CHAPTER 5
SEQUENTIAL ESTIMATION OF
THE MEAN OF RCAR(1) PROCESS

5.1 Introduction

There are two basic reasons why sequential methods are used in Statistics. Firstly, it is possible to reduce the sample size on an average as compared to corresponding fixed sample size procedure. Secondly to solve certain problems which cannot be solved by any procedure based on a predetermined sample size. Some of the examples to this effect are discussed in Section 1.5. The discussion in the present chapter focuses on the first aspects of the subject and deals in particular with Random Coefficient Autoregressive Processes of order one RCAR(1). The main problems discussed in this chapter are the sequential point estimation, and interval estimation. We have already discussed in detail the properties of this model in Chapter 4.

The problem of sequential estimation of the parameters of AR(1) model are studied by Sriram (1987, 1988). Recently Sriram’s results have been extended to AR(p) model and linear processes by Fakhre-Zakeri and Lee (1992) and Lee (1992).
2 Sequential Point Estimation

We study the problem of sequential point estimation of mean of RCAR(1) process in this section.

Since \{X_i, i \geq 0\} defined in (4.2.1) is a stationary and ergodic sequence, a natural estimator for \( \mu = E(X_i) \) is the sample mean

\[
\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.
\]

Suppose that we want to estimate \( \mu \) by \( \bar{X}_n \) using the loss function

\[
I_{n, \lambda} = A(\bar{X}_n - \mu)^2 + \lambda n, \quad A, \lambda > 0
\]

(5.2.1)

here \( A \) is a known constant and \( \lambda \) is the cost per observation. The loss function defined by (5.2.1) is the weighted error plus cost of inspection. An approximate expression of the risk \( R_{n, \lambda} \) can be calculated using (4.2.13) and is given by

\[
R_{n, \lambda} = A E(I_{n, \lambda}) = A E(\bar{X}_n - \mu)^2 + \lambda n
\]

\[
\approx An^2 \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} + \lambda n.
\]

(5.2.2)

Let \( n_0 \) be the value of \( n \) for which \( R_{n, \lambda} \) is minimum. Treating \( n \) as a continuous variable, we differentiate \( R_{n, \lambda} \) w.r.t. \( n \) and obtain \( n_0 \). Thus

\[
\frac{\partial R_{n, \lambda}}{\partial n} = 0 \implies
\]

\[-An^2 \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} + \lambda = 0.
\]
This gives

\[ n_0 \approx A^{1/2} \lambda^{1/2} \sigma[1-(b^2+\gamma)]^{1/2} \left( \frac{1+b}{1-b} \right)^{1/2} \]  

(5.2.3)

Clearly \( \frac{\partial^2 R_{\alpha,\lambda}}{\partial n^2} \) is positive at \( n_0 \).

We refer \( n_0 \) as the best fixed sample size procedure. The corresponding minimum value of risk \( R_{n_0,\lambda} \) can be obtained from (5.2.2) and is given by

\[ R_{n_0,\lambda} = 2\lambda n_0. \]  

(5.2.4)

If at least one of the parameters \( b, \sigma^2, \gamma \) in (5.2.3) is unknown, there does not exist any best fixed sample size procedure that will achieve the minimum risk \( R_{n_0,\lambda} \).

As a remedy we go for sequential procedure to estimate \( \mu \) by choosing a sample size such that the associated risk will be close to \( R_{n_0,\lambda} \), as cost per observation becomes small. Towards this end we use the least squares estimators of \( b, \sigma^2 \) and \( \gamma \). Properties of these estimators are already discussed in Section 4.3.

Let us define a stopping time \( T \) by

\[ T = \inf \left\{ n \geq m: n \geq A^{1/2} \lambda^{1/2} \left[ \sigma(1-(b^2+\gamma))^{1/2} \left( \frac{1+b}{1-b} \right)^{1/2} + n^h \right] \right\}, \]  

(5.2.5)

where \( m \) is an initial sample size, \( h>0 \) is a suitable constant to be defined later. Based on this stopping rule the sequential point estimator of \( \mu \) is \( \bar{X}_T \) and the associated risk is

\[ R_{T,\lambda} = AE(\bar{X}_T - \mu)^2 + \lambda E(T). \]  

(5.2.6)
The main theorem in this chapter is stated below. This theorem establishes the optimal properties of the sequential procedure for estimating $\mu$ using the stopping rule (5.2.5).

