Appendix I: The Radon Transform

Introduction

The Radon transform is a fundamental tool in many areas. For example, in reconstruction of an image from its projections (CT Scanning). An important problem in digital image processing is to reconstruct a cross-section of an object from several images of its projections. A projection is a shadowgram obtained by illuminating an object by penetrating radiation. The below figure shows a typical method for obtaining projections. Each horizontal line shown in this figure is a one-dimensional projection of a horizontal slice of the object. Each pixel of the projected image represents the total absorption of the X-ray along its path from the source to the detector. By routing the source-detector assembly around the object, projections for several different angles can be obtained. The goal of image reconstruction from projections is to obtain an image of a cross-section of the object from these projections. Imaging systems that generate such slice views are called CT (computerized tomography) scanners [Deans, 2007].

An X-ray CT scanning system and 3D projection system
The Radon transform is the underlying fundamental concept used for CT scanning, as well for a wide range of other disciplines, including radar imaging, geophysical imaging, nondestructive testing and medical imaging.

The 3D Radon transform is defined using 1D projections of a 3D object $f(x, y, z)$ where these projections are obtained by integrating $f(x, y, z)$ on a plane, whose orientation can be described by a unit vector $\vec{a}$. Geometrically, the continuous 3D Radon transform maps a function in $\mathbb{R}^3$ into the set of its plane integrals in $\mathbb{R}^3$. A 3D function $f(\vec{x}) \triangleq f(x, y, z)$ and a plane whose representation is given using the normal $\vec{a}$ and the distance $s$ of the plane from the origin, the 3D continuous Radon transform of $f$ for this plane is defined by [Helgason, 1999] [Alexander & Ramm, 1996]

$$
\mathcal{R}f(\vec{a}, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{x}) \delta(\vec{x}^T \vec{a} - s) d\vec{x}
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) \delta(x \sin \theta \cos \varphi + y \sin \theta \sin \varphi + z \cos \theta - s) dx \, dy \, dz
$$

(1)

where $\vec{x} = [x, y, z]^T$, $a = [\sin \theta \cos \varphi, y \sin \theta \sin \varphi, z \cos \theta]^T$, and $\delta$ is Dirac’s delta function defined by $\delta(x) = 0$, $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x) dx = 1$.

The Radon transform maps the spatial domain $(x, y, z)$ to the domain $(\vec{a}, s)$. Each point in the $(\vec{a}, s)$ space corresponds to a plane in the spatial domain $(x, y, z)$. The 3D Radon transform satisfies the 3D slice theorem, which states that the central slice $\hat{f}(\vec{a})$ in the direction $\vec{a}$ of the 3D Fourier transform of $f(\vec{x})$ equals $\mathcal{R}f(\vec{a}, \xi)$, that is
\[ \hat{Rf}(\vec{a}, \xi) = \hat{f}(\xi \vec{a}) = \hat{f}(\xi \sin \theta \cos \varphi, \xi \sin \theta \sin \varphi, \xi \cos \theta) \] (2)

For modern application it is important to have a discrete analogues of $\mathcal{R}f$ for 3D digital images $I = (I(u,v,w) : -n/2 \leq u,v,w < n/2)$. The definition of the 3D discrete Radon transform should satisfy the following properties:

i. **Algebraic exactness**: The transform should be based on a clear and rigorous definition.

ii. **Geometric Fidelity**: The transform should be based on true geometric planes rather than planes which wrap around or are otherwise non-geometric.

iii. **Speed**: The transform should be rapidly computable.

iv. **Inevitability**: The transform should be invertible on its range. Moreover, there should be a fast reconstruction algorithm.

v. **Parallels with continuum theory**: The transform should obey relations which parallel with those of the continuum theory.

The 3D discrete Radon transform is defined by summing the interpolated samples of a discrete array $I(u,v,w)$ lying on planes which satisfy certain constraints. Formally, given a plane whose explicit equation is $z = s_1 x + s_2 y + t$, we define the operator $R_3 I$ for the plane given in above equation by:

\[
R_3 I(s_1, s_2, t) = \sum_{u=-n/2}^{n/2-1} \sum_{v=-n/2}^{n/2-1} \tilde{I}_3(u, v, s_1 u + s_2 v + t),
\] (3)

where $\tilde{I}_3(u, v, z) = \sum_{w=-n/2}^{n/2-1} I(u, v, w) D_m(z - w), \ u, v = -n2, ..., n2-1, z \in \mathbb{R}$ and $D_m$ is the Dirichlet kernel given by $D_m(t) = \frac{\sin(\pi t)}{m \sin(\pi t/m)}$, $m = 3n + 1$. The notation $\tilde{I}_3$ used, since the we interpolate in the z-direction, which we consider the third direction. We refer to the $x-$, $y-$, $z-$ directions, as the first, second and third directions, respectively. By observation a definition of three types of planes "x-planes","y-planes"", and "z-planes" are defined as below:

i. A plane of the form $x = s_1 y + s_2 z + t$ where $|s_1| \leq 1, |S_2| \leq 1$ is called x-plane.
ii. A plane of the form \( y = s_1 x + s_2 z + t \) where \( |s_1| \leq 1, |S_2| \leq 1 \) is called y-plane.

iii. A plane of the form \( z = s_1 x + s_2 y + t \) where \( |s_1| \leq 1, |S_2| \leq 1 \) is called z-plane.

We define three summation operators, one for each type of plane (x-plane, y-plane, z-plane). Each summation operator takes a plane and an image \( I \) and calculates the sum of the samples from \( I \) on the plane. More precisely, it calculates the sum of the interpolated samples of \( I \) on the plane.