.

Consider $g(t_1, t_2, t_3, t_4, t_5) = h_1 \max \{ t_1, t_2, t_3, t_4, t_5 \}$ for all $t_1, t_2, t_3, t_4, t_5 \in [0, \infty)$, where $0 \leq h_1 < 1$. So, $g$ satisfies all the required conditions. In particular, we take $h_1 = 6/7$.

Now, four cases arise:

Case I. If $x, y \in [1/3, 2/3)$, then $d(Ax, By) = 1/4$ and $d(By, Sx) = 7/20$.

Case II. If $x \in [1/3, 2/3)$ and $y \in [2/3, 1)$, then $d(Ax, By) = 1/6$ and $d(By, Sx) = 4/15$.

Case III. If $y \in [1/3, 2/3)$ and $x \in [2/3, 1)$, then $d(Ax, By) = 1/12$ and $d(By, Ty) = 1/4$.

Case IV. If $x, y \in [2/3, 1)$, then $d(Ax, By) = 0$ and $d(By, Sx) = \left| 3(x + y) - 4 \right| / 6$.

In all cases, we obtain

$$d(Ax, By) \leq \frac{6}{7} \max \{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx) \}.$$
Now, we have $Ax = Sx$ iff $x = 2/3$, and $Bx = Tx$ iff $x = 2/3$. We see that $A(2/3) = 2/3 = S(2/3)$, $B(2/3) = 2/3 = T(2/3)$, $AS(2/3) = A(2/3) = 2/3$ and $SA(2/3) = S(2/3) = 2/3$, the pair $\{A, S\}$ is owc. One can ascertain that the pair $\{B, T\}$ is owc. Thus, all the conditions of Theorem 3.2.3 are satisfied and 2/3 is the unique common fixed point of $A, B, S$ and $T$.

### 3.3. Characterization of occasionally weakly compatible maps with weakly biased maps

In this section, we show that weakly $A$-biased and weakly $S$-biased (respectively, weakly $S$-biased of type $(A)$) do not imply each other, and weakly $A$-biased of type $(A)$ and weakly $S$-biased (respectively, weakly $S$-biased of type $(A)$) do not imply each other by giving examples. We attempt to characterize the concept of occasionally weakly compatible maps with that of weakly biased. In dealing with such characterization, we prove that occasionally weakly compatible relates directly to one of the weakly $A$-biased and weakly $S$-biased.

We furnish the following example as the manifestation of the fact that weakly $A$-biased does not imply weakly $S$-biased of type $(A)$.

**Example 3.3.1.** Let $X = [0,1]$ endowed with the usual metric. Define maps $A, S : X \to X$ by

\[
Ax = \begin{cases} 
1 - 2x, & 0 \leq x < 1/2 \\
1/3, & 1/2 \leq x \leq 1
\end{cases}
\quad \text{and} \quad
Sx = \begin{cases} 
2x, & 0 \leq x < 1/2 \\
0, & 1/2 \leq x \leq 1.
\end{cases}
\]

We see that $Ax = Sx$ iff $x = 1/4$. Now, we obtain $A(1/4) = 1/2 = S(1/4)$, $SS(1/4) = S(1/2) = 0$, $SA(1/4) = S(1/2) = 0$ and $AS(1/4) = A(1/2) = 1/3$. As $d(AS(1/4), A(1/4)) = 1/6$, $d(SA(1/4), S(1/4)) = 1/2$, $d(SS(1/4), A(1/4)) = 1/2$ and $d(AS(1/4), S(1/4)) = 1/6$, the pair $\{A, S\}$ is weakly $A$-biased, but not weakly $S$-biased of type $(A)$. 


On the other hand, weakly $S$-biased does not imply weakly $A$-biased of type $(A)$. The following example supports this fact.

**Example 3.3.2.** Let $X = [0,1]$ be as in Example 3.3.1. Define maps $A, S : X \to X$ by

$$A_x = \begin{cases} 1 - x, & x \in [0,1/2) \cup (1/2,1] \\ 2/3, & x = 1/2 \end{cases} \quad \text{and} \quad S_x = \begin{cases} x, & x \in [0,1/2) \cup (1/2,1] \\ 2/3, & x = 1/2 \end{cases}.$$

We see that $A_x S_x = x$ iff $x = 1/2$. Now, we obtain $A(1/2) = 2/3 = S(1/2)$, $SA(1/2) = S(2/3) = 2/3$, $AS(1/2) = A(2/3) = 1/3$ and $AA(1/2) = A(2/3) = 1/3$. As $d(SA(1/2), S(1/2)) = 0$, $d(AS(1/2), A(1/2)) = 1/3$, $d(AA(1/2), S(1/2)) = 1/3$ and $d(SA(1/2), A(1/2)) = 0$, the pair $\{A, S\}$ is weakly $S$-biased, but not weakly $A$-biased of type $(A)$.

