A1.1  **HURWITZ CRITERION**

Consider a polynomial of degree $n$,

$$\delta(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots + \alpha_n s^n$$

$\delta(s)$ is said to be Hurwitz polynomial if and only if all its roots lie in the open left half of the complex plane. It has the two following properties.

**Property A.1** If $\delta(s)$ is said to be Hurwitz polynomial then all its coefficients are non-zero and have the same sign.

**Property A.2** If $\delta(s)$ is a Hurwitz polynomial of degree $n$, then $\arg[\delta(j\omega)]$, also called the phase of $\delta(j\omega)$, is a continuous and strictly increasing function of $\omega$ on $(-\infty, \infty)$. Moreover the net increase in phase from $-\infty$ to $\infty$ is

$$\arg[\delta(+j\infty)] - \arg[\delta(+j\infty)] = n\pi.$$

A1.2  **MIKHAILOV’S THEOREM**

Consider a polynomial of degree $n$,

$$\delta(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots + \alpha_n s^n \quad (A1.1)$$
When \( n \) is even and \( n=2m \), the real and imaginary part of the polynomial are defined as

\[
\delta_r(\omega) = a_0 - a_2 \omega^2 + a_4 \omega^4 \ldots \ldots \ldots (\omega)^m \alpha_{2m} \omega^{2m} \quad (A1.2)
\]

\[
\delta_i(\omega) = a_1 \omega - a_3 \omega^3 + a_5 \omega^5 \ldots \ldots \ldots (\omega)^{m-1} \alpha_{2m-1} \omega^{2m-1} \quad (A1.3)
\]

When \( n \) is odd and \( n=2m+1 \), the real and imaginary part of the polynomial are defined as

\[
\delta_r(\omega) = a_0 - a_2 \omega^2 + a_4 \omega^4 \ldots \ldots \ldots (\omega)^{m} \alpha_{2m} \omega^{2m} \quad (A1.4)
\]

\[
\delta_i(\omega) = a_1 \omega - a_3 \omega^3 + a_5 \omega^5 \ldots \ldots \ldots (\omega)^{m} \alpha_{2m+1} \omega^{2m+1} \quad (A1.5)
\]

According to Mikhailov criterion, the polynomial (A1.1) is stable if and only if

1. The vector \( \delta(j\omega) \) rotates through an angle \( \theta = \frac{n\pi}{2} \) as \( \omega \) varies from 0 to +\( \infty \), (The reason is that \( \xi = -\cos\theta \). At critical stable condition \( \xi = 0 \) and therefore \( \theta = 90^\circ \). If the order of the system is \( n \), then the argument of vector \( \delta(j\omega) \) is increased from 0 and approaches an angle of \( \frac{n\pi}{2} \) i.e. the vector makes \( n/4 \) revolutions counterclockwise.

2. The vector \( \delta(j\omega) \) does not pass through the origin of coordinates as \( \omega \) varies from 0 to \( \infty \).
A real polynomial \( \delta(s) \) is Hurwitz if and only if the frequency plot moves strictly counterclockwise and goes through \( n \) quadrants.

### A1.3 INTERLACING THEOREM

Consider a polynomial of degree \( n \),

\[
\delta(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots + \alpha_n s^n
\]

Here the even and odd parts of the real polynomial are defined as:

\[
\begin{align*}
\delta_{\text{even}}(s) &= \alpha_0 + \alpha_2 s^2 + \alpha_4 s^4 + \ldots, \\
\delta_{\text{odd}}(s) &= \alpha_1 s + \alpha_3 s^3 + \alpha_5 s^5 + \ldots.
\end{align*}
\]

Define,

\[
\begin{align*}
\delta^r(\omega) &= \delta_{\text{real}}(j\omega) = \alpha_0 - \alpha_2 \omega^2 + \alpha_4 \omega^4 + \ldots, \\
\delta^i(\omega) &= \delta_{\text{imaginary}}(j\omega) = \frac{\delta_{\text{imaginary}}(j\omega)}{j\omega} = \alpha_1 - \alpha_3 \omega^2 + \alpha_5 \omega^4 + \ldots.
\end{align*}
\]
**Definition 1:** Consider the degree of polynomial is even (i.e., \( n \) is even and \( n=2m \)). The real and imaginary part of the polynomial is defined as,

\[
\delta^r(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 + \ldots + (-1)^m a_{2m}\omega^{2m}
\]

\[
\delta^i(\omega) = a_1 - a_3\omega^2 + a_5\omega^4 + \ldots + (-1)^{m-1} a_{2m-1}\omega^{2m-2}
\]

