3.1 Introduction

In many statistical inference problems, some predetermined accuracy is required and usually the optimal fixed sample size to meet this accuracy depends on some nuisance parameters. For example, if we wish to construct a confidence interval for the unknown mean \( \theta \) of a normal population, \( N(\theta, \sigma^2) \) with preassigned accuracy width \( 2d \) and confidence level \( \gamma \) for given \( d > 0 \) and \( \gamma \in (0, 1) \), the optimal fixed sample size procedure requires a sample of size \( n_0 = \left( \frac{z \sigma}{d} \right)^2 \), where \( z = \Phi^{-1}\left(\frac{1+\gamma}{2}\right) \) and \( \Phi \) is the cumulative distribution function of \( N(0,1) \) r.v. Note however that, the sample size \( n_0 = \left( \frac{z \sigma}{d} \right)^2 \), depends on \( \sigma^2 \) which is often unknown. To solve such problems it is necessary to use a sequential scheme.

The most frequently used sequential sampling scheme is the fully sequential scheme due to Anscombe (1953), Robbins (1959) and Chow and Robbins (1965). In this scheme of sampling a sample of size \( m \) is drawn first and then observations are taken one by one. It renews the estimates of the unknown parameter and the total sample size after each new observation and checks whether enough observations have already been drawn. Not surprisingly this scheme is very efficient in terms of sample size.
The purpose of present chapter is to extend the sequential estimation techniques to minification processes.

The general minification processes and its probabilistic properties are studied in Chapter 2. Compared to i.i.d cases, the literature on sequential estimation in time series emerged somewhat recently. See Sriram (1987, 1988), Basawa, McCormick and Sriram (1990) for the history of sequential estimation in dependent cases.

The present chapter is organised into four sections. In Section 3.2 we propose sequential procedure to deal with point estimation of mean. Sequential estimation for $k$ in exponential minification process is given in Section 3.3. Section 3.4 contains sequential interval estimation for mean and $k$.

### 3.2. Sequential point estimation of mean

Let $X_1, X_2, \ldots, X_n$ be the $n$ observation from the model (1.3.1) and our aim is to estimate $\mu = \mathbb{E}(X_i)$. As one can see in the literature on sequential estimation the loss function is often the sum of quadratic loss for the discrepancy between the target parameters and their estimates. Thus here the loss function is

$$L_\alpha(\mu) = \frac{A}{\sigma^2} (\bar{X}_n - \mu)^2, \quad A > 0$$

(3.2.1)

where $\sigma^2 = \text{Var}(X_i)$. The loss function defined in (3.2.1) have the property that for a given $\mu$, the loss increases as the difference between $\bar{X}_n$ and $\mu$ increases in either direction. Also this loss function is easy to handle mathematically compared to other loss functions. The expected value of a loss function is called risk function. The aim is to find an estimator for the unknown parameter which have minimum risk under any loss function. Such estimation procedures are known as minimum risk estimation method. Thus in minimum risk estimation problem, minimization of risk w.r.t the choice of sample size
leads to the minimum risk estimator (MRE). Here we estimate the parameter \( \mu \) such that the expected value of \( I_\alpha(\mu) \) in (3.2.1) is less than some prescribed value \( u_\mu \).

\[
E I_\alpha(\mu) = R_\alpha(\mu) = \frac{A}{\sigma^2} E (\bar{X}_n - \mu)^2
\]

Now using Theorem 2.3.1 we have \( E (\bar{X}_n - \mu)^2 \approx \frac{\sigma^2 B(k)}{n} \), where \( \approx \) means asymptotically equal. Hence

\[
R_\alpha(\mu) = A n^{-1} B(k) + o(n^{-1}), \quad (3.2.2)
\]

where \( B(k) \) is a continuous function of \( k \) for \( k>1 \) and \( f = o(g) \) means that \( \frac{f}{g} \to 0 \).

Let \( n_\alpha(\mu) \) be the smallest integer \( n \) such that

\[
R_\alpha(\mu) \leq u_\mu.
\]

ie,

\[
A n^{-1} B(k) \leq u_\mu.
\]

Thus

\[
n_\alpha(\mu) \approx A B(k) u_\mu. \quad (3.2.3)
\]

It is clear from the sample size defined in (3.2.3) that \( n_\alpha(\mu) \) depends on the parameter \( k \). When the parameter \( k \) is unknown, nonsequential optimal solutions may not exist in general. As a remedy we go for sequential method of estimation by defining a stopping rule \( T_\mu \) in analogy with \( n_\alpha(\mu) \) by

\[
T_\mu = \inf \{ n \geq m_u, \ n \geq u_\mu^{-1} A B(\tilde{k}_n) \} \quad (3.2.4)
\]

where \( m_u \) is an initial sample size imposed to avoid stopping too soon and that depends on the risk bound \( u_\mu \). \( B(\tilde{k}_n) \) is obtained by replacing \( k \) by \( \tilde{k}_n \) [See (2.4.1) and (2.4.2)] in
The estimator $\bar{k}_n$ and its properties are discussed in Theorem 2.4.1. Now the sequential point estimator for $\mu$ is $\bar{X}_{T_n}$ with corresponding risk

$$R_{T_n} = \frac{A}{\sigma^2} E(\bar{X}_{T_n} - \mu)^2$$

The efficiency of sequential procedure is measured in terms of the convergence properties of the following quantities, under some regularity conditions as cost per observations tends to zero. The quantities of interest are

(i) $\frac{T}{n_0}$

(ii) $\frac{E(T)}{n_0}$

(iii) $\frac{R_T}{R_{n_0}}$

Here $T$ denotes the stopping time, $n_o$ the fixed sample size $R_T$ denotes the risk under sequential setup and $R_{n_0}$ the risk under fixed sample size procedure. If $\frac{T}{n_0}$ converges to 1, then we say that the sequential procedure is asymptotically consistent and if $\frac{E(T)}{n_0} \to 1$ we say that the sequential procedure is asymptotically efficient. As a measure of relative efficiency of sequential estimator w.r.t fixed sample size estimator we consider the ratio $\frac{R_T}{R_{n_0}}$. The sequential point estimator is risk efficient if $\frac{R_T}{R_{n_0}}=1$. However, this is not true in general. But under some conditions if $\frac{R_T}{R_{n_0}}$ converges to 1, then we term the sequential procedure as asymptotically risk efficient.

The main results of this section are summarised in the following Theorem.

**Theorem 3.2.1:** If for $p > 2$, $E|Z_1|^{2p} < \infty$ and $m_u$ is such that $u^{-1/(h+1)} = m_u = O(u^{-1})$ for $h \in (0, p-2)$ then as $u_{\mu} \to 0$

i. $\frac{T_{\mu}}{n_{\mu}(\mu)} \to 1$, a.s
We need some lemmas to prove this theorem and we introduce the following notations for easy reference

\[ n_1 = n_d(\mu) (1-\varepsilon) \quad n_2 = n_d(\mu) (1+\varepsilon) \quad 0<\varepsilon<1 \]

\[ (\cdot' = [T_\mu < n_1] \quad I) = [T_\mu \geq n_2] \quad H = [n_1 < T_\mu < n_2]. \]

**Lemma 3.2.1:** If \( E|Z_1|^{2p} < \infty, p \geq 1 \) then

\[ \| \bar{X}_n - \mu \|_{2,p} = O(n^{1/2}). \]

**Proof:** From the definition of the model (1.3.1) we can write

\[ \bar{X}_n - \mu \leq n^{-1} \sum_{i=1}^{n} (kZ_i - \mu) \]

\[ = n^{-1} k \left[ \sum_{i=1}^{n} (Z_i - \alpha) + (\alpha - \mu / k) \right] \quad (3.2.5) \]

where \( \alpha = E(Z_i) \)

Thus using Minkowski inequality (See Result 1.6.1)

\[ \| \bar{X}_n - \mu \|_{2,p} \leq n^{1/2} k \left\| \sum_{i=1}^{n} (Z_i - \alpha) \right\|_{2,p} + k \| (\alpha - \mu / k) \|_{2,p} \]
Now since \( Z_i \)'s are iid with \( \mathbb{E}|Z_1|^{2p} < \infty \), we can use Marcinkiewicz-Zygmund inequality [See Result 1.6.10] to the first term to get
\[
\left\| \sum_{i=1}^{n} (Z_i - \alpha) \right\|_{2p} = O(n^{1/2}).
\]

Thus
\[
\left\| \overline{X}_n - \mu \right\|_{2p} = n^{1/2} O(n^{1/2}) = O(n^{1/2})
\]

The lemma is proved.

**Lemma 3.2.2**: If \( \{Z_n\} \) is a sequence of nonnegative and non-degenerate r.v.s and \( m_u \) is such that \( \mu_{u^{(n-k-1)}} \leq m_u = O(u_{-1}^{-1}) \), \( h \in (0, p-2) \) for \( p > 2 \). Then as \( u_{\mu} \to 0 \),

1. \( P[T_{\mu} \leq n_1] = O(u_{\mu}^{(p+1-h-1)})^{1} \)
2. \( \sum_{n=n_1}^{\infty} P[T_{\mu} > n] = O(u_{\mu}^{(p+1-h-1)})^{2} \).

**Proof**: From the definition of stopping time (3.2.4) if \( T_{\mu} \leq n_1 \) then

\[
u_{\mu}^{-1} A.B(\bar{k}_n) \leq n_1 \text{ for some } m_u \leq n \leq n_1.
\]

Thus
\[
P[T_{\mu} \leq n_1] \leq P[nu_{\mu}^{-1} A. B(\bar{k}_n) \leq n_1 \text{ for some } m_u \leq n \leq n_1]
\]
\[
\leq P[B(\bar{k}_n) - B(k) \leq -\varepsilon B(k) \text{ for some } m_u \leq n \leq n_1]
\]
\[
\leq \sum_{n=n_1}^{\infty} P[B(\bar{k}_n) - B(k) > \varepsilon'], \text{ where } \varepsilon' = \varepsilon B(k)
\]

\[ \leq \sum_{n=\infty}^{c} P\left|\tilde{k}_n - k\right| > \eta \], for \( \eta > 0 \) \hspace{1cm} (3.2.6) \]

The last inequality is due to the fact that \( B(.) \) is a continuous function of \( k \).