**Remark 3.3.3.** In view of the above Example 3.3.1 and Example 3.3.2, one can conclude in general that weakly $A$-biased and weakly $S$-biased (respectively, weakly $S$-biased of type $(A)$) do not imply each other. Moreover, weakly $A$-biased of type $(A)$ and weakly $S$-biased (respectively, weakly $S$-biased of type $(A)$) do not imply each other.

We need the following lemma to our proposition.

**Lemma 3.3.4.** Let $\{A, S\}$ be a pair of self-maps of a metric space $(X, d)$. If $x \in C(A, S)$, then the pair $\{A, S\}$ is either weakly $A$-biased or weakly $S$-biased.

**Proof.** Since $x \in C(A, S)$, we have $A_x = S_x$. Two cases arise:

**Case I.** If $x \in C(A, S)$ such that $A_x S_x = S_x A_x$, then $d(SA_x, S_x) \leq d(SA_x, A_x) + d(A_x, S_x)$. So, $d(SA_x, S_x) \leq d(AS_x, Ax)$. Similarly, interchanging $A$ and $S$ in the above, we obtain $d(AS_x, Ax) \leq d(SA_x, S_x)$.

**Case II.** If $x \in C(A, S)$ such that $A_x S_x \neq S_x A_x$, then either $A_x S_x < S_x A_x$ or $S_x A_x < A_x S_x$. 

Suppose that \( r_1 \) and \( r_2 \) are two positive real numbers such that \( ASx < r_1 < SAx < r_2 \). Then we obtain two open spheres \( S(Ax, r_1) \) and \( S(Sx, r_2) \) such that \( S(Ax, r_1) \subseteq S(Sx, r_2) \). Therefore, \( d(ASx, Ax) \leq d(SAx, Sx) \).

On the other hand, suppose that \( \delta_1 \) and \( \delta_2 \) are two positive real numbers such that \( SAx < \delta_1 < ASx < \delta_2 \). Then there exists two open spheres \( S(Sx, \delta_1) \) and \( S(Ax, \delta_2) \) such that \( S(Sx, \delta_1) \subseteq S(Ax, \delta_2) \). So, \( d(SAx, Sx) \leq d(ASx, Ax) \).

From both cases, we see that if \( x \in C(A, S) \), then either \( d(ASx, Ax) \leq d(SAx, Sx) \) or \( d(SAx, Sx) \leq d(ASx, Ax) \). Hence, if \( x \in C(A, S) \), then the pair \( \{A, S\} \) is either weakly \( A \)-biased or weakly \( S \)-biased.

The following proposition conveys that occasionally weakly compatible maps relate directly to one of the weakly \( A \)-biased and weakly \( S \)-biased maps.

**Proposition 3.3.5.** Let \( \{A, S\} \) be a pair of self-maps of a metric space \( (X, d) \). If the pair \( \{A, S\} \) is occasionally weakly compatible, then it is either weakly \( A \)-biased or weakly \( S \)-biased.

**Proof.** Since the pair \( \{A, S\} \) is occasionally weakly compatible, there exists a point \( x \in X \) such that \( Ax = Sx \) at which \( SAx = ASx \). Moreover, there may exist a point \( y \in X \) such that \( Ay = Sy \) at which \( SAy \neq ASy \). By Lemma 3.3.4, the pair \( \{A, S\} \) is either weakly \( A \)-biased or weakly \( S \)-biased. Hence, occasionally weak compatibility of \( \{A, S\} \) implies either weakly \( A \)-biased or weakly \( S \)-biased.

We furnish an example to show the utility of Proposition 3.3.5.

**Example 3.3.6.** Let \( X = [1, \infty) \) endowed with the usual metric. Define maps \( A, S : X \to X \) by \( Ax = 3x - 2 \) and \( Sx = x^2 \).
We see that $Ax = Sx$ iff $x = 1$ or $x = 2$. Now, we obtain $A(1) = 1 = S(1)$, $A(2) = 4 = S(2)$ and $AS(1) = 1 = SA(1)$. But $AS(2) \neq SA(2)$ as $AS(2) = 10$ and $SA(2) = 16$. Therefore, the pair $\{A, S\}$ is occasionally weakly compatible.