The polynomial satisfies interlacing property if

a) \( a_{2m} \) and \( a_{2m-1} \) have same sign.

b) All the roots of real and imaginary part are real and distinct and the \( m \) positive roots of real part together with \( m-1 \) positive roots of imaginary part interlace in the following manner

\[
0 < \omega_{r,1} < \omega_{i,1} < \omega_{r,2} < \ldots < \omega_{r,m-1} < \omega_{i,m-1} < \omega_{r,m} < \omega_{i,m}
\]

**Definition 2:** Consider the degree of polynomial is odd (i.e., \( n \) is odd and \( n=2m+1 \)). The real and imaginary part of the polynomial is defined as,

\[
\delta^r(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 + \ldots + (-1)^m a_{2m}\omega^{2m}
\]

\[
\delta^i(\omega) = a_1 - a_3\omega^2 + a_5\omega^4 + \ldots + (-1)^m a_{2m+1}\omega^{2m}
\]

The polynomial satisfies interlacing property if

a) \( a_{2m+1} \) and \( a_{2m} \) have same sign.

b) All the roots of real and imaginary part are real and distinct and the \( m \) positive roots of real part together with \( m \) positive roots of imaginary part interlace in the following manner

\[
0 < \omega_{r,1} < \omega_{i,1} < \omega_{r,2} < \ldots < \omega_{r,m-1} < \omega_{i,m-1} < \omega_{r,m} < \omega_{i,m}
\]
An alternate description of the interlacing property is as follows:

a) The leading coefficients of $\delta^{\text{even}}(s)$ and $\delta^{\text{odd}}(s)$ are of the same sign, and

b) All the zeros of $\delta^{\text{even}}(s)=0$ and of $\delta^{\text{odd}}(s)=0$ are distinct, lie on the imaginary axis and alternate along it.

Further, the polynomial with time delay is considered for investigation. In this context, Bhattacharyya et al (1995) proposed an algorithm applicable to quasi polynomial, which is described below:

In control problems involving time delays, we often deal with characteristic equation of the form

$$\delta(s) = d(s) + e^{sT_1}n_1(s) + e^{sT_2}n_2(s) + \ldots + e^{sT_m}n_m(s)$$  \hspace{1cm} (A1.6)

where $d(s), n_i(s)$ for $i = 1, 2, \ldots, m$, are polynomials with real coefficients.

Instead of (A1.6), the quasi polynomial (A1.7) is considered for further investigation.

$$\delta^*(s) = e^{sT_m}\delta(s) = e^{sT_m}d(s) + e^{s(T_m-T_1)}n_1(s) + e^{s(T_m-T_2)}n_2(s) + \ldots + e^{s(T_m-T_m)}n_m(s)$$  \hspace{1cm} (A1.7)

**Theorem 1:** $\delta^*(j\omega) = \delta_r(j\omega) + j\delta_i(j\omega)$ where $\delta_r(\omega)$ and $\delta_i(\omega)$ represent, respectively, the real and imaginary parts of $\delta^*(j\omega)$.

Under the assumption that A1: deg $[d(s)] = n$ and deg $[n_i(s)] < n$ and

A2: $0 < T_1 < T_2 < \ldots < T_m$,

$\delta^*(s)$ is stable if and only if,
i) $\delta'_{r}(\omega_{0})\delta_{i}(\omega_{0})-\delta'_{i}(\omega_{0})\delta_{r}(\omega_{0})>0$; for some $\omega_{0} \in (-\infty, \infty)$

ii) The roots of $\delta_{r}(\omega)$ and $\delta_{i}(\omega)$ should be real and interlacing.

We note that the interlacing condition ii) needs to be verified up to a finite frequency. This follows from the fact that the phasors of $\frac{n_{i}(j\omega)}{d(j\omega)}$ tend to zero as $\omega$ tend to $\infty$. This ensures that the quasi-polynomial has the monotonic phase property for sufficiently large $\omega$. Therefore, the interlacing condition needs to be verified only for low frequency range only, as the required limits of controller parameters cannot lie in the high frequency range.

The crucial step in applying the above theorem 1 to check the stability is to ensure that $\delta_{r}(\omega)$ and $\delta_{i}(\omega)$ have only real roots. Such a property can be ensured by using the following result, also due to Pontryagin.