Now consider

\[ P\left|\tilde{k}_n - k\right| > \eta \] = \( P[(\tilde{k}_n - k) > \eta] + P[(\tilde{k}_n - k) < -\eta] \]

\[ = 0 + P[(\tilde{k}_n - k) < -\eta] \]

\[ \leq P\left[ \max_{1 \leq i \leq n} \left( \frac{Z_i}{Z_{1,1}} \right) < k - \eta \right] \]

\[ = P\left[ \frac{Z_1}{Z_{0,1}} < k - \eta, \frac{Z_2}{Z_{1,1}} < k - \eta, \frac{Z_3}{Z_{2,1}} < k - \eta, \ldots, \frac{Z_{n-1}}{Z_{n-2,1}} < k - \eta \right] \]

\[ \leq P\left[ \frac{Z_2}{Z_{1,1}} < k - \eta, \frac{Z_3}{Z_{2,1}} < k - \eta, \ldots, \frac{Z_{n-1}}{Z_{n-2,1}} < k - \eta \right] \]

\[ \leq P\left[ \frac{Z_2}{Z_{1,1}} < k - \eta \right]^{[1]} = a^{[1]-1}, \]

where \( a = P\left[ \frac{Z_2}{Z_{1,1}} < k - \eta \right] \)

Since \( \{Z_n\} \) is a sequence of nondegenerate r.v.s it is true that \( 0 \leq P\left[ \frac{Z_2}{Z_{1,1}} < x \right] < 1 \) for \( x > 0 \).

Consider

\[ \frac{P\left|\tilde{k}_n - k\right| > \eta}{n^p} \leq \frac{a^{[1]}-1}{n^p} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Therefore,

\[ \lim_{n \rightarrow \infty} \frac{a^{[1]}-1}{n^p} \leq \lim_{n \rightarrow \infty} \frac{a^{1/2}}{n^p} = \lim_{n \rightarrow \infty} \frac{n^p}{a^{1/2}} = \infty. \]
Therefore,

\[ P\left| \bar{k}_n - k \right| > \eta = o(n^\rho). \]

This implies that

\[ P\left| \bar{k}_n - k \right| > \eta = O(n^\rho). \quad (3.2.7) \]

Now using (3.2.7) in (3.2.6) we have

\[ P[T_m \leq n_1] = \sum_{n=m_0}^{n_1} O(n^\rho) \]

\[ = c \left[ \frac{1}{m_0^\rho} + \frac{1}{(m_0 + 1)^\rho} + \ldots \right] \text{ for some } 0 < c < \infty. \]

Now we have the following relations

\[ \frac{1}{m_0^\rho} + \frac{1}{(m_0 + 1)^\rho} + \frac{1}{(2m_0 - 1)^\rho} < \frac{m_0}{m_0^\rho} \]

\[ < \frac{2m_0}{(2m_0)^\rho} \]

\[ < \frac{1}{(2m_0)^\rho - 1} \]
\[
\frac{1}{(4m_u)^p} + \frac{1}{(4m_u + 1)^p} + \cdots + \frac{1}{(8m_u - 1)^p} < \frac{4m_u}{(4m_u)^p} = \frac{1}{(4m_u)^{p-1}}
\]

Hence
\[
\frac{1}{m_u^p} + \frac{1}{(m_u + 1)^p} + \cdots + \frac{1}{(2m_u)^p} + \frac{1}{(4m_u)^p} + \frac{1}{(4m_u)^{p-1}}
\]

\[
= \frac{1}{m_u^{p-1}} \left[ 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \cdots \right]
\]

Now the series \[1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \cdots\] is a geometric series whose common ratio is \(\frac{1}{2^{p-1}}\) is less than unity since \(p>2\). Hence the sum of this geometric series is finite. Thus we have
\[
P[T_{\mu} \leq n_1] = O(m_u^{p-1})
\]
\[
= O(u_{\mu}^{(p \log_{1+h})^{-1}}),
\]

where we used the condition \(u_{\mu}^{1/(h-1)} \leq m_u\).

Hence first part of the Lemma is proved.

For the second part, from the definition of \(T_{\mu}\) it follows that for \(n \geq n_2\)
\[
P[T_{\mu} > n] \leq P[u_{\mu}^{-1} A B(\bar{k}_n) > n]
\]
\[
= P[B(\bar{k}_n) > n, A' u_{\mu}]
\]
\[
= P[B(\bar{k}_n) - B(k) \geq \varepsilon B(k)]
\]
\[
= P[|\bar{k}_n - k| \geq \varepsilon'] = O(n^p)
\]

As in the proof of part (1) here we can prove
\[
\sum_{n \geq n_2} P[T_{\mu} > n] = O(n_{\mu}^{(p-1)/2}).
\]
This completes the proof.

**Lemma 3.2.3:** Under the conditions of Lemma 3.2.2, \( \left\{ u_\mu^1 (X_{\tau_\mu} - \mu)^2 I_{(n_1, \tau_\mu, n_2)} \right\} \), \( 0 < u_\mu < u_\theta, u_\theta < 1 \) is uniformly integrable.

**Proof:** From the definition of uniform integrability [See Definition 1.6.4] it is enough to prove that

\[
\sup_{\tau_\mu} \mathbb{E} \left| u_\mu^1 (X_{\tau_\mu} - \mu)^2 I_{(n_1, \tau_\mu, n_2)} \right| < \infty.
\]

Using (3.2.5) we can write

\[
\mathbb{E} \left| u_\mu^1 (X_{\tau_\mu} - \mu)^2 I_H \right| \leq u_\mu^1 \mathbb{E} \left| \max_{n_1 < n_2} (X_n - \alpha) + (\alpha - \mu / k) \right|^2 I_H
\]

\[
\leq u_\mu^1 k \mathbb{E} \left| \max_{n_1 < n_2} \sum_{i=1}^n (Z_i - \alpha) + (\alpha - \mu / k) \right|^2 I_H
\]

\[
\leq u_\mu^1 k n_1^2 \mathbb{E} (F_{n_1}^2 I_H) + 2 n_1^1 u_\mu^1 k (\alpha - \mu / k) \mathbb{E} (F_{n_1} I_H)
\]

\[
+ u_\mu^1 k \mathbb{E} (I_H) (\alpha - \mu / k)^2. \quad (3.2.8)
\]

where \( I_H = \max_{n_1, n_2} \left| \sum_{i=1}^n (Z_i - \alpha) \right| \) is a submartingale w.r.t. \( F_n = \sigma\{X_0, Z_1, ..., Z_n\} \). Now using Schwartz inequality, Maximal inequality for submartingales and M-Z inequality, the first term in (3.2.8) can be written as

\[
u_\mu^1 k n_1^2 \mathbb{E} (F_{n_1}^2 I_H) \leq u_\mu^1 k n_1^2 \mathbb{E}^{1/2} (F_{n_1}^4) P^{1/2}(H)
\]

\[
\leq u_\mu^1 k n_1^2 P^{1/2}(H) \mathbb{E}^{1/2} \left( \max_{n_1 < n_2} \sum_{i=1}^n (Z_i - \alpha) \right)^4 \]

\[
\leq u_{\mu}^{-1} k n_1^{-1} P^{1/2}(H) E^{1/2} \left\{ \sum_{i=1}^{\infty} (Z_i - \alpha)^4 \right\}
\leq u_{\mu}^{-1} k n_1^{-1} P^{1/2}(H) O(n) 
\]

Now using Lemma 3.2.2, we have

\[
u_{\mu}^{-1} k n_1^{-1} E(F_n^2 I_{\mu}) < \infty.
\]

Similarly the second term in (3.2.8) can be written as

\[
2 n_1^{-1} \nu_{\mu}^{-1} k (\alpha - \mu k) E(F_n I_{\mu}) \leq n_1^{-1} \nu_{\mu}^{-1} k (\alpha - \mu k) n_1^{-1} E^{1/2}(F_n^2) P^{1/2}(H).
\]

Repeating the same arguments as above we have

\[
n_1^{-1} \nu_{\mu}^{-1} k E(F_n I_{\mu})(\alpha - \mu k) < \infty.
\]

Similarly the third term in (3.2.8) is finite. Thus we have proved the lemma.

**Lemma 3.2.4:** If \( E|Z|^p < \infty, \ p \geq 1 \) then \( \{ \sqrt{n}(X_n - \mu), \ n \geq 1 \} \) is uniformly continuous in probability.

**Proof:** We have \( \sqrt{n}(X_n - \mu) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu) / \sqrt{n} \)

Letting \( \sum_{i=1}^{n} (X_i - \mu) = Q_n \) and following Woodroofe (1982) (cf. pp.11)

we can write

\[
\left| \frac{Q_{n+j}}{\sqrt{n+j}} - \frac{Q_n}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{n}} \left| Q_{n+j} - Q_n \right| + \left[ 1 - \left( \frac{n}{n+j} \right)^{1/2} \right] \left| \frac{Q_n}{\sqrt{n}} \right|.
\]  

\[(3.2.9)\]
If \( j \leq n\delta \) the second term on the right hand side (3.9) is stochastically bounded by

\[
\left[ 1-(1+\delta)^{1/2} \right] \left\| \bar{X}_n - \mu \right\| \sqrt{n} = \left[ 1-(1+\delta)^{1/2} \right] O(1) \text{ by Lemma 3.2.1}
\]

which tends to zero as \( \delta \to 0 \) uniformly in \( n \).

Thus we have

\[
P \left[ \max_{0 \leq j \leq n \delta} \left| \frac{Q_{n,j}}{\sqrt{n}} \right| \geq \frac{\varepsilon}{2} \right] \to 0, \text{ as } \delta \to 0
\]

Now consider

\[
P \left[ \max_{0 \leq j \leq n \delta} \frac{1}{\sqrt{n}} \left| Q_{n,j} - Q_n \right| > \frac{\varepsilon}{2} \right] = P \left[ \max_{0 \leq j \leq n \delta} \sum_{i=j}^{n} (X_i - \mu) > \frac{\varepsilon \sqrt{n}}{2} \right].
\]

Using (3.2.5) we can write

\[
P \left[ \max_{0 \leq j \leq n \delta} \sum_{i=j}^{n} (X_i - \mu) > \frac{\varepsilon \sqrt{n}}{2} \right] \leq P \left[ \max_{0 \leq j \leq n \delta} \sum_{i=j}^{n} (Z_i - \alpha) + jk(\alpha - \frac{\mu}{\varepsilon}) > \frac{\varepsilon \sqrt{n}}{2} \right].
\]

Note that \( k \left\{ \sum_{i=j}^{n} (Z_i - \alpha) + jk(\alpha - \frac{\mu}{\varepsilon}) \right\} \) is a submartingale w.r.t \( G_n = \sigma \{ X_0, Z_i \ldots Z_n \} \).

