CHAPTER 3

HAAR WAVELET SERIES

3.1 INTRODUCTION

This chapter presents a brief introduction to Haar wavelet series algorithm and also includes the formulation of the proposed method. Certain useful properties of Haar wavelet method are also discussed. At the end of this chapter, Single Term Haar Wavelet Series (STHWS) method is formulated to solve dynamic equation of motion of the manipulator.

3.2 HAAR WAVELET SERIES METHOD

The Haar wavelet was initiated and independently developed by Haar in the early nineteen tens. In recent years, the Haar theory has been innovated and applied to various fields in engineering and science. Haar wavelets are the simplest wavelets among various types of wavelets. Haar wavelets are step functions (piecewise constant functions) on the real line that can take only three values. Haar wavelets, like the well-known Walsh functions, form an orthogonal and complete set of functions representing discretized functions and piecewise constant functions. A function is said to be piecewise constant if it is locally constant in connected regions. The Haar transform is one of the earliest examples of compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various
generalizations have been published and used. They were intended to adopt this concept in some practical applications as well as to extend its applications to different classes of signals. Haar functions appear to be very effective in many applications like, image coding, edge extraction and binary logic design. After discretizing the differential equations in a conventional manner like the finite difference approximation, wavelets can also be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

Recently, Haar wavelets have been applied in signal processing in communication research and also in physics research proving to be excellent mathematical tools. The initial work in state analysis via Haar wavelets was done by Chen and Hsiao (1997) who derived a Haar operational matrix for the integrals of the Haar wavelets vector and paved the way for the Haar analysis of the dynamic system. Later, Hsiao (1997) developed the Haar product matrix and coefficient which have been applied to various problems such as the state analysis of linear time-delayed systems and linear stiff systems; the related algorithms can be implemented easily. The main characteristic of this technique is that it converts a differential equation into an algebraic equation; hence, the solution identification and optimization procedures are either reduced or simplified.

The orthogonal set of Haar wavelets $h_i(t)$ is shown in Figures 3.1-3.8. This is a group of square waves with magnitude ±1 in some intervals and magnitude 0, elsewhere. The presence of the zeros makes the Haar transform faster than the transforms associated with the other square functions such as the Walsh functions. In Figures 3.1-3.4, the top line $h_0(t) = 1, 0 \leq t < 1$ is called the scaling function. The second line $h_1(t)$ is the fundamental square wave, which spans the whole interval (0,1). The third and fourth lines $h_2(t)$ and $h_3(t)$ are generated from $h_1(t)$ via two operations: translation and dilation.
Specifically $h_2(t)$ is obtained from $h_1(t)$ via dilation, meaning that $h_1(t)$ is compressed from the whole interval (0,1) to the half interval (0,1/2) to generate $h_2(t)$. $h_3(t)$ is the same as $h_2(t)$ but is shifted (translated) to the right by $\Delta t = 1/2$.

In general, $h_n(t) = h_1(2^j t - k), \quad n = 2^j t + k, \quad j \geq 0, \quad 0 \leq k < 2^j$ (3.1)

Therefore, it can be stated that the Haar set contains a family of single square wavelets. This orthogonal basis is not a recent invention. It is reminiscent of the Walsh basis. Each Walsh function contains many wavelets to fill completely in the interval (0,1) and form a global basis; each Haar wavelet contains just one wavelet during some subinterval of time and remains zero elsewhere. Therefore, the Haar set forms a local basis.

### 3.2.1 Preliminary Works

Any function $y(t)$ which is square integrable in the interval (0,1) can be expanded into a Haar series of infinite terms.

$$y(t) = \sum_{0}^{\infty} c_i h_i(t), \quad t \in (0,1)$$ (3.2)

Using the orthogonality relationship of Haar wavelets,

$$\int_{0}^{1} h_i(t) h_j(t) dt = 2^{-j} \delta_{ij} = \begin{cases} 2^{-j} & i = l = 2^j + k \\ 0 & i \neq l \end{cases}$$ (3.3)