**Theorem 5.2.1:** For $p>2$, if $E|\varepsilon_i|^{2p}<\infty$, $E|b + \beta_i|^{2p}<1$ and $h\in(0, (p-2)/4)$ then as $\lambda \to 0$

i. $\frac{T}{n_0} \to 1$, a.s

ii. $E\left| \frac{T}{n_0} - 1 \right| \to 0$

iii. $\frac{R_{r,\lambda}}{R_{n_0,\lambda}} \to 1$

iv. $\sqrt{T}(\bar{X}_T - \mu) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{1 - (b^2 + \gamma)(1 - b)}\right)$.

The proof of this Theorem depends on some lemmas, which are proved below. The following notations are introduced for easy reference.

$$n_1 = \left(\frac{A}{\lambda}\right)^{1/2(1+b)} \quad n_2 = n_0(1-\varepsilon) \quad n_3 = n_0(1+\varepsilon), \quad 0<\varepsilon<1, \quad K = \left(\frac{A}{\lambda}\right)^{1/2},$$

$$E = [n_2<T<n_3] \quad B = [T \leq n_2] \quad C = [T \geq n_3]$$

$I_F$ and $F^C$ denote the indicator and complement of a set $F$ respectively.

**Lemma 5.2.1:** Suppose that $E|\varepsilon_i|^{2p}<\infty$ and $E|b + \beta_i|^{2p}<1$ for $p>2$, then for every $\varepsilon>0$
Proof: Using Lemma 4.3.2, Lemma 4.3.3, Lemma 4.3.4 and Result 1.6.12 we can write

\[
P \left[ \left| \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \hat{\gamma}_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} - \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \right| > \varepsilon \right] = O(n^{p/2}).
\]

Once again use Result 1.6.12 to obtain

This completes the proof of Lemma 5.2.1.

Lemma 5.2.2: Suppose that \( E|\varepsilon_i|^p < \infty \) and \( E|b + \beta|^p < 1 \) for \( p > 2 \), then for every \( \varepsilon > 0 \)

i. \( P[T \leq n] = O \left( \lambda^{\frac{p}{p-1}} \right) \)

and

ii. \( \sum_{n \geq n_1} P[T > n] = O \left( \lambda^{\frac{p}{p-1}} \right) \).

Proof: From the definition of stopping rule (5.2.5), we have

\[ T \geq \left( \frac{4}{\lambda} \right)^{1/2} T^{-k} \]
That is

\[ T \geq \left( \frac{4}{\lambda} \right)^{1/2} = n_1. \]  

(5.2.7)

Now from (5.2.5) and (5.2.7),

\[ \Pr[T \leq n_2] \leq \Pr \left[ \left( \frac{4}{\lambda} \right)^{1/2} \left[ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \hat{\gamma}_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} \right]^{1/2} \leq n \text{ for some } n_1 < n \leq n_2 \right] \]

\[ \leq \Pr \left[ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \hat{\gamma}_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} \leq K^{-2}n_2^2 \text{ for some } n_1 < n \leq n_2 \right] \]

\[ \leq \frac{\max_{n \leq n_2} \left| \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \hat{\gamma}_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} - \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \right|}{\sigma^2 (2 - \epsilon) \epsilon \frac{1 + b}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b}} \]

\[ \sum_{n \leq n_2} \Pr \left[ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \hat{\gamma}_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} - \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \right] > \sigma^2 (2 - \epsilon) \epsilon \left( \frac{1 + b}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \right) \]

Now from Lemma 5.2.1 we have

\[ \Pr[T \leq n_2] = O(n_2^{(\frac{2}{3} - 1)}) = O \left( \lambda^{\frac{2}{3} - 1} \right). \]

This proves the first part of the Lemma.