Using maximal inequality for this submartingale we have

\[
P \left[ \max_{0 \leq j \leq n \delta} \frac{1}{\sqrt{n}} \left| Q_{n,j} - Q_n \right| > \frac{\varepsilon \sqrt{n}}{2} \right] \leq \frac{4}{\varepsilon^2 n} \mathbb{E} \left[ \sum_{i=j}^{n} k(Z_i - \alpha) + n \delta k(\alpha - \frac{\mu}{\varepsilon}) \right]
\]

\[
\leq \frac{4}{\varepsilon^2} k \delta \mathbb{E} [(Z_i - \alpha) + (\alpha - \frac{\mu}{\varepsilon})]
\]
Thus \( \{ \sqrt{n}(\bar{X}_n - \mu) \}, n > 1 \) is u.c.i.p.

Now we are in position to prove the theorem.

**Proof of Theorem 3.2.1**

Since \( \tilde{k}_n \) is a strongly consistent estimator of \( k \) and \( B(k) \) is a continuous function of \( k \), it follows from (3.2.4) that \( T_\mu < \infty \) and \( T_\mu \to \infty \) as \( u_\mu \to 0 \). Also \( B(\tilde{k}_{T_\mu}) \to B(k) \).

From the definition of stopping rule (3.2.4) we have

\[
T_\mu \leq m_u
\]

\[
T_\mu \geq u_\mu^{-1} A. B(\tilde{k}_{T_\mu})
\]

and

\[
T_\mu < u_\mu^{-1} A. B(\tilde{k}_{T_\mu}).
\]

Thus

\[
u^{-1}_\mu A. B(\tilde{k}_{T_\mu}) \leq T_\mu \leq u_\mu^{-1} A. B(\tilde{k}_{T_\mu}) + m_u. \tag{3.2.10}
\]

Dividing (3.2.10) by \( n_\delta(\mu) \) and using the above arguments it follows that

\[
\frac{T_\mu}{n_\delta(\mu)} \overset{a.s.}{\to} 1
\]

if \( m_u \) is such that \( \frac{m_u}{n_\delta(\mu)} \to 0 \).
For part (ii) we can write

\[ E\left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) = E\left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) + E\left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) \]  

(3.2.11)

But \( \left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) = \left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) I_{c} + \left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) I_{D} + \left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) I_{H} \)

and \( \left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) I_{c} = 0 \)

Now using Lemma 3.2.2,

\[ E\left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) = E\left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) I_{H} + E\left( \frac{T_{\mu} - n_{2}}{n_{0}(\mu)} \right) I_{D} + E\left( \frac{n_{2} - 1}{n_{0}(\mu)} \right) P(D) \]

\[ = \varepsilon P(H) + n_{0}^{-1}(\mu) E\left[ (T_{\mu} - n_{2})^{-1} I_{H} \right] + n_{0}^{-1}(\mu) (n_{2} - n_{0}^{-1}(\mu))^{-1} P(D) \]

\[ = \varepsilon + \varepsilon + o(1), \]

since \( \varepsilon \) is arbitrary, as \( u_{\mu} \to 0 \), we have

\[ E\left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) \to 0. \]  

(3.2.12)

Now dominated convergence theorem can be applied to the second term in (3.2.11), since

\[ \left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) \leq 1. \]

Thus

\[ \lim_{u_{\mu} \to 0} E\left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) = E \lim_{u_{\mu} \to 0} \left( \frac{T_{\mu}}{n_{0}(\mu)} - 1 \right) = 0. \]  

(3.2.13)
Now part (ii) of the Theorem follows from (3.2.11), (3.2.12) and (3.2.13).

For part (iii) recall from Theorem 2.3.1 that

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma_1^2).$$

We need to show that this result continues to hold when $n$ is replaced by the stopping time $T_\mu$. In Lemma 3.2.4 we have verified that $\{\sqrt{n}(\overline{X}_n - \mu), n \geq 1\}$ is u.c.i.p. Then one can conclude, using Anscombe's Theorem [See Result 1.6.5], (2.3.2) and from part (i) that

$$\sqrt{T_\mu(\overline{X}_{T_\mu} - \mu)} \xrightarrow{d} \mathcal{N}(0, \sigma_1^2), \text{ as } u_\mu \to 0.$$

In part (iv) we will show that the risk of the sequential procedure $\overline{X}_{T_\mu}$ defined by

$$R_{T_\mu} = A \mathbb{E}(\overline{X}_{T_\mu} - \mu)^2$$

is close to $R_{n(\mu)}$, as $u_\mu \to 0$.

Consider

$$\frac{R_{T_\mu}}{R_{n(\mu)}} = \frac{A \mathbb{E}(\overline{X}_{T_\mu} - \mu)^2}{u_\mu \sigma^2} I_{[\mathbb{C} \cup \mathbb{H} \cup \mathbb{D}]}, \quad (3.2.14)$$

where the events $\mathbb{C}, \mathbb{D}, \mathbb{H}$ are as defined earlier.

Using Schwartz inequality, Lemma 3.2.1 and Lemma 3.2.2, we write

$$\frac{A \mathbb{E}(\overline{X}_{T_\mu} - \mu)^2}{u_\mu \sigma^2} \leq \frac{A}{u_\mu \sigma^2} E^{1/p} \left[ \max_{m, n \geq m \wedge n} (\overline{X}_n - \mu)^2 \right] P^{1/q}(\mathbb{C}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$

$$= \frac{A}{u_\mu \sigma^2} \left[ \sum_{n=m}^{n_0} n^{-p} \right]^{1/p} P^{1/q}(\mathbb{C}).$$
\[ \frac{A}{\sigma^2} u^{-1} \mathbb{O}(m_{\mu}^{(p-1)\cdot p}) \mathbb{O}(u_{\mu}^{(p-1)(h-1)q}) = \frac{A}{\sigma^2} u^{-1} \mathbb{O}(u_{\mu}^{(p-1)\cdot (h-1)p}) \mathbb{O}(u_{\mu}^{(p-1)(h-1)q}) = o(1) \text{ as } u_{\mu} \to 0, \text{ since } h \in (0,p-2). \] (3.2.15)

Similarly using Schwartz inequality, Lemma 3.2.1 and part (2) of Lemma 3.2.2 we have

\[ \frac{A}{u_{\mu}^{-1}\sigma^2} \mathbb{E} (X_{\tau_{\mu}} - \mu)^2 I_{\mathcal{I}} = o(1) \] (3.2.16)

Now we will prove

\[ \frac{A}{\sigma^2} u^{-1} \mathbb{E} (X_{\tau_{\mu}} - \mu)^2 I_{\mathcal{H}} \to 1 \text{ as } u_{\mu} \to 0. \]

Using part (iii) and Lemma 3.2.3 it follows that

\[ \mathbb{E} (X_{\tau_{\mu}} - \mu)^2 I_{\mathcal{H}} \to \frac{\sigma^2}{T_{\mu}^2} \text{ as } u_{\mu} \to 0. \]

But we have \( \sigma_{\mu}^2 = B(k) \sigma^2 \) and \( n_0(\mu) \approx AB(k) u_{\mu} \) to write

\[ \frac{A}{\sigma^2} u^{-1} \mathbb{E} (X_{\tau_{\mu}} - \mu)^2 I_{\mathcal{H}} \to \frac{A}{\sigma^2} u^{-1} \sigma_{\mu}^2 n_0^{-1}(\mu) \approx \frac{A}{u_{\mu} n_0(\mu)} \to 1. \] (3.2.17)

Now the asymptotic risk efficiency follows from (3.2.14), (3.2.15), (3.2.16) and (3.2.17).

Hence the theorem is completely proved.
3.3 Sequential Estimation of $k$ in Exponential Minification processes

Exponential minification process have many nice features compared to other minification processes. This section deals with sequential estimation of $k$ of the exponential minification process.

Even though $\tilde{k}_n$ defined in (2.4.2) is consistent for $k$ in a general minification process, it is not CAN. However, for the exponential case the estimator suggested is $\hat{k}_n$ given by

$$\hat{k}_n = \frac{\overline{U}_n}{2\overline{U}_n - 1}$$

(3.3.1)

is CAN as discussed in Theorem 2.4.1.

For the sake of algebraic simplicity we consider a loss function of the form

$$L_{nk} = C[(2k-1)\overline{U}_n - k]^2, \quad C > 0.$$  

(3.3.2)

for estimating $k$ using (3.3.1). Using Theorem 2.4.1 (See (2.4.8) and (2.4.20)) the corresponding risk is given by

$$R_{nk} = E[L_{nk}] = C(2k-1)^2 E[\overline{U}_n - (k(2k-1))^2]$$

$$\leq Cn^4 \left[ k(k-1) - 2\sum_{n=1}^t \frac{(k-1)^3}{(k-1) + k^{k-1}(2k-1)} \right]$$

$$\leq Cn^4 H(k) \text{ (say)},$$

where $H(k) = k(k-1) - 2\sum_{n=1}^t \frac{(k-1)^3}{(k-1) + k^{k-1}(2k-1)}$  

(3.3.3)

Note that $H(k)$ defined by (3.3.3) is a continuous function of $k$ for $k>1$. As in the case of population mean here also we calculate the sample size such that the risk is less than some prescribed limit say $u_k$. That is,
Let $n_{ok}$ be the smallest integer $n$ such that (3.3.4) holds. That is,

$$C' n^{-1} H(k) \leq u_k.$$  

Thus

$$n_{ok} \geq C' n^{-1} H(k).$$  

(3.3.5)

Note that this fixed sample size procedure depends on the unknown parameter $k$.

Let us define a stopping time by

$$T_k = \inf \{ n \geq m_k : n \geq C' u_k^{-1} H(\hat{k}_n) \}. \quad (3.3.6)$$

where $m_k$ is an initial sample size that may dependent on $u_k$, $H(\hat{k}_n)$ is obtained by replacing $k$ by $\hat{k}_n$ in (3.3.3)

Based on this stopping rule the sequential point estimator of $k$ is $\hat{k}_{r_k}$ with corresponding risk $R_{r_k}$. The optimal properties of this sequential procedure are summarised in the following Theorem.

**Theorem 3.3.1:** If $E |Z_i|^{2p} < \infty$ for $p > 2$, and $m_k$ is such that $u_k^{-1/(h+1)} \leq m_k = O(u'')$ for $h \in (0, p-2)$, then as $u_k \to 0$

1. \( \frac{T_k}{n_{ok}} \overset{a.s.}{\to} 1 \)

2. \( E \left| \frac{T_k}{n_{ok}} - 1 \right| \to 0 \)

3. \( \sqrt{T_k} (\hat{k}_{r_k} - k) \overset{d}{\to} N(0, \sigma_2^2) \), where $\sigma_2^2$ is defined by (2.4.6)

4. \( \frac{R_{r_k}}{R_{n_{ok}}} \to 1 \)
The following lemma's are needed to prove this theorem and hence we prove them first.