Besides, we get $d(AS(2), A(2)) \leq d(SA(2), S(2))$ as $d(AS(2), A(2)) = 6$ and $d(SA(2), S(2)) = 12$. One can substantiate that $d(AS(1), A(1)) \leq d(SA(1), S(1))$. Therefore, the pair $\{A, S\}$ is weakly $A$-biased. Hence, the pair $\{A, S\}$ is occasionally weakly compatible as well as weakly $A$-biased.

However, the converse of the above proposition is not necessarily true. The following example reveals the fact that the pair $\{A, S\}$ is weakly $A$-biased, but not occasionally weakly compatible.

**Example 3.3.7.** Let $X = [1, \infty)$ be as in Example 3.3.6. Define maps $A, S : X \to X$ by $Ax = 5x$ and $Sx = x^2 + 6$.

We see that $Ax = Sx$ iff $x = 2$ or $x = 3$. Now we observe that $A(2) = 10 = S(2)$ and $A(3) = 15 = S(3)$. It follows that $AS(2) = A(10) = 50$, $SA(2) = S(10) = 106$, $AS(3) = A(15) = 75$ and $SA(3) = S(15) = 231$.

Since $d(AS(2), A(2)) = 40$, $d(SA(2), S(2)) = 96$, $d(AS(3), A(3)) = 60$ and $d(SA(3), S(3)) = 216$, we observe that $d(AS(2), A(2)) \leq d(SA(2), S(2))$ and $d(AS(3), A(3)) \leq d(SA(3), S(3))$. But, $AS(2) \neq SA(2)$ and $AS(3) \neq SA(3)$. So, the pair $\{A, S\}$ is weakly $A$-biased, but not occasionally weakly compatible.

One can verify the fact that the pair $\{A, S\}$ is weakly $S$-biased, but not occasionally weakly compatible, with Example 3.2.2.

We conclude here by noting that the concept of weakly biased maps appears to be an effective generalization of occasionally weakly compatible maps in metric space.
3.4. Fixed points of weakly biased maps satisfying generalized Fisher type contraction

We extend the result of M. R. Singh and Mahendra [111, Theorem 2.11] by employing a generalized Fisher type contraction. Further, we obtain the results of Pathak et al. [87, Theorem 3.5] and Pathak [86, Theorem 2.5] as particular cases.

For the existence of common fixed point theorems in metric space, we need the following.

Let $F$ be a family of all functions $\varphi: (\mathbb{R}_0^+)^5 \to \mathbb{R}_0^+$ such that $\varphi$ is upper semi-continuous, non decreasing in each coordinate variable, and for any $t > 0$

$$\varphi(t, t, 0, 0, 0) \leq \beta t, \quad \varphi(t, t, 0, 0, 0) \leq \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$. $\gamma(t) = \varphi(t, t, a_1, a_2, a_3, a_4) < t$, where $\gamma: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a map and

$$a_1 + a_2 + a_3 = 4.$$

We prove the following lemma before going to main theorem.

**Lemma 3.4.1.** Let $A, B, S$ and $T$ be self-maps of a metric space $(X, d)$ satisfying the following conditions:

(i) $AX \subset TX$ and $BX \subset SX$;

(ii) $d^2p(Ax, By) \leq \varphi\{d^{2p}(Sx, Ty), d^{p}(Sx, Ax), d^{p}(Ty, By),

\begin{align*}
&d^{p}(Sx, By).d^{p}(Ty, Ax), d(Sx, Ax).d^{2p-1}(Ty, Ax),
&d^{2p-1}(Sx, By).d(Ty, By)\}
\end{align*}$

for all $x, y \in X$ where $p \geq 1$ and $\varphi \in F$;

(iii) one range of the maps $A, B, S$ and $T$ is a closed subspace of $X$;

(iv) one of the pairs $\{A, S\}$ and $\{B, T\}$ satisfies the property (E.A).

Then each pair (viz. $\{A, S\}$ and $\{B, T\}$) has a coincidence point.
Proof. Suppose that the pair \( \{ B, T \} \) satisfies the property (E.A). Then there exists a sequence \( \{ x_n \} \) in \( X \) such that \( \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). Since \( BX \subset SX \), there exists a sequence \( \{ y_n \} \subset X \) such that \( Bx_n = Sy_n \). So, \( \lim_{n \to \infty} Sy_n = t \).