For the second part, from the definition of T it follows that for \( n \geq n_3 \),

\[ \Pr[T > n] = \Pr \left[ \left( \frac{4}{\lambda} \right)^{1/2} \left[ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \hat{\gamma}_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} \right]^{1/2} + n^{-\lambda} \right] > n \]

\[ = \Pr \left[ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \hat{\gamma}_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} > K^{-1}n - n^{-\lambda} \right] \]
\[
\begin{align*}
&= P \left\{ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \gamma_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} \right\}^{1/2} - \left[ \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \right] > K^{-1}(n_3 - n_0) - n_3^{-h} \right\}. 
\end{align*}
\]

But

\[
K^{-1}(n_3 - n_0) - n_3^{-h} = \frac{\varepsilon \sigma^2 (1 + b)}{[1 - (\hat{b}_n^2 + \gamma_n)(1 - b)]^{1/2}} \left( \frac{\lambda}{\alpha} \right)^{1/2} \frac{1}{1 + \varepsilon}. 
\]

Choose \( \lambda \) small enough so that the above expression for \( K^{-1}(n_3 - n_0) - n_3^{-h} \) is greater than

\[
\left[ \frac{\sigma^2 (1 + b)}{4[1 - (\hat{b}_n^2 + \gamma_n) (1 - b)]} \right]^{1/2}
\]

Thus we can write

\[
P[\tau > n] \leq P \left\{ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \gamma_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} \right\}^{1/2} - \left[ \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \right]^{1/2} > \left[ \frac{\sigma^2 (1 + b)}{4[1 - (b^2 + \gamma) (1 - b)]} \right]^{1/2}
\]

\[
\leq P \left\{ \frac{\hat{\sigma}_n^2}{1 - (\hat{b}_n^2 + \gamma_n)} \frac{1 + \hat{b}_n}{1 - \hat{b}_n} - \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \right\} > \varepsilon^2 \sigma^2 (1 + b) \frac{4[1 - (\hat{h}_n^2 + \gamma) (1 - b)]}{4[1 - (h^2 + \gamma) (1 - b)]}. 
\]

Now using Lemma 5.2.1 and repeating the same argument as in the first part we get the result.
Proof of Theorem 5.2.1

In section 4.3 we have proved that as $n \to \infty$, $\hat{b}_n \overset{a.s.}{\to} b$, $\hat{\sigma}^2_n \overset{a.s.}{\to} \sigma^2$, $\hat{\gamma}_n \overset{a.s.}{\to} \gamma$. Also we have noted in (5.2.7) that

$$T \geq \left( \frac{\lambda}{\lambda} \right)^{1/2(1-k)}$$

Thus $T \to \infty$ as $\lambda \to 0$.

Hence it follows that as $\lambda \to 0$

$$\hat{b}_T \overset{a.s.}{\to} b$$

$$\hat{\sigma}^2_T \overset{a.s.}{\to} \sigma^2$$

and

$$\hat{\gamma}_T \overset{a.s.}{\to} \gamma.$$  \hfill (5.2.8)

From the definition of stopping rule $T$ we can write

$$\left( \frac{\lambda}{\lambda} \right)^{1/2} \left[ \frac{\hat{\sigma}^2_T}{1 - (\hat{b}^2_T + \hat{\gamma}_T) 1 - \hat{b}_T} \right]^{1/2} \leq T$$

$$\leq \left( \frac{\lambda}{\lambda} \right)^{1/2} \left[ \left\{ \frac{\hat{\sigma}^2_{T-1}}{1 - (\hat{b}^2_{T-1} + \hat{\gamma}_{T-1}) 1 - \hat{b}_{T-1}} \right\}^{1/2} + (T - 1)^{1-k} \right] + m.$$  \hfill (5.2.9)

Hence dividing (5.2.9) by $n_0$ and using (5.2.8) and then letting $\lambda \to 0$, we obtain

$$\frac{T}{n_0} \overset{a.s.}{\to} 1.$$  \hfill (5.2.10)

As for part (ii) we have the result

$$E|X| = EX^+ + EX^-$$

where

$$X^- = \text{Max} \ (X, 0)$$
and

\[ X^- = \text{Max} (-X, 0). \]