**Lemma 3.3.1:** Let \( \{X_n\} \) be an exponential minification sequence defined in section 2.4 and if \( \mathbb{E} |Z_i|^{2p} < \infty \) for \( p \geq 1 \) then

\[
\left\| (2k - 1)\overline{U}_n - k \right\|_{2p} = O(n^{1/2}),
\]

where \( \overline{U}_n = (U_1 + U_2 + \ldots + U_n)/n \) and \( U_i \) is defined by (2.4.4)

**Proof:** Consider

\[
\left\| (2k - 1)\overline{U}_n - k \right\|_{2p} = (2k-1) \left\| \overline{U}_n - \frac{k}{(2k - 1)} \right\|_{2p}
\]

\[
= (2k-1)n \left\| \sum_{i=1}^{n} (U_i - \frac{k}{(2k - 1)}) \right\|_{2p}.
\]

Note that \( \sum_{i=1}^{n} (U_i - \frac{k}{(2k - 1)}) \) is a zero mean martingale w.r.t \( F_n = \sigma \{X_0, Z_1, \ldots, Z_n\} \). Then by applying Burkholder inequality (See Result 1.6.1) and moment inequality we have

\[
B_p^{-1}n^{1/2} \left\| \sum_{i=1}^{n} (U_i - \frac{k}{(2k - 1)}) \right\|_{2p} \leq \left\| n \sum_{i=1}^{n} (U_i - \frac{k}{(2k - 1)})^p \right\|_{2p}^{1/2}
\]

\[
\leq \left\| n \sum_{i=1}^{n} (U_i - \frac{k}{(2k - 1)})^p \right\|_{2p}^{1/2}
\]

\[
= O(1),
\]

where \( B_p = 18p^{-2} (p-1)^{-1/2} \). Hence

\[
\left\| \sum_{i=1}^{n} (U_i - \frac{k}{(2k - 1)}) \right\|_{2p} = O(n^{1/2}).
\]
Therefore
\[(2k-1)^{\tilde{\alpha}_n} - k \in O(n^{1/2})\]
\[= O(n^{1/2}).\]

The lemma is proved.

For the following lemma we need to introduce some notations

Let \( n_1 = n_0(1-\varepsilon), n_1 = n_0(1+\varepsilon), A = [T_k \leq n_1], D = [T_k \geq n_2] \) and \( E = [n_1 < T_k < n_2] \).

**Lemma 3.3.2**: If \( E \mid Z_f \mid^{2p} < \infty \) for \( p > 2 \) and \( m_k \) is such that \( u_k^{-\varepsilon (n+1)} \leq m_k = O(u_k^{-1}) \), where \( u_k \) is as defined in Section 3.3. Then for \( h \in (0, p-2) \),

1. \( P[T_k \leq n_1] = O(u_k^{(p-2)(n+1)}) \)

2. \( \sum_{n \sim n_2} [T_k \geq n_2] = O(u_k^{p-1}) \)

**Proof**: From the definition of stopping time \( T_k \) (3.3.6)

\( T_k \leq n_1 \) implies

\[ C u_k^{-1} H(\hat{k}_n) \leq n_1 \] for some \( m_k < n \leq n_1 \).

Thus using the definition of \( n_1 \)

\[ P[T_k \leq n_1] \leq P[C u_k^{-1} H(\hat{k}_n) \leq n_1 \text{ for some } m_k < n \leq n_1] \]

\[ = P[H(\hat{k}_n) - H(k) \leq -\varepsilon H(k) \text{ for some } m_k < n \leq n_1] \]

\[ \leq P \left[ \max_{m_k, n \sim n_1} |H(\hat{k}_n) - H(k)| > \varepsilon \right] \text{ where } \varepsilon' = \varepsilon H(k) \]
\[ \leq \sum_{n \in n} P \left[ |H(k_n) - H(k)| > \varepsilon \right] \]

\[ \leq \sum_{n \in n} P \left[ |k_n - k| > \eta \right] \quad (3.3.8) \]

Now we will prove

\[ P \left[ |k_n - k| > \eta \right] = O(n^{-p}) \quad (3.3.9) \]

In view of relation (3.3.1) and Result 1.6.12 it is enough to prove

\[ P \left[ \left| \frac{U_n - k}{2k-1} \right| > \varepsilon \right] = O(n^p). \]

Consider

\[ \left\| U_n - \frac{k}{2k-1} \right\|_{2p}. \]

We have already proved in Lemma 3.3.1 that

\[ \left\| U_n - \frac{k}{2k-1} \right\|_{2p} = O(n^{-1/2}). \]

Using Markov inequality for \( \varepsilon > 0 \), we get

\[ P \left[ \left| \frac{U_n - k}{2k-1} \right| > \varepsilon \right] \leq \frac{E \left| \frac{U_n - k}{2k-1} \right|^p}{\varepsilon^p} = O(n^{p-2}). \]

Now using the Result 1.6.12,

\[ P \left[ \left| \frac{U_n}{2U_n - 1} - \frac{k}{2k-1} \right| > \varepsilon \right] = O(n^{p-2}). \]

Hence the required result (3.3.9) follows from the above equation.
Combining (3.3.8) and (3.3.9)

\[ P[T_k \leq n_k] \leq \sum_{n=n_k}^{\infty} O(n^{p/2}) \]

\[ = O(m_k^{(p/2)\cdot 1}) \]

\[ = O(n_k^{(p/2)\cdot (b+1)}) \]

This proves the first part of the lemma.

On similar lines the second part can be proved. We have already provided a similar result in lemma 3.3.2. Hence we omit the details.

**Lemma 3.3.3:** Under the conditions of lemma 3.3.2,

\[ \{ u_k^1 [(2k-1) \overline{U}_{\tau_k} - k]^2 I_{(n, n_1 \cdot n_2)}, 0 < u_k < u_0 \} \]

is uniformly integrable.

**Proof:** By the definition of uniform integrability it is enough to show that

\[ \sup_{\tau_k} E |u_k^1 [(2k-1) \overline{U}_{\tau_k} - k]^2 I_E| < \infty. \]

Consider

\[ E |u_k^1 [(2k-1) \overline{U}_{\tau_k} - k]^2 I_E| \leq u_k^1 E \left[ \max_{n, n_k} \left( (2k-1) \overline{U}_n - k \right) \right]^2 \]

\[ \leq u_k^1 (2k-1)^2 E \left[ \max_{n, n_k} \left( \frac{U_n - \frac{k}{2k-1}}{2k-1} \right) \right]^2 I_E \]

\[ \leq u_k^1 \frac{(2k-1)^2}{n_k^2} E \left[ \max_{n, n_k} \sum_{j=1}^n \left( U_j - \frac{k}{2k-1} \right) \right]^2 I_E \]
Note that \( \sum_{i=1}^{n} [U_i - \frac{1}{2k-1}] \) is a martingale w.r.t. \( G_n = \sigma\{X_0, Z_1, \ldots, Z_n\} \) and hence

\[
V_n = \left\{ \max_{i, n} \sum_{i=1}^{n} [U_i - \frac{1}{2k-1}] \right\} \text{ is a submartingale.}
\]

Now using Schwartz inequality and maximal inequality for submartingales,

\[
E \left| u_i^{-1} \left[ (2k-1) \left\{ \bar{U}_{j_i} - k \right\}^2 I_E \right] \right| \leq u_i^{-1} \left( \frac{2k-1}{n_i^2} \right)^2 \quad E^{1/2} \left( \frac{\nu_1^4}{\nu_1^2} \right) \quad (3.3.10)
\]

Consider \( E(V_n^2) = E \left\{ \sum_{i=1}^{n} \left| U_i - \frac{1}{2k-1} \right|^4 \right\} \)

Using M-Z inequality we have

\[
E(V_n^4) = O(n_1^2) \quad \text{[See Lemma 3.3.1]}
\]

Using (3.3.10), lemma 3.3.2 and above arguments we have

\[
E \left| u_i^{-1} \left[ (2k-1) \left\{ \bar{U}_{j_i} - k \right\}^2 I_E \right] \right| \leq u_i^{-1} \left( \frac{2k-1}{n_i^2} \right)^2 \quad O(n_1) \quad P^{1/2}(E) < \infty.
\]

Hence the proof of the lemma is complete.

**Lemma 3.3.4:** \( \{ \sqrt{n}(\hat{k}_n - k), n \geq 1 \} \) is stochastically bounded and uniformly continuous in probability.

**Proof:** Using (3.3.1), \( \sqrt{n}(\hat{k}_n - k) \) can be written as

\[
\sqrt{n}(\hat{k}_n - k) = \sqrt{n} \left( \frac{\bar{U}_n}{2\bar{U}_n - 1} - k \right)
\]

\[
= \frac{\sqrt{n}[(1 - 2k)\bar{U}_n + k]}{2\bar{U}_n - 1}
\]
We will prove that the terms in numerator and denominator of (3.3.11) are u.c.i.p and stochastically bounded and then use the Remark 1.6.3 to get the required result.

As \( \overline{U}_n \leq 1 \), it follows that \{ \[ (1 - 2k)\overline{U}_n + k \] / \sqrt{n} \} and \{ (2\overline{U}_n - 1) / n \} converge to zero almost surely as \( n \to \infty \). Thus by Remark 1.6.2 these terms are u.c.i.p and stochastically bounded. Since any continuous function of u.c.i.p and stochastically bounded sequences is again u.c.i.p (cf. Remark 1.6.3) lemma 3.3.4 now follows easily.

The proof of Theorem 3.3.1 is skipped as it is parallel to the proof of Theorem 3.2.1. In the next section we will consider sequential interval estimation for the mean and \( k \).