The Haar coefficients $c_i$ can be determined by

$$c_i = 2^j \int_{0}^{1} y(t) h_i(t) dt$$ (3.4)
Usually, the series expansion Equation (3.2) contains infinite terms for a general function \( y(t) \). If \( y(t) \) is either piecewise constant or may be approximated by piecewise constant segments then Equation (3.2) will be terminated at a finite number of terms; that is,

\[
y(t) \approx \sum_{i=0}^{m-1} c_i \varphi_i(t) = \tilde{c}_m^T \tilde{h}_m(t) \hat{\gamma}(t), \quad t \in [0,1)
\]  

(3.5)

where the subscript \( T \) means transposition and

\[
\tilde{c}_m \triangleq [c_0, c_1, \ldots, c_{m-1}]^T
\]

(3.6)

\[
\tilde{h}_m \triangleq [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T
\]

(3.7)

\( m \) is chosen to be \( 2^j \) for the positive integer \( j \). Define the square Haar matrix of dimension as \( m \times m \) as

\[
H_{m,m} \triangleq [\tilde{h}_m(1/2m), \tilde{h}_m(3/m), \ldots, \tilde{h}_m((2m-1)/2m)].
\]

(3.8)

Therefore, Equation (3.5) can be represented as

\[
[\tilde{\gamma}(1/2m), \tilde{\gamma}(3/m), \ldots, \tilde{\gamma}((2m-1)/2m)] = \tilde{c}_m^T H_{m,m}.
\]

(3.9)

It obvious that

\[
\tilde{c}_m^T = [\tilde{\gamma}(1/2m), \tilde{\gamma}(3/m), \ldots, \tilde{\gamma}((2m-1)/2m)] H_{m,m}^{-1}.
\]

(3.10)

Equation (3.10), called the forward transform, transforms the time function \( \tilde{\gamma}(t) \) into the coefficient vector \( \tilde{c}_m^T \); Equation (3.9), called the inverse transform, recovers \( \tilde{\gamma}(t) \) from \( c_m^T \). Since \( H_{m,m} \) and \( H_{m,m}^{-1} \) contain many zeros,
the Haar transform is much faster than the Fourier transform, and even faster than the Walsh transform.

For example, consider the case \( m = 4 \). The Haar wavelets can be expressed as

\[
\begin{align*}
    h_0(t) &= \langle 1, 1, 1, 1 \rangle \\
    h_1(t) &= \langle 1, 1, -1, -1 \rangle \\
    h_2(t) &= \langle 1, -1, 0, 0 \rangle, \\
    h_3(t) &= \langle 0, 0, 1, -1 \rangle,
\end{align*}
\]

where \( \langle \alpha_0, \alpha_1, \ldots, \alpha_{m-1} \rangle \) means that the function has the value \( \alpha_i \) at \( t \in [i/m, (i+1)/m) \), \( i = 0, 1, \ldots, m-1 \). Suppose that \( \tilde{y}(t) = \langle 8, 6, 7, 3 \rangle \). Then it can be represented by

\[
\begin{align*}
    \tilde{y}(t) &= 6h_0(t) + h_1(t) + h_2(t) + 2h_3(t) = \tilde{c}_4^T H_{4,4}, \\
    H_{4,4} &= [\tilde{h}_4(1/8), \tilde{h}_4(3/8), \tilde{h}_4(5/8), \tilde{h}_4(7/8)].
\end{align*}
\]

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & 1 & -1 & -1 \\
    1 & -1 & 0 & 0 \\
    0 & 0 & 1 & -1
\end{bmatrix}
\]

The Haar coefficient \( c_i \), can be obtained by applying Equation (3.10) directly,

\[
\tilde{c}_4^T [c_0, c_1, c_2, c_3] = \tilde{y}(t)H^{-1}_{m,m} = [6, 1, 1, 2].
\]
In practical applications, a small number of terms increase the calculation speed and saves memory storage; a large number of terms improves resolution accuracy. Therefore, a trade-off between calculation speed, memory saving, and resolution accuracy must be considered in this analysis.