**Theorem 2:** Let $M$ and $N$ denote the highest powers $s$ and $e^{s}$ respectively in $\delta^{r}(s)$. Let $\eta$ be an appropriate constant such that the coefficients of terms of the highest degree in $\delta_{r}(\omega)$ and $\delta_{i}(\omega)$ do not vanish at $\omega=\eta$. Then for equations $\delta_{r}(\omega) = 0$ or $\delta_{i}(\omega) = 0$ to have only real roots, it is necessary and sufficient that in the interval $[-2l\pi+\eta, \omega \leq (2l\pi+\eta)]$ for $l=1,2,3\ldots$ $\delta_{i}(\omega)$ or $\delta_{r}(\omega)$ have exactly $[4/N]+M$ real roots with a sufficiently large $l$.

**A1.4 STOJIC’S RELATIVE STABILITY INVESTIGATION BASED ON DAMPING COEFFICIENT SPECIFICATION**

This approach first transforms the regional pole constraints problem into stability problem based on the coordinate rotational transform, (i.e., rotate
the imaginary axis by a fixed positive angle $\varphi$ about the origin as shown in Figure 3.2(b) using the concepts of Chebyshev function.

The characteristic equation is

$$\delta(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots + \alpha_n s^n = 0$$  \hspace{1cm} (A1.8)

The equation (A1.8) can be represented as,

$$\delta(s) = \sum_{k=0}^{n} \alpha_k s^k$$  \hspace{1cm} (A1.9)

where the coefficients $\alpha_k$ are real.

If the root of the equation (A1.9) is located in desired dominant place, it can be stated as,

$$s = -\omega_n \xi + j \omega_n \sqrt{1-\xi^2} = \omega_n \left( -\xi + j \sqrt{1-\xi^2} \right)$$

$$= \omega_n \left( \cos \theta + j \sin \theta \right)$$

$$s = \omega_n e^{j\theta}$$  \hspace{1cm} (A1.10)

Substitute (A.10) in equation (A1.9)

$$\delta(s) = \sum_{k=0}^{n} \alpha_k \left( e^{j\theta} \omega_n \right)^k$$  \hspace{1cm} (A1.11)

The coefficients $\alpha_k e^{jk\theta}$ can be converted to the form

$$\alpha_k e^{jk\theta} = b_k + j c_k$$  \hspace{1cm} (A1.12)

where

$$b_k = \alpha_k T_k (-\xi) = (-1)^k \alpha_k T_k (\xi)$$

$$c_k = \alpha_k \sqrt{1-\xi^2} U_k (-\xi) = (-1)^{k+1} \alpha_k \sqrt{1-\xi^2} U_k (\xi)$$
These functions are obtained according to the following recurrence formulae:

\[ T_{k+1}(\xi) - 2\xi T_k(\xi) + T_{k-1}(\xi) = 0 \]

\[ U_{k+1}(\xi) - 2\xi U_k(\xi) + U_{k-1}(\xi) = 0 \]

The initial values are \( T_0(\xi) = 1, T_1(\xi) = \xi, U_0(\xi) = 0, U_1(\xi) = 1 \).

For pertinent of \( \xi \), the chebyshev functions may be computed.

By substituting equation (A1.11) into equation (A1.10), one can obtain the function \( \delta(s) \) and it can be expressed as,

\[ \delta(\omega_n, \xi) = \delta_r(\omega_n, \xi) + j\delta_i(\omega_n, \xi) \]  \hspace{1cm} (A1.12)

where \( \delta_r(\omega_n, \xi) = \sum_{k=0}^{n} (-1)^k \alpha_k \omega_n^k T_k(\xi) \) and

\[ \delta_i(\omega_n, \xi) = \sqrt{1 - \xi^2} \sum_{k=0}^{n} (-1)^{k+1} \alpha_k \omega_n^k U_k(\xi) \]

**A1.5 SILJAK’S RELATIVE STABILITY INVESTIGATION**

This method transforms the regional pole constraints problem into stability problem based on the coordinate parallel transform, (i.e., shift the imaginary axis parallel to itself by an appropriate number of units \( \alpha \) to the left as shown in Figure 3.2(a)) using the concepts of modified version of Chebyshev function.