### 3.4 Sequential Interval Estimation

This section is devoted to the study of sequential interval estimation for the mean of general minification processes defined by (2.1.1). In the iid setup Chow and Robbins (1965) proposed a sequential confidence interval for the mean \( \theta \) of a population with finite variance as described below. They consider a situation where \( \{X_n\} \) is a sequence of iid observations and \( \hat{\theta}_{Ln}, \hat{\theta}_{Un} \) (both based on \( X_1, X_2, ..., X_n \)) such that \( \hat{\theta}_{Ln} \leq \hat{\theta}_{Un} \) and \( \mathbb{P}[\hat{\theta}_{Ln} \leq \theta \leq \hat{\theta}_{Un}] \geq 1 - \alpha \). In this case \( 1 - \alpha \) is referred to as the confidence coefficient or the coverage probability and \( \alpha \in (0, 1) \). In the confidence interval, \( \hat{\theta}_{Ln} \) and \( \hat{\theta}_{Un} \) are the lower and upper confidence limits and the width of this interval is equal to \( \hat{\theta}_{Un} - \hat{\theta}_{Ln} \). In many problems of practical interest one wants to provide such a confidence interval for a parameter of interest satisfying the additional condition that for some preassigned \( d(>0) \).

\[
0 < \hat{\theta}_{Un} - \hat{\theta}_{Ln} \leq 2d.
\]
Assume the estimator $T_n$ for $\theta$ is strongly consistent and $\sqrt{n} (T_n - \theta)$ is asymptotically normally distributed as $n \to \infty$ say N(0, $\sigma^2$).

Then

$$\lim_{n \to \infty} P_n (T_n - n^{1/2} \sigma Z_{1 - \alpha/2} \leq \theta \leq T_n + n^{1/2} \sigma Z_{1 - \alpha/2}) = 1 - \alpha. \quad (3.4.1)$$

where $Z_{1 - \alpha/2} = \Phi^{-1} (1 - \frac{\alpha}{2})$, $\Phi$ being the standard normal distribution function. Consider the interval

$$I_{n_d} = [T_n - d, T_n + d] \quad (3.4.2)$$

as a possible confidence interval for $\theta$. Its length is $2d$ and if $\sigma^2$ is known the best fixed sample size which minimizes the length can be obtained from (3.4.1) and (3.4.2) which is given by (cf. Chow and Robbins (1965)).

$$n_d = d^2 Z_{1 - \alpha/2}^2 \sigma^2. \quad (3.4.3)$$

and

$$\lim_{d \to 0} P_n (\theta \in I_{n_d}) = 1 - \alpha.$$

For small $d$, $I_{n_d}$ provides a bounded length confidence interval for $\theta$ with asymptotic convergence probability $1 - \alpha$.

However, when $\sigma^2$ is unknown, $\sigma^2$ in (3.4.3) it can be replaced by an estimator

$$S_n^2 = S_n (X_1, X_2 \ldots X_n),$$

but then we cannot use the above fixed sample size procedure. So we replace $n_d$ by a random sample or a stopping rule

$$N_d = \min \{n \geq n_0 : n \geq d^{-2} Z_{1 - \alpha/2}^2 S_n^2 \},$$
where, \( n_0 \) is an initial sample size. Then we use the confidence interval
\[
I_{N_d} = \{T_{Nd} - d, T_{Nd} + d\}
\]
for estimating \( \theta \).

For the stopping rule \( N_d \) and the interval \( I_{N_d} \) Chow and R. 1-bins (1965) have proved the following properties.
1. \( N_d \) is non decreasing in \( d \)
2. \( N_d \) is finite with probability one for every \( d > 0 \)
3. \( N_d \) \( d \rightarrow 1 \) as \( d \rightarrow 0 \)
4. \( \lim_{n \rightarrow \infty} P_{\theta}(\theta \in I_{N_d}) = 1 - \alpha. \)

Our problem here in this section is to find a confidence interval for \( \mu = \text{E}(X_i) \) for the model (2.1.1) having prescribed width \( 2d \) and a converge probability \( 1 - \alpha \).
That is to find \( I_{N_d} \) such that \( P[\mu \in I_{N_d}] = 1 - \alpha, 0 \leq \alpha < 1 \). We have proved in Theorem 2.3.1 [See (2.3.2)] that
\[
\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma_i^2),
\]
where \( \sigma_i^2 \) is as defined in (2.3.3) and
\[
\sigma_i^2 = \frac{\sigma^2}{n} B(k),
\]
\[
B(k) = 1 + 2 \sum_{j=1}^{n} c(k^j).
\]

Based on the above result an approximate confidence interval for \( \mu \) when \( \sigma^2 \) and \( k \) are known are constructed in Section 2.3.

Let \( I_{K_i} \) be the required confidence interval. Then
\[ I_{k_0} = [\bar{X}_{k_0} - d, \bar{X}_{k_0} + d] \]

where

\[ K_0 = [d^{-2} Z^2_{1-\alpha/2} \sigma^2 B(k)] \quad (3.4.4) \]

and \( Z_{1-\alpha/2} \) is such that

\[ \frac{1}{\sqrt{2\pi}} \int_{Z_{1-\alpha/2}}^{Z_{1-\alpha/2}} \exp\left\{-\frac{u^2}{2}\right\} du = 1-\alpha. \]

Note that from (3.4.4) that \( K_0 \to \infty \) when \( d \to 0 \) and

\[ P[\mu \in I_{k_0}] = P\left[ \sqrt{K_0} \left| \bar{X}_{k_0} - \mu \right| \leq \frac{d\sqrt{K_0}}{\Delta} \right] \to 1-\alpha. \]

where

\[ \Delta = [\sigma^2 B(k)]^{1/2} \quad (3.4.5) \]

When at least one of the parameters \( \sigma^2, k \) is unknown we proposes a sequential confidence interval. For that we define a stopping rule as in the case of point estimation,

\[ N = \inf\{n \geq n_0 : n \geq d^{-2} Z^2_{1-\alpha/2} \left[ S^2_n B(\tilde{k}_n) + n^{-h} \right]\}, \quad (3.4.6) \]

where

\[ n_0 \text{ is an initial sample size} \]

\[ S^2_n = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \]

\( \tilde{k}_n \) is as defined by (2.4.2) and \( h \) is a suitable constant to be defined later. Note that from the above definition of stopping rule \( N \geq d^2 Z^2_{1-\alpha/2} N^h \)

That is \( N \geq \left( \frac{Z_{1-\alpha/2}}{d} \right) \bar{Z}^{1.5} \)
Thus when \( d \to 0, N \to \infty. \)

The performance of the above stopping time \( N \) and the corresponding confidence interval \( I_N \) are discussed in the following Theorem.

**Theorem 3.4.1** If \( E \left| Z_i \right|^{2p} < \infty \) for \( p > 2 \) and \( h \in (0, p-2) \) then as \( d \to 0 \)

i \[ \frac{N}{K_0} \xrightarrow{d} 1 \]

ii \[ E \left( \frac{N}{K_0} \right) \to 1 \]

iii \[ P[\mu \in I_N] \to 1 - \alpha. \]

The following lemmas are needed to prove this theorem and hence we prove them first.

**Lemma 3.4.1.** If \( E \left| Z_i \right|^{2p} < \infty \) for \( p > 2 \)

\[ P[ \left| S_n \alpha^2 - A(\bar{k}_n) - \alpha^2 A(k) \right| > \varepsilon] = O(n^{p/2}). \]

**Proof:** We have proved in Section 3.2 that

\[ P[ \left| A(\bar{k}_n) - A(k) \right| > \varepsilon] = O(n^p), \]

which implies

\[ P[ \left| A(\bar{k}_n) - A(k) \right| > \varepsilon] = O(n^{p/2}). \]

In view of the Result 1.6.12 here it is enough to prove

\[ P[ \left| S_n^2 - \sigma^2 \right| > \varepsilon] = O(n^{p/2}). \]

As for (3.4.7) consider

\[ \left\| S_n^2 - \sigma^2 \right\|_p = \left\| n^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 - \sigma^2 \right\|_p \]

\[ = \left\| n^{-1} \sum_{i=1}^{n} (X_i - \mu)^2 + 2(\mu - \bar{X}_n)n^{-1} \sum_{i=1}^{n} (X_i - \mu) + (\mu - \bar{X}_n)^2 - \sigma^2 \right\|_p \]

Now using (3.2.5), Minkowski inequality and Schwarz inequality we can write
\[ \left\| S_n^2 - \sigma^2 \right\|_p \leq n \left( k^2 \sum_{i=1}^{n} (Z_i - \alpha)^2 + 2kn (k\alpha - \mu) \sum_{i=1}^{n} (Z_i - \alpha) + (k\alpha - \mu)^2 \right) \]

\[ + 2kn (\mu - \bar{X}_n) \sum_{i=1}^{n} (Z_i - \alpha) + 2(\mu - \bar{X}_n) (k\alpha - \mu) + (\mu - \bar{X}_n)^2 - \sigma^2 \right\|_p \]

\[ \leq n \left( k^2 \sum_{i=1}^{n} \left[ (Z_i - \alpha)^2 - \theta \right] \right)_p + 2n \left( k\alpha - \mu \right) \sum_{i=1}^{n} (Z_i - \alpha) \right\|_p \]

\[ + 2n \left[ \sum_{i=1}^{n} (Z_i - \alpha) \right\|_p \left\| (\mu - \bar{X}_n) \right\|_{2p} + 2(k\alpha - \mu) \left\| (\mu - \bar{X}_n) \right\|_p \]

\[ + \left\| (\mu - \bar{X}_n)^2 \right\|_p + \left\| k^2 \theta - \sigma^2 \right\|_p , \tag{3.4.8} \]

where \( \theta = E(Z_i - \alpha)^2 \).

Note that \( \sum_{i=1}^{n} [(Z_i - \alpha)^2 - \theta] \) is a mean zero martingale w.r.t \( F_n = \sigma \{ X_0, Z_1, ..., Z_n \} \) and hence from M-Z inequality the first term in (3.4.8) can be calculated as

\[ \left\| n \left( k^2 \sum_{i=1}^{n} [(Z_i - \alpha)^2 - \theta] \right) \right\|_p = O(n^{1/2}) \]

Now by applying Schwartz inequality, Lemma 3.2.1 and M-Z inequality each term in (3.4.8) can shown to be of \( O(n^{1/2}) \). Thus we have

\[ \left\| S_n^2 - \sigma^2 \right\|_p = O(n^{1/2}) \]

Now from Markov inequality

\[ \text{Pr} \left[ \left| S_n^2 - \sigma^2 \right| > \varepsilon \right] \leq \frac{E \left| S_n^2 - \sigma^2 \right|^p}{\varepsilon^p} \]

\[ = O(n^{p/2}) \]

Hence we have the required result (3.4.7). The lemma is proved.

Proof of the following lemma is omitted as it is similar to that of Lemma 3.2.2.

**Lemma 3.4.2** If \( E \left| Z_i \right| ^{2p} < \infty \) for \( p > 2 \) and \( h \in (0, \varphi - 2) \) then

i. \[ \text{Pr} \left[ N \leq K(1 - \varepsilon) \right] = O(\left( \frac{d_{2p}}{2(1+h)} \right)^{p/2}) \]
ii \[ \sum_{n_k \geq 1} P(N > n) = O(d^{\frac{r}{2}}) \]

**Proof of Theorem 3.4.1**

We can prove part (i) and part (ii) using Lemmas 3.2.1 and 3.4.2. The proof is parallel to that of part (i) and part (ii) of Theorem 3.2.1. Hence we omit the details.