### 3.3 Properties of Haar Wavelets

In the wavelet analysis for a dynamic system, all relevant functions need to be transformed into Haar series. Since differentiation of Haar wavelets results always in impulse functions, this must be avoided; instead, the integration of Haar wavelets is preferred. In turn the integration of Haar wavelets should be expandable into Haar series with Haar coefficient matrix $P$

\[
H_{4,4}^{-1} = \begin{bmatrix}
1 & 1 & 2 & 0 \\
1 & 1 & -2 & 0 \\
1 & -1 & 0 & 2 \\
1 & -1 & 0 & -2
\end{bmatrix}
\]

The $m \times m$ square matrix $P$ is called the operational matrix of integration and $\tilde{h}_m(t)$ is defined in Equation (3.7), with

\[
\int_0^t \tilde{h}_m(\tau) d\tau \approx P_{m,m} \tilde{h}_m(t), \quad t \in [0,1)
\]

(3.11)

And $H_{m,m}$ defined in Equation (3.8)
In the integration of the adjoint equations, it is necessary to integrate Haar wavelets from 1 to \( t \). Figure 3.1 shows the backward integration functions \( \int_1^t \tilde{h}_m(\tau)d\tau \).

In general,

\[
\int_1^t \tilde{h}_m(\tau)d\tau \approx S_{m,m} \tilde{h}_m(t) \quad t \in [0,1]
\]

(3.13)

where

\[
S_{m,m} = (1/2m) \begin{bmatrix} 2mS_{(m/2),(m/2)} & -H_{(m/2),(m/2)} \\ H_{(m/2),(m/2)}^{-1} & 0_{(m/2),(m/2)} \end{bmatrix}, \quad S_{1,1} = 1/2
\]

(3.14)

With \( H_{m,m} \) defined in Equation (3.8). From the comparison of \( P_{m,m} \) in Equation (3.12) with \( S_{m,m} \) in Equation (3.14), it can be seen that these two matrices are the same for any \( m \), except \( m = 2 \); indeed, \( P_{1,1} = 1/2 \), while \( S_{1,1} = -1/2 \). \( S_{m,m} \) called the operational matrix of backward integration. Figures 3.1-3.8 also show that

\[
\int_1^t h_0(\tau)d\tau = \int_1^t h_0(\tau)d\tau - 1
\]

(3.15)

\[
\int_1^t h_i(\tau)d\tau = \int_1^t h_i(\tau)d\tau \quad i = 1,2,\ldots,m-1
\]

(3.16)

In the study of time-varying systems via Haar wavelets, it is needed to evaluate \( \tilde{h}_m(t)\tilde{h}_m^T(t) \). Let the product of \( \tilde{h}_m(t) \) and \( \tilde{h}_m^T(t) \) be called the Haar product matrix \( M_{m,m}(t) \) (ie),

\[
\tilde{h}_m(t)\tilde{h}_m^T(t) \Delta M_{m,m}(t)
\]

(3.17)
The basic multiplication properties of Haar wavelets are as follows:

(i) For any two Haar wavelets $h_n(t)$ and $h_l(t)$, with $n < l$,

$$h_n(t)h_l(t) = \rho h_l(t), \quad (3.18)$$

$$\rho = h_n(2^{-j}(q + 1/2))$$

$$= \begin{cases} 
1, & 2^{i-j}k \leq q < 2^{i-j}(k+1/2) \\
-1, & 2^{i-j}(k+1/2) \leq q < 2^{i-j}(k+1) \\
0, & \text{otherwise} 
\end{cases} \quad (3.19)$$

$$n = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^i,$$

$$l = 2^i + q, \quad i \geq 0, \quad 0 \leq q < 2i. \quad (3.20)$$

(ii) The square of any Haar wavelet is a block pulse with magnitude of 1 during both the positive and negative half waves of the Haar wavelet

3.3.1 Remark

Equation (3.18) means that, when $n < l$ the product $h_n(t)h_l(t)$ equals $h_l(t)$ if $h_l(t)$ occurs during the first positive half wave of $h_n(t)$; and it equals $-h_l(t)$ if $h_l(t)$ occurs during the second negative half wave of $h_n(t)$. The product $h_n(t)h_l(t)$ must be zero when these two wavelets have no overlaps.