The characteristic equation can be written as,

\[ \delta(\sigma, \omega) = \delta_r(\sigma, \omega) + j\delta_i(\sigma, \omega) \]  \hspace{1cm} (A1.13)
The real and imaginary part of equation (A1.13) is,

\[ \delta_r (\sigma, \omega) = \sum_{k=0}^{n} a_k X_k \]

\[ \delta_i (\sigma, \omega) = \sum_{k=0}^{n} a_k Y_k \]

Let \( X_k = (-1)^k \omega_n^k T_k (\xi) \), \( Y_k = (-1)^{k+1} \omega_n^k U_k (\xi) \sqrt{1-\xi^2} \)

The values of \( X_k \) and \( Y_k \) are obtained according to the following recurrence formulae:

\[ X_{k+1} = 2X_k Y_k - \left[ X_1^2 + Y_1^2 \right] X_{k-1} \]

\[ Y_{k+1} = 2X_k Y_k - \left[ X_1^2 + Y_1^2 \right] Y_{k-1} \]

The initial values are \( X_0 = 1, X_1 = \sigma, Y_0 = 0, Y_1 = \omega \).
APPENDIX 2

CASE STUDIES

A2.1 VALIDATION OF UNIFIED PROCEDURE FOR SELECTED TITO SYSTEMS

In this section, two typical cases are considered to illustrate the benefits and limitations of our systematic procedure for computation of controller parameters. The main limitation of the proposed approach is the low dimensionality of the parameter space (one or two).

Example A2.1: This example is taken from the paper authored by Keel et al (2000).

![Multivariable system with two masses and two springs](image)

Figure A2.1 Multivariable system with two masses and two springs
Consider the transfer function matrix given as

\[
G_p(s) = \begin{pmatrix}
\frac{s^2+4}{2s^4+8s^2+2} & \frac{s^2+5}{2s^4+8s^2+2} \\
\frac{s^3+4s}{2s^4+8s^2+2} & \frac{s^3+5s}{2s^4+8s^2+2}
\end{pmatrix}
\]

The decentralized controllers may not give the satisfactory responses for the above system. The reason is that the poles and zeros of the system transfer function \(G_{11}(s) = \frac{s^2+4}{2s^4+8s^2+2}\) should lie on the imaginary axis. Hence it is not possible to stabilize the above-mentioned system using two-loop controller. For such a system, the centralized controller will give satisfactory response. Here the centralized controller can be designed by assuming the controller structure as a full matrix with each element as constant gain controller.

The controller structure is \(G_c(s) = \begin{pmatrix} K_{c1} & K_{c2} \\ K_{c3} & K_{c4} \end{pmatrix}\)

The characteristic equation is written as, \(\text{det}(I+G_pG_c) = 0 \quad (A2.1)\)

The equation (A2.1) can be rewritten as \(|I+G_cG_p| = 0;\)

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} K_{c1} & K_{c2} \\ K_{c3} & K_{c4} \end{pmatrix} \begin{pmatrix}
\frac{s^2+4}{2s^4+8s^2+2} & \frac{s^2+5}{2s^4+8s^2+2} \\
\frac{s^3+4s}{2s^4+8s^2+2} & \frac{s^3+5s}{2s^4+8s^2+2}
\end{pmatrix}
= 0
\]
The characteristic equation can be rewritten as,

\[ \delta(s) = 4s^8 + s^7(2K_{c_2} + 2K_{c_4}) + s^6(32 + 2K_{c_1} + 2K_{c_3}) + s^5(16K_{c_2} + 18K_{c_4}) + s^4(72 + 16K_{c_1} + 18K_{c_3}) + s^3(34K_{c_2} + 42K_{c_4}) + s^2(32 + 34K_{c_1} + 42K_{c_3}) + s(8K_{c_2} + 10K_{c_4}) + (4 + 8K_{c_1} + 10K_{c_3}) = 0 \]

The real and imaginary part can be written as,

\[ \delta_r(\omega) = (4 + 8K_{c_1} + 10K_{c_3}) - (32 + 34K_{c_1} + 42K_{c_3})\omega^2 + (72 + 16K_{c_1} + 18K_{c_3})\omega^4 - (32 + 2K_{c_1} + 2K_{c_3})\omega^6 + 4\omega^8 = 0 \]

\[ \delta_i(\omega) = (4K_{c_2} + 5K_{c_4})\omega - (17K_{c_2} + 21K_{c_4})\omega^3 + (8K_{c_2} + 9K_{c_4})\omega^5 - (K_{c_2} + K_{c_4})\omega^7 = 0 \]

Here, the systematic procedure for finding the controller gain matrix is as follows:

**Step 1:** For fixed values of \( K_{c_2} \) and \( K_{c_4} \), the roots of imaginary part can be obtained.

\[ \omega = 0 \text{ and } (4K_{c_2} + 5K_{c_4}) - (17K_{c_2} + 21K_{c_4})\omega^2 + (8K_{c_2} + 9K_{c_4})\omega^4 - (K_{c_2} + K_{c_4})\omega^6 = 0. \]