Here observe that

\[ \left( \frac{T}{n_0} - 1 \right)^+ \leq 1. \]

Therefore, by dominated convergence theorem and part (i) of the Theorem 5.2.1 we have

\[ \mathbb{E}\left( \frac{T}{n_0} - 1 \right) \to 0 \text{ as } \lambda \to 0. \]

Now we write

\[ \left( \frac{T}{n_0} - 1 \right) = \left( \frac{T}{n_0} - 1 \right) I_B + \left( \frac{T}{n_0} - 1 \right) I_E + \left( \frac{T}{n_0} - 1 \right) I_C \]  \hspace{1cm} (5.2.10)

and hence

\[ \mathbb{E}\left( \frac{T}{n_0} - 1 \right) \leq (1 - \varepsilon) P(B) + \varepsilon + n_0 \sum_{n \leq n_0} P(T > n) + P(C). \]  \hspace{1cm} (5.2.11)

Since \( 0 < \varepsilon < 1 \) is arbitrary, from Lemma 5.2.2 we have

\[ \mathbb{E}\left( \frac{T}{n_0} - 1 \right) \to 0 \text{ as } \lambda \to 0. \]

So part (ii) of the theorem is also proved.

In order to prove the part (iii), (that is \( T \) is asymptotically risk efficient) assume without loss of generality that \( \mu = 0 \).

Now using (5.2.4) and (5.2.6)

\[ \frac{R_{T,\lambda}}{R_{n_0,\lambda}} = \frac{AE \bar{X}_T^2}{2 \lambda n_0} + \frac{\lambda ET}{2 \lambda n_0}. \]
Since we have already proved (ii) it is enough to show

$$\frac{AE \overline{X}_T^2}{\lambda n_0} \rightarrow 1, \text{ as } \lambda \rightarrow 0. \quad (5.2.12)$$

Instead of proving (5.2.12) we will prove

$$\frac{AE \overline{X}_T^2 I_E^2}{\lambda n_0} \rightarrow 0 \quad (5.2.13)$$

and

$$\frac{AE(\overline{X}_T - \overline{X}_n)^2 I_E}{\lambda n_0} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0. \quad (5.2.14)$$

Towards that end consider (5.2.13) and write

$$\frac{AE \overline{X}_T^2 I_E^2}{\lambda n_0} = \frac{AE \overline{X}_T^2 I_B}{\lambda n_0} + \frac{AE \overline{X}_T^2 I_C}{\lambda n_0} \quad (5.2.15)$$

using (4.2.7) we can write

$$E \overline{X}_T^2 I_B \leq E \max_{n_5 \leq n \leq n_2} \overline{X}_n^2 I_B$$

$$\leq E \max_{n_5 \leq n \leq n_2} \left[ \sum_{j=0}^{\infty} Y_{nj} \right]^2 I_B$$

$$\leq E \left[ \sum_{j=0}^{\infty} M_{nj} \right]^2 I_B$$

$$\leq \sum_{j=0}^{\infty} EM_{nj}^2 I_B + 2 \sum_{j < j'} EM_{nj} M_{nj'} I_B, \quad (5.2.16)$$

where,

$$Y_{nj} = n^{-1} \sum_{i=1}^{n} \left[ \sum_{k=0}^{j-1} (b + \beta_{i-k}) \right] e_{i+j} \quad (5.2.17)$$
and

\[ M_{n_j} = \max_{n_i \leq n \leq n_{i+1}} Y_{n_j}. \]

Observe that for \( j \geq 0 \), the sequence \( \{Y_{n_j}\} \) is a reverse martingale w.r.t. \( \{G_n\} \), where

\( G_n = \sigma\{(\beta_i, \varepsilon_i), k \geq n\}. \)