In the case of $n$ and $l$ defined in Equation (3.20), with $i = j$, but $q \neq k$, meaning that $h_n(t)$ and $h_l(t)$ have the same dilations but different shifts, then,

$$h_n(t)h_l(t) = 0. \quad (3.21)$$
For notation simplification, let
\[
\tilde{h}_a(t) = \sum [h_0(t), h_1(t), \ldots, h_{m/2-1}(t)]^T = \tilde{h}_{m/2}(t),
\] (3.22)
\[
\tilde{h}_b(t) = \sum [h_{m/2}(t), h_{m/2+1}(t), \ldots, h_{m-1}(t)]^T.
\] (3.23)

The matrix \( M_{m,m}(t) \) in Equation (3.17) can be derived easily as follows
\[
M_{m,m}(t) = \begin{bmatrix}
M_{(m/2), (m/2)}(t) & H_{(m/2), (m/2)} \text{diag}[\tilde{h}_b(t)] \\
\text{diag}[\tilde{h}_a(t)]H^{-T}_{(m/2), (m/2)} & \text{diag}[H^{-1}_{(m/2), (m/2)} \tilde{h}_a(t)]
\end{bmatrix},
\] (3.24a)
\[
M_{1,1}(t) = h_0(t).
\] (3.24b)

With the above recursive formulas, it is possible to evaluate \( M_{m,m}(t) \) for any \( M = 2^j, j \) a positive integer. The matrix \( M_{m,m}(t) \) satisfies
\[
M_{m,m}(t)\tilde{c}_m = C_{m,m}(t)\tilde{h}_m(t)
\] (3.25)
where the coefficient vector \( C_m \) is defined in Equation (3.6). By Equation (3.24) and Equation (3.25), the coefficient matrix \( C_{m \times m} \) has the following form:
\[
C_{m,m}(t) = \begin{bmatrix}
C_{(m/2), (m/2)} & H_{(m/2), (m/2)} \text{diag}[\tilde{c}_b] \\
\text{diag}[\tilde{c}_a]H^{-1}_{(m/2), (m/2)} & \text{diag}[\tilde{c}_b^T H_{(m/2), (m/2)}]
\end{bmatrix}
\] (3.26a)
\[
C_{1,1}(t) = C_0,
\] (3.26b)
where
\[
\tilde{c}_a = \sum [c_0, c_1, \ldots, c_{m/2-1}]^T = \tilde{c}_{m/2},
\] (3.27)
Equation (3.25) is an important relationship for studying time-varying systems.
3.4 SINGLE TERM HAAR WAVELET SERIES

With the STHWS approach, in the first interval, the unknown function and its derivative are expanded as STHWS in the normalized interval \( \tau \in [0, 1] \) which corresponds to \( t \in [0, 1/m] \) by taking \( \tau = mt \), \( m \) being any integer. In STHWS, the operational matrix \( P \) in Equation (3.12) becomes \( P = 1/2 \). The single-term Haar wavelets method is an extension of the single-term algorithm \( P_{1,1} = 1/2 \), which avoid the inverse of the big matrix induced by the Kronecker product. This approach is applicable for any transform with
piecewise constant basis and one can take the advantages of its fast, local, and multiplicative properties to solve any kind of problems.

3.5 CONCLUDING REMARKS

The Haar wavelet orthogonal function and their integration matrixes have been introduced in this chapter. The Haar wavelets approach is a powerful tool for numerical analysis. The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. Of all well the known wavelets, the Haar wavelet is the simple one. This new approach, formulated for Haar wavelet, can also be extended to other wavelets. Compared with other numerical methods, the Haar wavelet has two advantages, namely

(i) High accuracy, fast transformation and possibility of implementing of fast algorithms when compared with other known methods.

(ii) It is a computer – oriented method, because no imaginary numbers are involved in the calculation.

It is worth mentioning that the Haar solution provides excellent results even for small values of \( m \) (\( m = 16 \)). For larger number of terms in Haar series ‘\( m \)’ (i.e. \( m = 32, m = 64, m = 128 \) and \( m = 256 \)) gives accurate results. The proposed research uses the STHWS algorithm by taking \( P = \frac{1}{2} \) to solve the dynamic equation of motion of the robot manipulator. The proposed research uses the above advantages, applied to solve the dynamic equations of motion. The formulation of dynamic equations of motion has been discussed in the subsequent Chapter 4.