**Step 2:** By substituting the smallest non-negative value of \( \omega \) in the real part of the characteristic polynomial, the value of \( K_{c_3} \) can be obtained for known values \( K_{c_1} \).

\[ K_{c_3} = \frac{\left[ -(4 + 8K_{c_1}) + (32 + 34K_{c_1})\omega^2 - (72 + 16K_{c_1})\omega^4 + (32 + 2K_{c_1})\omega^6 - 4\omega^8 \right]}{(10 - 42\omega^2 + 18\omega^4 - 2\omega^6)} \]
Step 3: In order to ensure the stability, the phase angle contribution of each root of the polynomial and net phase angle contribution of the characteristic polynomial is to be determined. The phase property can be investigated by finding the number of real roots that cross the real and imaginary axis of the S plane for the all-possible combination of controller matrix.

For a particular range of \( G_c(s) = \begin{pmatrix} K_{c_1} & K_{c_2} \\ K_{c_3} & K_{c_4} \end{pmatrix} \) only, the roots of real and imaginary part of characteristic polynomial are real and distinct and also they satisfy the phase property.

Due to the dimensionality of the controller parameter, the proposed algorithm is conservative.

Figure A2.2 The stabilizing set of \( K_{c_4} \) and \( K_{c_3} \) for \( K_{c_1} = 0.0718 \) and \( K_{c_2} = 5.0 \)
Figure A2.3 The stabilizing set of $K_{c_4}$ and $K_{c_3}$ for $K_{c_1} = 0.0718$ and $K_{c_2} = 4.011$

Figure A2.4 The stabilizing set of $K_{c_4}$ and $K_{c_3}$ for $K_{c_1} = 0.0718$ and $K_{c_2} = 3.0$

**Example A2.2:** The system transfer function mentioned in the book authored by Marlin (2000) is considered to illustrate the benefits of the proposed approach.
The system transfer function considered is

\[ G_p(s) = \begin{pmatrix}
\frac{1e^{-s}}{2s+1} & \frac{0.75e^{-s}}{2s+1} \\
\frac{0.75e^{-s}}{2s+1} & \frac{1e^{-s}}{2s+1}
\end{pmatrix} \]

Marlin has found out the allowable range of tuning constants that yield the stable system. The author has assumed the integral time of the controllers constant at 3.0 min to simplify the calculations. For fixed values of \( T_{i1} \) and \( T_{i2} \) \( (T_{i1} = T_{i2} = 3.0) \), the same results are obtained. The stabilizing set is shown in Figure A2.5.

![Figure A2.5 Stable controller gain regions with \( T_{i1} = T_{i2} = 3.0 \)](image)

This procedure provides the solution for a recycling process. Consider the system transfer function for the recycling process given as

\[ G_p(s) = \frac{0.5e^{-10s}}{\frac{5s+1}{0.5}e^{-20s}} \]

(Mohammad Bozorg et.al 2005)

The multivariable Nyquist stability criterion and stability investigation based on characteristic loci plot are the well-known stability...
theories. Each approach has its own merits and limitations. We need to point out here the limitations in existing stability theories. First, we have presented the brief overview of stability theories.

**The Nyquist stability criterion for a multivariable open loop stable process is:**

If a plot of $\text{Det}[I+G_c(s)G_p(s)]$ encircles the origin, the system is closed loop stable. These curves can be quite complex, particularly with higher order systems and with multiple dead times.

A different kind of plot, called a ‘characteristic loci plot’, is sometimes easier to understand. The method is as follows:

i) Specify the controller $G_c(s)=\begin{bmatrix} k_{c1}s+k_{i1} & 0 \\ s & k_{c2}s+k_{i2} \\ 0 & s \end{bmatrix}$

(both structure and tuning)

ii) Pick a specific numerical value of frequency $\omega$. Calculate the complex matrix $Q(j\omega)=G_c(j\omega)G_p(j\omega)$.

iii) Calculate the Eigen values of $Q(j\omega)$. If the system is $N \times N$, (N controlled variables and N manipulated variables) there will be N complex eigen values.

iv) Plot the N Eigen values as frequency is varied from 0 to $\infty$.

v) If the any of the curves encircle the (-1,0) point, the system is closed loop unstable.
APPENDIX 3
DEAD TIME ESTIMATION FOR MODEL BASED FEEDBACK CONTROL SCHEME

A3.1 ESTIMATION OF DEAD TIME FOR CLOSED LOOP STABILITY WITH PROPORTIONAL CONTROLLER

Bequette has suggested the simplified procedure for finding the maximum allowable perturbation, which may occur in time delay. The procedure is outlined in chapter 2. The systematic procedure for finding the upper bound of L for FOPTD model with conventional feedback control scheme is presented in chapter 6. Here, the systematic procedure is extended to estimate the upper bound of time delay for integrating plus time delay system with model based feedback control configuration.