Since

\[
E[Y_{n-1,j} \mid G_n] = E\left\{ (n - 1)^{-1} \sum_{i=1}^{n-1} \left[ \sum_{k=0}^{j-1} (b + \beta_{i,k}) \right] \varepsilon_{i-j} \mid G_n \right\}
\]

\[
= (n - 1)^{-1} \sum_{i=1}^{n} E\left\{ \left[ \sum_{k=0}^{j-1} (b + \beta_{i,k}) \right] \varepsilon_{i-j} \mid G_n \right\}
\]

\[
= (n - 1)^{-1} (n - 1) E\left\{ \left[ \sum_{k=0}^{j-1} (b + \beta_{i,k}) \right] \varepsilon_{i-j} \mid G_n \right\}
\]

\[
= E\left\{ \left[ \sum_{k=0}^{j-1} (b + \beta_{i,k}) \right] \varepsilon_{i-j} \mid G_n \right\}. \quad (5.2.18)
\]

Letting \( Z_n = nY_{n_j} \) and using (5.2.17)

we have

\[
Z_n = E[Z_n \mid Z_n, Z_{n-1}, \ldots]
\]

\[
= E[Z_n \mid G_n]
\]

\[
= \sum_{i=1}^{n} E\left\{ \left[ \sum_{k=0}^{j-1} (b + \beta_{i,k}) \right] \varepsilon_{i-j} \mid G_n \right\}
\]

\[
= n E\left\{ \left[ \sum_{k=0}^{j-1} (b + \beta_{i,k}) \right] \varepsilon_{i-j} \mid G_n \right\}. \quad (5.2.19)
\]
Using (5.2.19) in (5.2.18) we get
\[ E[Y_{n-1} | G_n] = Y_{n}. \]

Hence \( \{M_{n}\} \) and \( \{M_{n}^{2}\} \) are reverse submartingales. By Schwarz inequality and Maximal inequality for reverse submartingale we have
\[ E \left| M_{n}^{2} I_B \right| \leq E^{1/2}(M_{n}^{4})^{1/2}(B) \]
\[ \leq \frac{16}{9} E^{1/2} \left| n^{-1} \sum_{i=1}^{n} \prod_{k=0}^{i-1} (b + \beta_{i-k}) \epsilon_{i-j} \right|^{4} P^{1/2}(B). \]

An application of M-Z inequality gives
\[ E^{1/2} \left| n^{-1} \sum_{i=1}^{n} \prod_{k=0}^{i-1} (b + \beta_{i-k}) \epsilon_{i-j} \right|^{4} = O(n^{-1}). \quad (5.2.20) \]

Application of lemma 5.2.2 and (5.2.20) leads to
\[ E M_{n}^{2} I_B = O(\lambda^{1/2(1+h)}) O(\lambda^{(p-2)/(2(p+h))}). \]

Since \( h < \frac{p^2}{4} \) we have
\[ \frac{EM_{n}^{2} I_B}{\lambda n_0} \rightarrow 0 \text{ as } \lambda \rightarrow 0. \quad (5.2.21) \]

Using Schwarz inequality for the second term in (5.2.16)
\[ E[M_{n} M_{n}^{2} I_B] \leq E^{1/4} (M_{n}^{4})^{1/4} (M_{n}^{2}) P^{1/2}(B). \]
So that

\[
\frac{A}{\lambda n_0} \mathbb{E}[M_{n_0} M_{n_0'} I_{B}] \to 0 \text{ as } \lambda \to 0. \tag{5.2.22}
\]

Using (5.2.21) and (5.2.22) in (5.2.16)

\[
\frac{AE\overline{X}^2_{I_{B}}}{\lambda n_0} \to 0 \text{ as } \lambda \to 0.
\]

Repeating the same arguments as above and using part (ii) of Lemma 5.2.2

\[
\frac{AE\overline{X}^2_{I_{C}}}{\lambda n_0} \to 0 \text{ as } \lambda \to 0.
\]

Thus we have proved (5.2.13)