For part (iii) consider,

\[ P(\mu \in I_k) = P \left( |\overline{X}_n - \mu| \leq d \right) = P \left[ \frac{\sqrt{N} |\overline{X}_n - \mu|}{\Delta} \leq \frac{d\sqrt{N}}{\Delta} \right] = P \left[ \frac{\sqrt{N} |\overline{X}_n - \mu|}{\Delta} \leq \frac{Z_1 \sqrt{N}}{\sqrt{K_0}} \right], \tag{3.4.9} \]

where \( \Delta \) is as defined in (3.4.5).

Recall from Section 3.2.1,

\[ \sqrt{N} (\overline{X}_n - \mu) \xrightarrow{d} N(0, \Delta^2). \]

That is,

\[ \Delta^1 \sqrt{N} (\overline{X}_n - \mu) \xrightarrow{d} N(0,1). \]

Also we have from part (i) of Theorem 3.4.1,

\[ \frac{N}{K_0} \xrightarrow{a.s.} 1 \]

and hence \[ \frac{\sqrt{N}}{\sqrt{K_0}} \xrightarrow{a.s.} 1. \]

Now part (iii) follows from (3.4.9) and the above argument.

This completes the proof of the Theorem.
CHAPTER 4

ESTIMATION IN RANDOM COEFFICIENT AUTOREGRESSIVE MODEL

4.1 Introduction

The rest of this thesis is about sequential estimation of first order random coefficient autoregressive model RCAR(1). Linear time series models such as autoregressive models have been widely and successfully used in many fields. The reasons are that these models can be easily analysed and they provide fairly good approximations for the underlying chance mechanisms of numerous real life time-series. However, in some particular situations one may ask if there exist other models which can provide better fits. One is then led to consider nonstationary or nonlinear models. RCAR model is one such class of nonlinear models which have been found useful in many areas. Some of the specific applications of RCAR(1) models are described in Section 1.2. In the present chapter we consider the properties of RCAR(1) model and properties of least squares estimators of its parameters.

4.2 The Model and its Properties

Let \{X_i\} be a sequence of r.v.s defined by an RCAR(1) model

\[ X_i - \mu = (b+ \beta_i)(X_{i-1} - \mu) + \varepsilon_i, \quad i = 1, 2, \ldots \tag{4.2.1} \]

where \( \mu = E(X_i) \) and the r.v.s satisfies the assumptions A1 - A4 in section 1.2 with \( p=1 \). They are

a1) \( \{\varepsilon_i, \quad i = \pm 1, \pm 2, \ldots \} \) is a sequence of iid r.vs with mean zero and variance \( \sigma^2 < \infty \)
a2) \( \{ \beta_i, i = \pm 1, \pm 2, \ldots \} \) is a sequence of iid r.v.s with mean zero and variance \( \gamma < \infty \)

a3) The sequence \( \{ \varepsilon_i \} \) and \( \{ \beta_i \} \) are statistically independent.

a4) \( X_i \) is independent of \( \varepsilon_i \) and \( \beta_i \) for \( j \neq i \).

Recursively using (4.2.1) we can express \( X_i - \mu \) as

\[
X_i - \mu = V_i + W_i, \quad i = 1, 2, \ldots
\]

(4.2.2)

where

\[
V_i = \varepsilon_i + \sum_{j=1}^{m} \prod_{k=0}^{j} (b + \beta_{j-k}) \varepsilon_{i-j}, \quad \text{for any } m
\]

(4.2.3)

and

\[
W_i = \prod_{k=0}^{m} (b + \beta_{j-k}) (X_{i-(m+1)} - \mu)
\]

(4.2.4)

Here \( \{ V_i \} \) defined in (4.2.3) is an \((m+1)\) dependent stationary process. [See Definition 1.6.6]

In the following we discuss the conditions required for the stationarity of \( \{ X_i \} \). Using (4.2.2) we can write

\[
(X_i - \mu) - V_i = W_i.
\]

Thus

\[
E[(X_i - \mu) - V_i]^2 = E[W_i^2].
\]

(4.2.5)

Now using the assumptions for the model (4.2.1) and (4.2.4)

\[
E[W_i^2] = E\left[\prod_{k=0}^{m} (b + \beta_{j-k})\right]^2 E\left(X_{i-(m+1)} - \mu\right)^2
\]
Now if \((b^2 + \gamma) < 1\) and \(\mathbb{E}(X_{t \rightarrow m} - \mu)^2 < \infty\), then as \(m \to \infty\) \(\mathbb{E}[W_t^2]\) converges to zero. Hence from (4.2.5) and from definition of convergence in mean square it follows that \(\mu\) converge in mean square and, hence in probability to \(V_i\). Thus we have

\[
\mathbb{P}(X_t - \mu - V_i) \to 0.
\]

(4.2.6)

Thus there exist a solution for the model (4.2.1) if \((b^2 + \gamma) < 1\). The solution is given by

\[
X_t - \mu = \varepsilon_t + \sum_{j=1}^{\infty} \left[ \prod_{k=0}^{j-1} (b + \beta_{t-k}) \right] \varepsilon_{t-j}.
\]

(4.2.7)

The solution for \(X_t - \mu\) defined by (4.2.7) contains only iid r.v.s \(\varepsilon_t\)'s and \(\beta_t\)'s. Hence this solution is stationary also. Nicholls and Quinn (1982) proved that the solution to \(X_t - \mu\) defined in (4.2.7) is ergodic. [See Theorem 2.7 of Nicholls and Quinn, (1982)].

We have noted in (4.2.6) that \(\mathbb{P}(X_t - \mu - V_i) \to 0\). Now asymptotic properties of \((X_t - \mu)\) is same as that of \(V_i\). [See Rao (1973), pp. 122]. Moreover, the asymptotic distribution of \(\sqrt{n}(X_n - \mu)\) is also same as that of \(\sqrt{n}(V_n - \mu)\) where \(\mu\) is the mean of \(V_i\) and \(\overline{V}_n = \frac{1}{n} \sum_{i=1}^{n} V_i\). Since \(\mathbb{E}(V_i^2)\) is finite, and \(V_i\) is \((m+1)\)-dependent r.v.s, we have for fixed \(m\) [See Result 1.6.13],

\[
\sqrt{n}(\overline{V}_n - \mu) \xrightarrow{d} N\left(0, \sum_{n=m}^{\infty} \text{Cov}(V_i, V_{i+k})\right).
\]

(4.2.8)
Also as \( m \to \infty \), \( V_i \) converges to \((X_i - \mu)\) in mean square. Hence \( \text{Cov}(V_i, V_{i+h}) \) also converges to \( \text{Cov}(X_i - \mu, X_{i+h} - \mu) \) [cf. Rohatgi(1976), pp.248]. Thus as \( m \to \infty \) the variance of the asymptotic distribution in (4.2.8) converges to

\[
\sum_{h=-\infty}^{\infty} \text{Cov}(X_i - \mu, X_{i+h} - \mu).
\]

That is,

\[
\text{A.V.} \left[ \sqrt{n}(\bar{V}_n - \mu) \right] = \sum_{h=-\infty}^{\infty} \text{Cov}(X_i - \mu, X_{i+h} - \mu). \quad (4.2.9)
\]

For the sequence defined by (4.2.1) using (4.2.7) and assumption on the model we have

\[
V(X_i) = \mathbb{E}\{X_i - \mu\}^2 = \mathbb{E}\left[ \varepsilon_i + \sum_{j=1}^{\infty} \prod_{l=1}^{j} (b + \beta_{i-k}) \varepsilon_{i-j} \right]^2
\]

\[
= \mathbb{E}[\varepsilon_i^2] + \mathbb{E}[b + \beta]^2 \mathbb{E}[\varepsilon_{i-1}^2] + \mathbb{E}[b + \beta]^2 \mathbb{E}[\varepsilon_{i-2}^2] + \ldots
\]

\[
+ \ldots + 2 \mathbb{E}[b + \beta] \mathbb{E}(\varepsilon_i) \mathbb{E}(\varepsilon_{i-1}) + \ldots
\]

\[
= \sigma^2 + (b^2 + \gamma) \sigma^2 + (b^2 + \gamma)^2 \sigma^2 + \ldots
\]

\[
\frac{\sigma^2}{1 - (b^2 + \gamma)} = V \text{ (say).} \quad (4.2.10)
\]

\( V = \text{Var}(X_i) \) defined in (4.2.10) is finite if \( \mathbb{E}[\varepsilon_i^2] = \sigma^2 < \infty \) and \( b^2 + \gamma < 1 \).

Now consider

\[
r(h) = \text{Cov}(X_i, X_{i+h}).
\]

Using the assumptions on the model we can write

\[
X_{i+h} - \mu = \varepsilon_{i+h} + (b + \beta_{i+h}) \varepsilon_{i+h-1} + \ldots + \prod_{k=1}^{h} (b + \beta_{i+h-k})(X_i - \mu).
\]
Thus

\[ r(h) = \mathbb{E}[(X_i - \mu)(X_{i+h} - \mu)] \]

\[ = \mathbb{E}[\epsilon_{i+h}(X_i - \mu)] + \mathbb{E}[(b + \beta_{i+h}) \epsilon_{i+h-1} (X_i - \mu)] + \ldots \]

\[ + \mathbb{E}\left[ \prod_{k=1}^{h} (b + \beta_{i+h-k}) (X_i - \mu)^2 \right] \]

\[ = \mathbb{E}[\epsilon_{i+h}] \mathbb{E}(X_i - \mu) + \mathbb{E}[(b + \beta_{i+h})] \mathbb{E}[\epsilon_{i+h-1}] \mathbb{E}(X_i - \mu) + \ldots \]

\[ + \mathbb{E}\left[ \prod_{k=1}^{h} (b + \beta_{i+h-k}) \right] \mathbb{E}(X_i - \mu)^2 = b^h \text{Var}(X_i) \]

On similar lines

\[ r(-h) = \text{Cov}(X_i, X_{i-h}) = b^h \text{Var}(X_i) = r(h) \quad (4.2.11) \]

Now using (4.2.9), (4.2.10) and (4.2.11)

\[ \text{A.V}[\sqrt{n}(\bar{X}_n - \mu)] = \sum_{h=-\infty}^{\infty} r(h) \]

\[ = \frac{\sigma^2}{1 - (b^2 + \gamma)} \left[ 1 + 2(b + b^2 + \ldots) \right] \]

\[ = \frac{\sigma^2}{1 - (b^2 + \gamma)} \left[ 1 + \frac{2b}{1 - b} \right] \]

\[ = \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b} \quad (4.2.12) \]

From (4.2.8) and from the above discussion we have

\[ \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N\left(0, \frac{\sigma^2}{1 - (b^2 + \gamma)} \frac{1 + b}{1 - b}\right) \quad (4.2.13) \]
The next section deals with least squares estimation of parameters of RCAR(1) model defined in (4.2.1).