The Plant transfer function considered is \( G_p(s) = \frac{1}{s}e^{-Ls} \), the model transfer function is \( G_m(s) = G_{m1}(s)e^{-s} \) (where \( G_{m1}(s) = \frac{1}{s} \)) and the controller transfer function is \( G_c(s) = 3 \).

Consider a model-based feedback control system as shown in Figure A3.1,

![Figure A3.1 Model-based control of a delayed process](image)

The objective is to determine all those values of L, if any, for which the closed loop system is Hurwitz.

The characteristic equation is
1+G_c(s)G_p(s)-G_c(s)G_m(s)+G_c(s)G_m1(s)=0

When the transfer function of plant, model and controller are substituted in the above equation, the characteristic equation becomes

\[ 1+3\frac{1}{s}e^{-Ls}-3\frac{1}{s}e^s+3\frac{1}{s} = 0 \]

The equation can be rewritten as, \((s+3)+3e^{-ls}-3e^s = 0\)

\[ (s+3)e^{(l+1)s}+3e^s-3e^ls = 0 \]

\[ (s+3)e^ls-3e^ls = -3e^s \]

\[ e^ls\left[(s+3)e^s-3\right] = -3e^s \]

\[ e^ls = \frac{-3e^s}{(s+3)e^s-3} \]

\[ e^ls = \frac{-3}{(s+3)-3e^s} \]

\[ \cos(L\omega)+jsin(L\omega) = \frac{-3}{(j(\omega+3)-3(\cos(\omega)-jsin(\omega)))} \]

\[ \cos(L\omega)+jsin(L\omega) = \frac{-3}{(3-3\cos(\omega))+j(\omega+3\sin(\omega)))} \]

\[ \cos(L\omega)+jsin(L\omega) = \frac{-3}{\left[(3-3\cos(\omega))+j(\omega+3\sin(\omega)))\right] \left[(3-3\cos(\omega))-j(\omega+3\sin(\omega)))\right] \]

\[ \cos(L\omega)+jsin(L\omega) = \frac{-3}{\left[(3-3\cos(\omega))^2+(\omega+3\sin(\omega))^2\right] \left[(3-3\cos(\omega))-j(\omega+3\sin(\omega))\right] \]
The real and imaginary parts can be written as,

\[
\cos(L\omega) = \left(3-3\cos(\omega)\right)\frac{-3}{\left[\left(3-3\cos(\omega)\right)^2+(\omega+3\sin(\omega))^2\right]} \tag{A3.1}
\]

\[
\sin(L\omega) = \left(\omega+3\sin(\omega)\right)\frac{3}{\left[\left(3-3\cos(\omega)\right)^2+(\omega+3\sin(\omega))^2\right]} \tag{A3.2}
\]

First, the real roots of imaginary part are determined. By substituting the roots in (A3.1), the range of L can be found. The equation (A3.2) has two unknown parameters. Hence it is difficult to solve the above equation. In order to make the imaginary part as a function of \(\omega\), one equality constraint is introduced.

The equation for phase cross over frequency is as follows,

\[
\arg\left[\begin{bmatrix}
G_c(j\omega) \\
1-G_c(j\omega)G_m(j\omega)+G_c(j\omega)\frac{1}{(j\omega)}
\end{bmatrix}G_p(j\omega)\right]_{\omega_{pc}} = -\pi
\]

\[
\arg\left[\begin{bmatrix}
3 \\
1-3\frac{1}{(j\omega)}e^{-j\omega}+3\frac{1}{(j\omega)}
\end{bmatrix}\frac{1}{(j\omega)}e^{-Lj\omega}\right]_{\omega_{pc}} = -\pi
\]

\[
\arg\left[\begin{bmatrix}
3 \\
(j\omega)-3e^{-j\omega}+3
\end{bmatrix}e^{-Lj\omega}\right]_{\omega_{pc}} = -\pi
\]