We now show that \( \lim_{n \to \infty} Ay_n = t \). Since \( d(Ay_n, t) \leq d(Ay_n, Bx_n) + d(Bx_n, t) \), it is enough to show that \( \lim_{n \to \infty} d(Ay_n, Bx_n) = 0 \). If not, then there exists a real number \( \varepsilon > 0 \) such that \( \lim_{n \to \infty} d(Ay_n, Bx_n) = \varepsilon \). This assures that there exists a subsequence \( \{ y_{n_k} \} \) of \( \{ y_n \} \) in \( X \) such that for each positive integer \( k \geq n, \lim_{k \to \infty} d(Ay_{n_k}, Bx_{n_k}) = \varepsilon \). From (ii), we have
\[
d(Ay_{n_k}, Bx_{n_k}) \leq [ \varphi \{ d^{2p}(Sy_{n_k}, Tx_{n_k}), d^p(Sy_{n_k}, Ay_{n_k}), d^p(Tx_{n_k}, Bx_{n_k}) \},

\quad d^p(Sy_{n_k}, Bx_{n_k}), d^p(Tx_{n_k}, Ay_{n_k}), d(Sy_{n_k}, Ay_{n_k}), d^{2p-1}(Tx_{n_k}, Ay_{n_k}),

\quad d^{2p-1}(Sy_{n_k}, Bx_{n_k}), d(Tx_{n_k}, Bx_{n_k}) )]^{1/2p}.
\]

Letting \( k \to \infty \), we obtain \( \varepsilon \leq [ \varphi(0, 0, 0, \varepsilon^{2p})]^{1/2p} \leq [ \gamma(\varepsilon^{2p})]^{1/2p} < \varepsilon \), a contradiction.

Therefore, \( \lim_{k \to \infty} d(Ay_{n_k}, Bx_{n_k}) = 0 \) and so, \( \lim_{n \to \infty} d(Ay_n, Bx_n) = 0 \) which implies \( \lim_{n \to \infty} Ay_n = t \).

Suppose that \( SX \) is closed subspace of \( X \). Then \( t = Su \) for some \( u \in X \). Subsequently, we have \( \lim_{n \to \infty} Ay_n = \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Sy_n = \lim_{n \to \infty} Tx_n = t = Su \). To prove that \( Au = t \), using (ii), we obtain
\[
d(Au, Bx_n) \leq [ \varphi \{ d^{2p}(Su, Tx_n), d^p(Su, Au), d^p(Tx_n, Bx_n), d^p(Su, Bx_n), d^p(Tx_n, Au),

\quad d(Su, Au), d^{2p-1}(Tx_n, Au), d^{2p-1}(Su, Bx_n), d(Tx_n, Bx_n) )]^{1/2p}.
\]

As \( n \to \infty \), it follows that \( d(Au, t) \leq [ \varphi(0, 0, 0, d^{2p}(Au, t), 0)]^{1/2p} \).
\[ \leq [\gamma (d^{2p}(Au,t))]^{1/2p} \]
\[ < d(Au,t) . \]

This is a contradiction and therefore, \( Au = t \). Consequently, \( u \) is a coincidence point of the pair \( \{A,S\} \). From \( AX \subseteq TX \), which gives \( t \in X \), we declare that there exists \( v \in X \) such that \( TV = t \). To prove that \( BV = t \), suppose that \( BV \neq t \). By (ii), we have
\[
d(t,Bv) = d(Au,Bv) \\
\leq [\varphi\{d^{2p}(Su,Tv), d^p(Su,Au), d^p(Tv,Bv), d^p(Su,Bv), d^p(Tv,Au), d(Su,Au), d^{2p-1}(Tv,Au), d^{2p-1}(Su,Bv), d(Tv,Bv)\}]^{1/2p} \\
= [\varphi\{0,0,0,0,d^{2p}(t,Bv)\}]^{1/2p} \\
\leq [\gamma(d^{2p}(t,Bv))]^{1/2p} \\
< d(t,Bv)
\]
which means that \( BV = t \). Therefore, the pair \( \{B,T\} \) has a coincidence point \( v \).

The same result holds if we suppose that one of \( AX, BX \) and \( TX \) is closed subspace of \( X \).