Next consider

\[
\mathbb{E}(\overline{X}_T - \overline{X}_{n_0})^2 I_E \leq \mathbb{E}\left[ \max_{n_1 \leq n \leq n_0} |\overline{X}_T - \overline{X}_{n_0}|^2 \right] I_E
\]

\[
\leq \mathbb{E}\left[ \sum_{j=0}^{\infty} \max_{n_1 \leq n \leq n_0} W_{n, n_j} \right]^2
\]

\[
\leq \sum_{j=0}^{\infty} \mathbb{E} \max_{n_1 \leq n \leq n_0} W^2_{n, n_j} + 2 \sum_{j<j'} \mathbb{E} \max_{n_1 \leq n \leq n_0} \max_{n_1 \leq n \leq n_0} W_{n, n_j} W_{n, n_{j'}}. \tag{5.2.23}
\]

where \( W_{n, n_j} = n^{-1} \sum_{i=1}^{j} \prod_{k=0}^{i-1} (b + \beta_{i-k}) \varepsilon_{i-j} - n_0^{-1} \sum_{i=1}^{n_0} \prod_{k=0}^{j-1} (b + \beta_{i-k}) \varepsilon_{i-j} \).

Note that for each fixed \( j \geq 0 \) \( \{ W_{n, n_j}, n_0 \leq n \leq n_3 \} \) is a reverse martingale w.r.t. \( \{ G_n \} \).

Consider

\[
\mathbb{E}\left\{ \max_{n_1 \leq n \leq n_0} W^2_{n, n_j} \right\} = \mathbb{E}\left\{ \max_{(n_1 \leq n \leq n_0) \cup (n_0 \leq n \leq n_1)} W^2_{n, n_j} \right\}. \tag{5.2.24}
\]
Now applying Schwarz and maximal inequalities for reverse submartingale \( \{ W_{n,n_0} \} \),

\[
\mathbb{E} \left\{ \max_{n_0 \leq n \leq n_0} \frac{W^2_{n,n_0}}{\lambda n_0} \right\} \to 0
\]

and

\[
\mathbb{E} \left\{ \max_{n_2 \leq n \leq n_0} W^2_{n,n_0} \right\} / \lambda n_0 \to 0.
\]

Thus from (5.2.24) we get

\[
\mathbb{E} \left\{ \max_{n_2 \leq n \leq n_0} W^2_{n,n_0} \right\} / \lambda n_0 \to 0.
\]

The second term in (5.2.23) can be handled similarly. This completes the proof of part (iii).

For part (iv) we have noted in Chapter 4 [See (4.2.13)] that as \( n \to \infty \)

\[
\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \frac{\sigma^2}{1-(b^2 + \gamma) \frac{1+b}{1-b}}).
\]

Now write

\[
\sqrt{n}(\bar{X}_n - \mu) = \sqrt{\frac{1}{\lambda n_0}} \sqrt{n_0} (\bar{X}_f - \bar{X}_{n_0}) + \sqrt{\frac{1}{\lambda n_0}} \sqrt{n_0} (\bar{X}_{n_0} - \mu).
\]

From (5.2.14) we have

\[
\frac{AE(\bar{X}_f - \bar{X}_{n_0})^2 I_E}{\lambda n_0} \to 0 \text{ as } \lambda \to 0.
\]

Using (5.2.3)

\[
\frac{\lambda n_0 [1-(b^2 + \gamma)](1-b)}{\sigma^2 (1+b)} \frac{E(\bar{X}_f - \bar{X}_{n_0})^2}{\lambda n_0} \to 0 \text{ as } \lambda \to 0.
\]
That is, as $\lambda \to 0$,

$$n_0 E(\bar{X}_r - \bar{X}_{n_0})^2 \to 0.$$ 

which implies that

$$\sqrt{n_0} (\bar{X}_r - \bar{X}_{n_0}) \xrightarrow{d} 0. \tag{5.2.26}$$

From part (i) we have

$$\frac{T}{n_0} \xrightarrow{d} 1 \text{ as } \lambda \to 0.$$ 

Application of (5.2.26), part (i) of Theorem 5.2.1, (4.2.13) and Slutsky’s Theorem in (5.2.25) we get (iv). This completes the proof the theorem.

5.3 Sequential Interval Estimation

In Section 3.4 we have discussed the general framework of sequential interval estimation. Our problem in this section is to find an interval $I_n$ for the population mean of RCAR(1) process having prescribed width $2d$ and a coverage probability $1-\alpha$.