\[
\arg\left[\begin{bmatrix}
3 \\
(j\omega+3)-3e^{-j\omega}
\end{bmatrix}e^{-Lj\omega}\right]_{\omega_{pc}} = -\pi
\]
\[
\text{arg} \left\{ \frac{3 e^{-Lj\omega}}{\left( (j\omega+3)(\cos(\omega)-j\sin(\omega)) \right)} \right\}_{\omega_{pc}} = -\pi
\]

\[
\text{arg} \left\{ \frac{3 e^{-Lj\omega}}{\left( (3-3\cos(\omega)+j(\omega+3\sin(\omega)) \right)} \right\}_{\omega_{pc}} = -\pi
\]

\[-L\omega_{pc} \cdot \text{atan} \left( \frac{\omega_{pc} + 3\sin(\omega_{pc})}{3-3\cos(\omega_{pc})} \right) = -\pi
\]

\[L\omega_{pc} = \pi \cdot \text{atan} \left( \frac{\omega_{pc} + 3\sin(\omega_{pc})}{3-3\cos(\omega_{pc})} \right) \quad (A3.3)
\]

Substituting the equation (A3.3) in (A3.2),

\[\sin \left( \pi \cdot \text{atan} \left( \frac{\omega_{pc} + 3\sin(\omega_{pc})}{3-3\cos(\omega_{pc})} \right) \right) = \frac{3\omega_{pc} + 9\sin(\omega_{pc})}{\left( 3-3\cos(\omega_{pc}) \right)^2 + \left( \omega_{pc} + 3\sin(\omega_{pc}) \right)^2}
\]

The smallest positive real zero is \(\omega_{pc} = 0.7752\); substitute the value of \(\omega_{pc}\) in equation (A3.3).

\[\pi \cdot \text{atan} \left( \frac{0.7752 + 3\sin(0.7752)}{3-3\cos(0.7752)} \right) = L\omega_{pc};
\]

The value of \(L\) is 2.4; substitute the value of \(L\) and \(\omega_{pc}\) in equation (A3.1) and (A3.2) and investigate the stability criterion which has been highlighted in section 6.3.

Alternate approach based on Bequette’s concept is as follows:
By solving the above equation, the value of $\omega_{gc}$ is obtained as 0.7752 rad/sec. At the critically stable condition, both gain cross over and phase cross over frequencies are same. By substituting the value of $\omega_{pc} = 0.7752$ rad/sec in the equation (A3.2), L is obtained as 2.4. The
proposed systematic approach provides same results compared to Bequette’s concept.

APPENDIX 4

STABILIZATION OF NON-COMMENSURATE FOS

In this section, the solution for non-commensurate fractional order system, which has been mentioned in section 5.6, is presented.

A4.1 BACKGROUND

Consider the system transfer function given as

\[ G_p(s) = \frac{1}{a s^\alpha + b s^\beta + c} \]

where a, b and c are real coefficients and \( \alpha \) and \( \beta \) are real.

The well-known DeMoivre’s theorem is,

\[
(j\omega)^\alpha = \omega^\alpha \cos\left(\frac{\alpha \pi}{2}\right) + j\omega^\alpha \sin\left(\frac{\alpha \pi}{2}\right);
\]

\[
(j\omega)^\alpha + b = b + \omega^\alpha \cos\left(\frac{\alpha \pi}{2}\right) + j\omega^\alpha \sin\left(\frac{\alpha \pi}{2}\right)
\]

The key idea is to use the results of well-known stability theories and DeMoivre’s theorem with appropriate modification to present a systematic procedure for the design of fractional order PD controller parameters for non-commensurate fractional order system.

A4.2 PROPOSED ALGORITHM
The procedure for finding the set of fractional order PD controller for which the non-commensurate fractional order polynomial meets gain margin specification is summarized as follows:

**Step 1:** First, the smallest non-negative real zero (excluding zero) of the equation (A4.2) is determined. By substituting the smallest non-negative real zero in equation (A4.1), the value of $K_d$ for which the non-commensurate fractional order polynomial meets gain margin specification is found.

\[
\lambda^2 r^2 + t^2 + A_m K_d \omega^\lambda = 0 \tag{A4.1}
\]

\[-mt + nr = 0 \tag{A4.2}\]

where \( r = \cos\left(\frac{\lambda \pi}{2}\right); t = \sin\left(\frac{\lambda \pi}{2}\right); \]
\[
n = \left( a \omega^a \sin\left(\frac{\alpha \pi}{2}\right) + b \omega^b \sin\left(\frac{\beta \pi}{2}\right) \right) \]

\[
u = \cos\left(\varphi_m\right); v = \sin\left(\varphi_m\right) \text{ and } m = \left( a \omega^a \cos\left(\frac{\alpha \pi}{2}\right) + b \omega^b \cos\left(\frac{\beta \pi}{2}\right) + c \right)\]