4.3 Properties of Least Square Estimators

The main objective of estimating the unknown parameters of a stationary time series \( \{X_n\} \) is to provide predictors of \( X_n \) given the past values of the process. The least squares estimators are those estimators which minimize the sum of squares of errors. Random coefficient autoregressive process are nonlinear in nature with two error components. Thus the least squares estimation procedure adopted here is a two step procedure. Many researchers have suggested estimators for regression parameter \( b \) in the model (4.2.1) that are efficient in the presence of nuisance parameters. For example see Koul and Schick (1996) and Schick (1996). The least squares estimators for \( b, \sigma^2 \) and \( \gamma \) obtained by Nicholls and Quinn (1982) are given below.

Assuming \( \mu = E(X_i) \) is known,

\[
\hat{b}_n = \frac{\sum_{i=1}^{n} (X_i - \mu)(X_{i+1} - \mu)}{\sum_{i=1}^{n} (X_{i+1} - \mu)^2}
\]

\[
\hat{\gamma}_n = \frac{\sum_{i=1}^{n} \hat{U}_i^2 (Z_i - \bar{Z})}{\sum_{i=1}^{n} (Z_i - \bar{Z})^2}
\]

\[
\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \hat{U}_i^2 - \hat{\gamma}_n \bar{Z}
\]

where,

\( U_i = \beta_i (X_i - \mu) + \varepsilon_i = (X_i - \mu) - b(X_{i+1} - \mu) \)

\( \hat{U}_i = (X_i - \mu) - \hat{b}_n (X_{i+1} - \mu) \)

\( Z_i = (X_i - \mu)^2 \)
\[ \bar{Z} = n^{-1} \sum_{i=1}^{n} Z_i, \]

when \( \mu \) is unknown the above estimators can be modified as

\[ \hat{b}_n = \sum_{i=1}^{n} (X_i - \bar{X}_n) (X_{i-1} - \bar{X}_{n-1}^*) / \sum_{i=1}^{n} (X_{i-1} - \bar{X}_{n-1}^*)^2 \quad (4.3.1) \]

\[ \hat{\gamma}_n = \sum_{i=1}^{n} \hat{U}_i^2 (Z_i - \bar{Z}) / \sum_{i=1}^{n} (Z_i - \bar{Z})^2 \quad (4.3.2) \]

\[ \hat{\sigma}_n^2 = n \sum_{i=1}^{n} \hat{U}_i^2 - \hat{\gamma}_n \bar{Z} \quad (4.3.3) \]

where,

\[ \bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i, \quad \bar{X}_{n-1}^* = n^{-1} \sum_{i=1}^{n} X_{i-1} \]

\[ U_i = \beta_i (X_i - \bar{X}_n) + \varepsilon_i = (X_i - \bar{X}_n) - b(X_{i-1} - \bar{X}_{n-1}^*) \]

\[ \hat{U}_i = (X_i - \bar{X}_n) - \hat{b}_n (X_{i-1} - \bar{X}_{n-1}^*) \]

\[ Z_i = (X_i - \bar{X}_{n-1}^*)^2 \]

\[ \bar{Z} = n^{-1} \sum_{i=1}^{n} Z_i. \]

Some of the properties of these estimators useful in sequential analysis are discussed in the following lemmas.

**Lemma 4.3.1:** If \( E|\varepsilon_i^s| < \infty \) and \( \| b + \beta_1 \|_4 < 1, \ s \geq 1, \) then \( \| \bar{X}_n - \mu \|_2 = O(n^{1/2}). \)

**Proof:** Assume without loss of generality \( \mu = 0. \) Then using (4.2.7)

\[ \| \bar{X}_n \|_4 = \left\| n^{-1} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \prod_{k=0}^{n} (b + \beta_{i-k}) \varepsilon_{i-j} + \varepsilon_i \right) \right\|. \]
Interchanging the order of summation and using Minkowski inequality we have

$$\|X_n\|_{4,5} \leq n^{1-\epsilon} \sum_{j=1}^{\infty} \left( \sum_{i=1}^{n} \left( \prod_{k=0}^{j-1} (b + \beta_i) \right) \varepsilon_{i-j} \right)_{4,5} + n^{-1} \sum_{i=1}^{n} \varepsilon_{i} \right)_{4,5}$$

(4.3.5)

By the Marcinkiewicz-Zygmund (M-Z) inequality (See Result 1.6.4),

$$\left\| \sum_{i=1}^{n} \left[ \prod_{k=0}^{j} (b + \beta_i) \right] \varepsilon_{i-j} \right\|_{4,5} = O(n^{1/2})$$

and

$$\left\| \sum_{i=1}^{n} \varepsilon_{i} \right\|_{4,5} = O(n^{1/2}).$$

Hence the required result.

The next lemma deals with $p^{th}$ moment convergence of $\hat{b}_n$.

Lemma 4.3.2: If $E|\varepsilon|^{2p} < \infty$ and $E|b + \beta_i|^{2p} < 1, p \geq 1$ then $P(|\hat{b}_n - b| > \varepsilon) = O(n^{1/2}).$

Proof: When $\mu = E(X_i)$ is known the estimator of $b$ is given by

$$\hat{b}_n = \sum_{i=1}^{n} (X_i - \mu)(X_{i-1} - \mu) / \sum_{i=1}^{n} (X_i - \mu)^2$$

Using (4.2.1) $\hat{b}_n$ can be written as

$$\hat{b}_n = \sum_{i=1}^{n} [(b + \beta_i)(X_{i-1} - \mu) + \varepsilon_i] / \sum_{i=1}^{n} (X_i - \mu)^2$$

$$= \frac{b \sum_{i=1}^{n} (X_{i-1} - \mu)^2 + \sum_{i=1}^{n} \beta_i (X_{i-1} - \mu)^2 + \sum_{i=1}^{n} \varepsilon_i (X_{i-1} - \mu)^2}{\sum_{i=1}^{n} (X_{i-1} - \mu)^2}$$
\[
\sum_{i=1}^{n} \beta_i (X_{i-1} - \mu)^2 + \sum_{i=1}^{n} \epsilon_i (X_{i-1} - \mu)^2
\]

\[
= \frac{\sum_{i=1}^{n} (X_{i-1} - \mu)^2}{\sum_{i=1}^{n} (X_{i-1} - \mu)^2} + b
\]

Thus

\[
\hat{b}_n - b = \frac{\sum_{i=1}^{n} \beta_i (X_{i-1} - \mu)^2 + \sum_{i=1}^{n} \epsilon_i (X_{i-1} - \mu)^2}{\sum_{i=1}^{n} (X_{i-1} - \mu)^2}.
\]

If \( \mu \) is unknown the estimator for \( \mu \) is \( \bar{X}_{n-1} = n^{-1} \sum_{i=0}^{n-1} X_i \). Then

\[
\hat{b}_n - b = \frac{n^{-1} J_n}{n^{-1} K_n},
\]

where,

\[
J_n = \sum_{i=1}^{n} \beta_i (X_{i-1} - \bar{X}_{n-1}^*)^2 + \sum_{i=1}^{n} \epsilon_i (X_{i-1} - \bar{X}_{n-1}^*)^2
\]

\[
K_n = \sum_{i=1}^{n} (X_{i-1} - \bar{X}_{n-1}^*)^2.
\]

The ergodic theorem (See Result 1.6.15) for \( \{X_i\} \) implies that as \( n \to \infty \), \( \bar{X}_{n-1} \to \mu \) a.s. The model (4.2.1) and the assumptions on that immediately imply that the sequences \( \{ \beta_i (X_{i-1} - \bar{X}_{n-1}^*)^2 \} \), \( \{ \epsilon_i (X_{i-1} - \bar{X}_{n-1}^*)^2 \} \) and \( (X_{i-1} - \bar{X}_{n-1}^*)^2 \) are stationary and ergodic [See Remark 1.6.1]. Now applying Ergodic Theorem for these sequences, it follows that \( n^{-1} J_n \to 0 \) and \( n^{-1} K_n \to V \) a.s as well as in the \( p \)th moment, where \( V \) is defined in (4.2.10). As a consequence we have \( (\hat{b}_n - b) \to 0 \) a.s as \( n \to \infty \).

Next we calculate \( \| n^{-1} J_n \|_p \) and \( \| n^{-1} K_n - V \|_p \).

Consider \( J_n \) defined in (4.3.7) and by some algebraic manipulations we can write

\[
n^{-1} J_n = J_{n_1} + J_{n_2} + J_{n_3} + J_{n_4} + J_{n_5},
\]
where

\[ J_n = n^j \sum_{i=1}^{n} \beta_i (X_{i-1} - \mu)^2 \]

\[ J_n = n^j (\mu - \bar{X}_{n-1}^*)^2 \sum_{i=1}^{n} \beta_i \]

\[ J_n = n^j \sum_{i=1}^{n} \varepsilon_i (X_{i-1} - \mu)^2 \]

\[ J_n = 2(\mu - \bar{X}_{n-1}^*) \sum_{i=1}^{n} \beta_i (X_{i-1} - \mu) \]

\[ J_n = n^j (\mu - \bar{X}_{n-1}^*) \sum_{i=1}^{n} \varepsilon_i \]

We write

\[ P[|J_n| > \varepsilon] \leq P[|J_{n_1}| > \varepsilon/5] + P[|J_{n_2}| > \varepsilon/5] + P[|J_{n_3}| > \varepsilon/5] + P[|J_{n_4}| > \varepsilon/5] \]

\[ + P[|J_n| > \varepsilon/5]. \quad (4.3.10) \]

Using Markov inequality we can write

\[ P[|J_n| > \varepsilon/5] \leq \frac{E[J_n]}{(\varepsilon/5)^p}. \quad (4.3.11) \]

If we define \(F_n\) as the \(\sigma\)-field induced by \((\beta_k, \varepsilon_k), k \leq n\) then

\[ E[\sum_{i=1}^{n} \beta_i (X_{i-1} - \mu)^2 | F_n-1] = \sum_{i=1}^{n-1} \beta_i (X_{i-1} - \mu)^2 + E[\beta_n (X_{n-1} - \mu)^2] \]

\[ = \sum_{i=1}^{n-1} \beta_i (X_{i-1} - \mu)^2. \]