That is to find an interval $I_n$ such that

$$P[\mu \in I_n] = 1 - \alpha.$$ 

Recall from section 4.2 (see (4.2.13)) that

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \frac{\sigma^2}{1-(b^2 + \gamma)} \cdot \frac{1+b}{1-b}).$$

Based on this result an appropriate confidence interval for $\mu$ when $b$, $\sigma^2$ and $\gamma$ are known is given by

$$I_{n_0} = [\bar{X}_{n_0} - d, \bar{X}_{n_0} + d].$$
where

$$n_0 = \left[ d^{-2} \frac{Z_{1-a/2}^2}{b^2 + \gamma} \frac{1+b}{1-b} \right]. \quad (5.3.1)$$

and $Z_{1-a/2}$ is such that

$$\frac{1}{\sqrt{2\pi}} \int_{-z_{1-a/2}}^{z_{1-a/2}} \exp\left(-\frac{u^2}{2}\right) \, du = 1-\alpha.$$ 

Note from (5.3.1) that $n_0 \to \infty$ when $d \to 0$ and

$$P[\mu \in I_{n_0}] = P \left[ \sqrt{n_0} \bar{X}_{n_0} - \mu \leq \frac{d\sqrt{n_0}}{\xi} \right] \to I-\alpha,$$

where

$$\xi = \left( \frac{\sigma^2}{1-(b^2 + \gamma)} \frac{1+b}{1-b} \right)^{1/2} \quad (5.3.2)$$

When at least one of the parameters $b$, $\sigma^2$ and $\gamma$ is unknown we proposes a sequential confidence interval. For that we define a stopping rule as in the case of point estimation,

$$N = \inf \left\{ n \geq m: n \geq d^{-2} Z_{1-a/2}^2 \left[ \left( \frac{\hat{\sigma}^2_n}{1-\hat{b}^2_n + \hat{\gamma}} \frac{1+\hat{b}_n}{1-\hat{b}_n} \right) + n/h \right] \right\}, \quad (5.3.3)$$

where $m$ is an initial sample size and $h$ is a suitable constant to be defined later. Note that from the above definition of stopping rule $N \geq d^{-2} Z_{1-a/2}^2 N^h$.

That is $N \geq \left( \frac{Z_{1-a/2}}{d} \right)^{2/(1+h)}$

Thus when $d \to 0, N \to \infty$. 

The performance of the above stopping time \( N \) and the corresponding confidence interval \( I_N \) are discussed in the following Theorem.

**Theorem 5.3.1** For \( p > 2 \), if \( E\left| e_1^{4p} \right| < \infty \), \( E\left| b + \beta_1 \right|^{4p} < 1 \) and \( h\varepsilon(0, \frac{p-2}{4}) \)

then as \( d \to 0 \)

(i) \( \frac{N}{n_0} \xrightarrow{d} 1 \)

(ii) \( E\left( \frac{N}{n_0} \right) \to 1 \)

(iii) \( P[\mu \in I_N] \to 1-\alpha. \)

**Proof:** Proof of part (i) and part (ii) are very much similar to the proof of part (i) and (ii) of Theorem 5.2.1 and hence we omit the details.

For part (iii)

\[
P[\mu \in I_N] = P\left[ \left| \frac{\sqrt{N}}{\xi} \bar{X}_N - \mu \right| \leq \frac{d\sqrt{N}}{\xi} \right]
\]

where \( \xi \) is as defined in as (5.3.2).

Recall from Section (5.2) that

\[
\sqrt{N}(\bar{X}_N - \mu) \xrightarrow{d} N(0, \xi^2).
\]

Now using the definition of \( n_0 \) (5.3.4) becomes

\[
P\left[ \left| \frac{\sqrt{N}}{\xi} \bar{X}_N - \mu \right| \leq Z \sqrt{\frac{n_0}{\xi^2}} \right].
\]
Also we have noted in Part(i) that \( \frac{N}{n_o} \rightarrow 1 \) a.s.

That is \( \sqrt{\frac{N}{n_o}} \rightarrow 1 \) a.s.

Now Part (iii) follows from (5.3.4) and the above arguments. The proof of the Theorem is complete.

The work of this chapter is summarised in Balakrishna and Jacob (1998). In the next chapter we discuss the sequential estimation of \( b \).