**Step 2:** Next, the smallest non-negative real zero of equation (A4.4) for all permissible values of $K_d$ is determined. The set of controller parameters that meet the following condition are the permissible values of $K_d$:

\[
1 + A_m G_p G_c(s) \text{ is Hurwitz, where } G_c(s) \text{ is } K_d s^\lambda.
\]

By substituting the root in equation (A4.3), the value of $K_c$ for all possible values of $K_d$ and given gain margin is found.
\[ m + A_m K_d \omega^\lambda r + A_m K_c = 0 \] (A4.3)

\[ n + A_m K_d \omega^\lambda t = 0 \] (A4.4)

The procedure for finding the set of fractional order PD controller for which the non-commensurate fractional order polynomial meets phase margin specification is summarized as follows:

\[ (mu-nv)r + t(mv+un) + K_d \omega^\lambda \left( r^2 + t^2 \right) = 0 \] (A4.5)

\[ -(mu-nv)t + (mv+un)r = 0 \] (A4.6)

Step 1: The procedure for finding the range of \( K_d \) based on phase margin specification is same as that discussed in section A4.2, except that equations (A4.5) and (A4.6) are used for finding the range of \( K_d \) instead of equations (A4.1) and (A4.2).

\[ (mu-nv) + K_c + K_d \omega^\lambda r = 0 \] (A4.7)

\[ (mv+un) + K_d \omega^\lambda t = 0 \] (A4.8)

Step 2: The procedure for finding the range of \( K_c \) based on phase margin specification for all possible values of \( K_d \) is same as that discussed in section A4.2, except that equations (A4.7) and (A4.8) are used for finding the range of \( K_c \) instead of equations (A4.3) and (A4.4).

A4.3 ILLUSTRATIVE EXAMPLES

The system transfer function mentioned in the paper authored by Nataraj et al (2004) is considered to illustrate the benefits of the proposed approach.
\[ G_p(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1}; \quad G_c(s) = K_c + K_d s^\lambda, \]

where \( \lambda = 1.15 \) and \( K_d = 3.7343 \). The value of proportional gain is to be determined for which phase margin of the system is 44.15°.

Solution: \( r = -0.23345; \ t = 0.97237; \ m = -0.7608\omega^{2.2} + 0.0782\omega^{0.9} + 1 \)
\( u = 0.71745; v = 0.696614 \) and \( n = -0.2472\omega^{2.2} + 0.4938\omega^{0.9} \)

From equation (A4.7) and (A4.8), the real and imaginary part is obtained.

\[ \begin{align*}
-0.37363\omega^{2.2} + 0.71745 - 0.28788\omega^{0.9} + K_c - 0.8718\omega^{1.15} &= 0 \\
-0.707333\omega^{2.2} + 0.696614 + 0.4087746\omega^{0.9} + 3.631121\omega^{1.15} &= 0 
\end{align*} \]

The nearest real root of imaginary part is 5.2123 rad/sec (obtained using MATLAB subroutine). By substituting \( \omega = 5.2123 \) rad/sec in real part of the characteristic polynomial, it is possible to find the upper bound of \( K_c \). The upper bound of \( K_c \) is 20.5.

Here, if we utilize DeMoivre’s theorem, computation time increases in an exponential manner with the order of the system being considered i.e., implementation of DeMoivre’s theorem is tedious especially for higher order system. Hence, this procedure is recommended to investigate stability conditions particularly for lower order systems (up to second order). This method is recommended for lower order system especially when the characteristic polynomial is non commensurate.

**A4.4 OPEN LOOP RESPONSE**

The open loop response of the fractional order system with transfer function
\[ G_p(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} \]

to step input is obtained using MATLAB subroutine. The response is plotted in Figure A4.1.

![Figure A4.1 Open loop response of fractional order system](image)

A4.5 FREQUENCY RESPONSE

In order to plot the frequency response, the fractional order system with fractional order controller are considered with transfer functions given by

\[ G_p(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} , \quad \text{and} \]

\[ G_c(s) = 20.5 + 3.73s^{1.15} \]
respectively. The frequency response of $G_p G_c(s)$ is plotted using MATLAB subroutine and the same is shown in Figure A4.2.

It may be observed that the computed phase margin from the frequency response is $44^\circ$, which validates the controller design.

Figure A4.2 Frequency response of fractional order system with fractional order controller