**Theorem 3.4.2.** Let \( A, B, S \) and \( T \) be self-maps as in Lemma 3.4.1 satisfying (i)-(iv). If the pairs \( \{A,S\} \) and \( \{B,T\} \) are weakly \( S \)-biased and weakly \( T \)-biased respectively, then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** Assume that \( SX \) is closed subspace of \( X \) and let \( t, u \) and \( v \) be as in Lemma 3.4.1. By that lemma, \( u \) and \( v \) are coincidence points of the pairs \( \{A,S\} \) and \( \{B,T\} \) respectively.
Since the pair \( \{A, S\} \) is weakly \( S \)-biased, \( A u = Su \) implies \( d(SAu, Su) \leq d(ASu, Au) \) which gives \( d(St, t) \leq d(At, t) \). On the other hand, \( A u = Su \) implies \( AAu = ASu \) and \( SAu = SSu \). Subsequently,

\[
\begin{align*}
d(SAu, AAu) & \leq d(SAu, Su) + d(Su, AAu) \\
& \leq d(ASu, Au) + d(AAu, Au)
\end{align*}
\]

which implies that \( d(St, At) \leq 2d(At, t) \).

To show that \( d(At, t) = 0 \), suppose that \( At \neq t \). From (ii), we have

\[
d(At, t) = d(At, Bv)
\]

\[
\leq \left[ \varphi \{d^{2p}(StTv), d^{p}(St, At).d^{p}(Tv, Bv), d^{p}(St, Bv).d^{p}(Tv, At),
\right. \\
\left. d(St, At).d^{2p-1}(Tv, At), d^{2p-1}(St, Bv).d(Tv, Bv)\} \right]^{1/2p}
\]

\[
= \left[ \varphi \{d^{2p}(St, t), 0, d^{p}(St, t).d^{p}(t, At), d(St, At).d^{2p-1}(At, t), 0\} \right]^{1/2p}
\]

\[
\leq \left[ \gamma \{d^{2p}(At, t), 0, 2d^{2p}(At, t), 0\} \right]^{1/2p}
\]

\[
< d(At, t)
\]

which leads to contradiction. Therefore, \( At = t \). Further, since \( d(St, t) \leq d(At, t) \), we obtain \( d(St, t) = 0 \) which yields that \( St = t \). Consequently, \( t \) is a common fixed point of the pair \( \{A, S\} \). Similarly, one can prove that \( t \) is a common fixed point of the pair \( \{B, T\} \). Thus, we conclude that \( t \) is a common fixed point of \( A, B, S \) and \( T \).

If \( z \in X \) is also a common fixed point of \( A, B, S \) and \( T \) with \( t \neq z \), then using (ii), we have

\[
d(t, z) = d(At, Bz)
\]

\[
\leq \left[ \varphi \{d^{2p}(St, Tz), d^{p}(St, At).d^{p}(Tz, Bz), d^{p}(St, Bz).d^{p}(Tz, At),
\right. \\
\left. d(St, At).d^{2p-1}(Tz, At), d^{2p-1}(St, Bz).d(Tz, Bz)\} \right]^{1/2p}
\]
\[
= \left[ \varphi \{ d^{2p}(t, z), 0, d^{2p}(t, z), 0, 0 \} \right]^{1/2p} \\
\leq \left[ \gamma \{ d^{2p}(t, z) \} \right]^{1/2p} \\
< d(t, z), \text{ a contradiction.}
\]

Therefore, \(d(t, z) = 0\) which implies that \(t = z\). This completes the proof.

From Theorem 3.4.2, we deduce the following corollary.

**Corollary 3.4.3.** Let \(A, B, S\) and \(T\) be self-maps as in Lemma 3.4.1 satisfying (i), (iii) and (iv), and condition (3.4.1)

\[
d^{2p}(Ax, By) \leq \alpha d^{2p}(Sx, Ty) + \frac{\beta}{2} \max \{ d^p(Sx, Ax).d^p(Ty, By), \\
\quad d^p(Sx, By).d^p(Ty, Ax), d(Sx, Ax).d^{2p-1}(Ty, Ax), \\
\quad d^{2p-1}(Sx, By).d(Ty, By) \} \quad \cdots (3.4.1)
\]

for all \(x, y \in X\), where \(p \geq 1\); \(\alpha, \beta > 0\) and \(\alpha + \beta < 1\).

If the pairs \(\{A, S\}\) and \(\{B, T\}\) are weakly \(S\)-biased and weakly \(T\)-biased respectively, then \(A, B, S\) and \(T\) have a unique common fixed point.

It is to be noted that condition (3.4.1) is a special case of condition (ii) with

\[
\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \frac{\beta}{2} \max \{ t_2, t_3, t_4, t_5 \}
\]

for all \(t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_0^+\), where \(p \geq 1\); \(\alpha, \beta > 0\) and \(\alpha + \beta < 1\), and then the above corollary follows immediately from Theorem 3.4.2.

**Remark 3.4.4.** Contractive condition (ii) of Theorem 3.4.2 is more general than that of Theorem 2.11 [111].

**Remark 3.4.5.** For \(p = 1\), contractive condition (ii) of Theorem 3.4.2, reduces to condition (3.2) (Theorem 3.5 of [87]).
Remark 3.4.6. For $p = 1$ and $A = B$, contractive condition (ii) of Theorem 3.4.2, reduces to condition (ii) (Theorem 2.5 of [86]).

In lieu of weakly biased in Theorem 3.4.2, we employ weakly biased of type (A) in the following theorem.

**Theorem 3.4.7.** Let $A, B, S$ and $T$ be self-maps as in Lemma 3.4.1 satisfying (i)-(iv). If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly $S$-biased of type (A) and weakly $T$-biased of type (A) respectively, then $A, B, S$ and $T$ have a unique common fixed point.

**Proof.** From [111], weakly biased and weakly biased of type (A) are invariant. Therefore, the result follows immediately from Theorem 3.4.2.

We obtain the following corollary.

**Corollary 3.4.8.** Let $A, B, S$ and $T$ be self-maps as in Lemma 3.4.1 satisfying (i), (iii), and (iv), and condition (3.4.1). If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly $S$-biased of type (A) and weakly $T$-biased of type (A) respectively, then $A, B, S$ and $T$ have a unique common fixed point.

Its proof parallels to that of Theorem 3.4.7 as condition (3.4.1) is a special case of condition (ii) with

$$\varphi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \frac{\beta}{2} \max\{t_2, t_3, t_4, t_5\}$$

for all $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_0^+$, where $p \geq 1; \alpha, \beta > 0$ and $\alpha + \beta < 1$.

Now, we furnish the following example in support of Theorem 3.4.2.

**Example 3.4.9.** Let $X = [0, 1]$ be as in Example 3.3.1. Define maps $A, B, S, T : X \to X$ by

$$Ax = Bx = \begin{cases} x/2 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

and

$$Sx = Tx = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Clearly $AX = BX = [0, 1/2] \subset [0,1] = SX = TX$ which is closed.
Consider \( \varphi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \frac{\beta}{2} \max\{t_2, t_3, t_4, t_5\} \) for all \( t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}^+_0 \), where \( p \geq 1; \) \( \alpha, \beta > 0 \) and \( \alpha + \beta < 1 \). Obviously, \( \varphi \) satisfies all the required conditions. In particular, we take \( \alpha = 1/3 \) and \( \beta = 1/2 \).

Now, if \( x, y \in [0, 1) \), then \( d^{2p}(Ax, By) = \frac{1}{2^{3p}} d^{2p}(Sx, Ty) \). And, if \( x \in [0, 1) \) and \( y = 1 \), then \( d^{2p}(Ax, By) = d^{2p}(Sx, Ax) \). Further, if \( x = 1 \) and \( y = 1 \), then \( d^{2p}(Ax, By) = 0 \).

In all cases, we obtain

\[
d^{2p}(Ax, By) \leq \frac{1}{3} d^{2p}(Sx, Ty) + \frac{1}{4} \max\{d^p(Sx, Ax), d^p(Ty, By), d^p(Sx, By) d^p(Ty, Ax), d(Sx, Ax) d^{2p-1}(Ty, Ax), d^{2p-1}(Sx, By) d(Ty, By)\}.
\]

Let \( \{x_n\} \subseteq X \) be a sequence such that \( Ax_n, Sx_n \rightarrow t \) for some \( t \in X \). For this, let \( x_n \rightarrow 0 \) and \( x_n > 0 \) for all \( n \in \mathbb{N} \). Then \( Ax_n = x_n / 2 \rightarrow 0 = t \) and \( Sx_n = x_n \rightarrow 0 = t \).

We see that \( Ax = Sx \) iff \( x = 0 \). Now, we get \( A(0) = 0 = S(0) \). As \( SA(0) = S(0) = 0 \) and \( AS(0) = A(0) = 0 \), \( d(SA(0), S(0)) \leq d(AS(0), A(0)) \). Therefore, the pair \( \{A, S\} \) is weakly \( S \)-biased. One can ascertain that the pair \( \{B, T\} \) is weakly \( T \)-biased. Thus, all the conditions of Theorem 3.4.2 are satisfied and 0 is the unique common fixed point of \( A, B, S \) and \( T \).

One can verify the validity of Theorem 3.4.7 with Example 3.4.9.