Since \(\beta_n\) is independent of \(X_j\) for \(j < n\) and \(E(\beta_i) = 0\). Thus \(\{\sum_{i=1}^{n} \beta_i (X_{i-1} - \mu)^2, n \geq 1\}\) is a zero mean martingle w.r.t \(F_n\). By using Burkholder inequality [See Result 1.6.11], moment inequalities, assumptions of the Lemma and independence of \(\beta_i\) and \(X_i\) we have
Thus
\[
B_p^{-1} n \left\| \sum_{i=1}^{n} \beta_i (X_i - \mu)^2 \right\|_p \leq \left\| n^{-1} \sum_{i=1}^{n} \beta_i^2 (X_i - \mu)^4 \right\|^{1/2}_p
\]
\[
\leq \left\| n^{-1} \sum_{i=1}^{n} \beta_i (X_i - \mu)^2 \right\|^{1/2}_p
\]
\[
\leq n^{-1} \sum_{i=1}^{n} \beta_i (X_i - \mu)^2 \right\|_p
\]
\[
= O(1),
\]
where \( B_p = 18p^{3/2} (p-1)^{1/2} \)

Thus
\[
\left\| \sum_{i=1}^{n} \beta_i (X_i - \mu)^2 \right\|_p = O(n^{1/2}).
\]

Hence from (4.3.11) it follows that
\[
P[ | J_n | > \varepsilon / \delta ] = O(n^{p/2}). \quad (4.3.12)
\]

As for
\[
P[ | J_n | > \varepsilon / \delta ] \leq P \left( n \left( \sum_{i=1}^{n} \beta_i \right)^{1/2} \left( \left| \mu - \bar{X}_{n-1} \right| > \sqrt{\delta} \right) \right)
\]
\[
\leq C \cdot n^{p/2} \mathbb{E} \left( \left( \sum_{i=1}^{n} \beta_i \right)^{p/2} \left| \mu - \bar{X}_{n-1} \right|^p \right)
\]
\[
= O(n^{p/2}). \quad (4.3.13)
\]
Where we used the moment inequality, Cauchy-Schwarz inequality, Lemma 4.3.1 with $s = p/2$ and the fact that $\left\| \sum_{i=1}^{n} \beta_i \right\|_p = O(n^{1/2}).$ Note that $\sum_{i=1}^{n} \pi_i (X_{i-1} - \mu)^2$ and $\sum_{i=1}^{n} \beta_i (X_{i-1} - \mu)$ are mean zero martingales w.r.t $\mathcal{F}_n^\pi$. Now using similar arguments as in the case of $J_n$, we can show that

$$P\left[ \left| J_{n, \pi} \right| > \varepsilon \right] = O(n^{p/2})$$

and

$$P\left[ \left| J_{n, \pi} \right| > \varepsilon \right] = O(n^{p/2}).$$

As for $J_n$, use Schwartz inequality, Lemma 4.3.1 and M-Z inequality to get

$$P\left[ \left| J_{n, \pi} \right| > \varepsilon \right] = O(n^{p/2}).$$

Hence from the above arguments and (4.3.10) we have for $\varepsilon > 0$

$$P\left[ \left| n^{1/2} J_n \right| > \varepsilon \right] = O(n^{p/2}). \quad (4.3.14)$$

By writing

$$n^{1/2} K_n - V = n^{1/2} \sum_{i=1}^{n} ((X_{i-1} - \mu)^2 + (\mu - \bar{X}_{n-1}^*)^2 + 2(X_{i-1} - \mu)(\mu - \bar{X}_{n-1}^*)$$

$$- E(X_{i-1} - \mu)^2$$

and repeating the similar arguments as in the case of $J_n$ we get

$$\left\| n^{1/2} K_n - V \right\|_p = O(n^{1/2})$$

and hence
\[ P[|n^{-1}K_n - V| > \varepsilon] = O(n^{p/2}). \]  \((4.3.15)\)

Now Lemma 4.3.2 follows from (4.3.14), (4.3.15), (4.3.7) and Result 1.6.12. This completes the proof.

In the expression (4.3.2) replacing \(\hat{U}_i\) by \(U_i\), we write

\[ \frac{n^{-1} \sum_{i=1}^{n} U_i^2(Z_i - \bar{Z})}{n^{-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2} = \gamma \]  \((4.3.16)\)

Now \(\gamma_n - \gamma\) can be written as

\[ \frac{n^{-1} \sum_{i=1}^{n} U_i^2(Z_i - \bar{Z})}{n^{-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2} - \gamma = \frac{n^{-1} \left[ \sum_{i=1}^{n} U_i^2(Z_i - \bar{Z}) - \gamma (Z_i - \bar{Z})^2 \right]}{n^{-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2} \]

\[ = \frac{n^{-1} \sum_{i=1}^{n} (U_i - \gamma)(Z_i - \bar{Z})}{n^{-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2} \]  \((4.3.17)\)

since \(\sum_{i=1}^{n} (Z_i - \bar{Z})Z_i = 0\).

Define \(\xi_i = U_i^2 - \sigma^2 - \gamma Z_i\) \((4.3.18)\) and write

\[ \gamma_n - \gamma = n^{-1}E_n n^{-1}R_n, \]  \((4.3.19)\)

where
\[ T_n = \sum_{i=1}^{n} (Z_i - \overline{Z}) \xi_i \]  
(4.3.20)

\[ R_n = \sum_{i=1}^{n} (Z_i - \overline{Z})^2 \]  
(4.3.21)

Now by repeating the arguments used to prove lemma 4.3.2 we can prove that as \( n \to \infty, \) \( n^{-1}T_n \to 0 \) and \( n^{-1}R_n \to R \) a.s and in \( p^{th} \) moment, where \( R = \text{Var}(Z) < \infty. \) Hence from (4.3.19) we have \( \hat{\gamma}_n - \gamma \to 0. \)

Let us write

\[ \hat{\gamma}_n - \overline{\gamma} = \frac{n^{-1} \sum_{i=1}^{n} (\hat{U}_i^2 - U_i^2)(Z_i - \overline{Z})}{\sum_{i=1}^{n} (Z_i - \overline{Z})^2} \]

\[ = \frac{n^{-1} \sum_{i=1}^{n} (Z_i - \overline{Z})(\hat{U}_i - U_i)(\hat{U}_i + U_i)}{\sum_{i=1}^{n} (Z_i - \overline{Z})^2} \]

\[ = \frac{n^{-1} \sum_{i=1}^{n} (Z_i - \overline{Z})(b - b_n)(X_i - 1 - \overline{X}_{i-1}^*)\{2U_i + (b - b_n)(X_i - 1 - \overline{X}_{i-1}^*)\}}{\sum_{i=1}^{n} (Z_i - \overline{Z})^2} \]

\[ = n^{-1} H_n n^{-1} R_n, \]  
(4.3.22)

where,

\[ H_n = \sum_{i=1}^{n} (Z_i - \overline{Z})(b - b_n)(X_i - 1 - \overline{X}_{i-1}^*)\{2U_i + (b - b_n)(X_i - 1 - \overline{X}_{i-1}^*)\} \]  
(4.3.23)

and \( R_n \) is as defined by (4.3.21). Using similar arguments as before we can prove \( \hat{\gamma}_n - \overline{\gamma} \to 0 \) a.s.

**Lemma 4.3.3:** Under the conditions of Lemma 4.3.2
Proof: We write

\[ P[|\hat{\gamma}_n - \gamma| > \epsilon] \leq P[|\hat{\gamma}_n - \tilde{\gamma}_n| > \epsilon'] + P[|\tilde{\gamma}_n - \gamma| > \epsilon']. \]  

(4.3.24)

Now let us calculate \( P[|\hat{\gamma}_n - \tilde{\gamma}_n| > \epsilon'] \) and \( P[|\tilde{\gamma}_n - \gamma| > \epsilon'] \).

Using Minkowski inequality and Schwartz inequality in (4.3.20) we can write

\[ n' \|T_n\|_p \leq \left\| n^{-1} \sum_{i=1}^{n} Z_i \xi_i \right\|_p + \left\| \overline{Z} \right\|_p \left\| n^{-1} \sum_{i=1}^{n} \xi_i \right\|_2. \]

(4.3.25)

Now using (4.3.18) and (4.3.4) each term in (4.3.25) can be written as

\[ \left\| n^{-1} \sum_{i=1}^{n} Z_i \xi_i \right\|_p = \left\| n^{-1} \sum_{i=1}^{n} [\beta_i^2 (X_{i-1} - \overline{X}_{n-1})^4 + \epsilon_i (X_{i-1} - \overline{X}_{n-1}) \right. \]

\[ + 2 \beta_i \epsilon_i (X_{i-1} - \overline{X}_{n-1})^3 - (X_{i-1} - \overline{X}_{n-1})^2 \sigma^2 - (X_{i-1} - \overline{X}_{n-1})^4 \gamma \right\|_p \]

\[ \|\overline{Z}\|_p = \left\| n^{-1} \sum_{i=1}^{n} (X_{i-1} - \mu)^2 + (\mu - \overline{X}_{n-1})^2 + 2(X_{i-1} - \mu)(\mu - \overline{X}_{n-1}) \right\|_2 \]

\[ \left\| n^{-1} \sum_{i=1}^{n} \xi_i \right\|_2 = \left\| n^{-1} \sum_{i=1}^{n} [\beta_i^2 (X_{i-1} - \overline{X}_{n-1})^2 + \epsilon_i + 2 \beta_i \epsilon_i (X_{i-1} - \overline{X}_{n-1}) - \sigma^2 \right. \]

\[ \left. - (X_{i-1} - \overline{X}_{n-1})^2 \gamma \right\|_2 \]

From the proofs of earlier lemmas it follows that

\[ n' \|T_n\|_p = O(n^{1/2}). \]

Using Markov inequality for \( \epsilon > 0 \)

\[ P[n' T_n - 0 | \epsilon] = O(n^{p/2}). \]

(4.3.26)
On similar lines it can be shown that

\[ P[|n^{-1} R_n - R| > \varepsilon] = O(n^{p/2}). \] (4.3.27)

Hence from (4.3.19), (4.3.26), (4.3.27) and Result 1.6.12 we get

\[ P[|\tilde{\gamma}_n - \bar{\gamma}_n| > \varepsilon_2] = O(n^{p/2}). \] (4.3.28)

\(H_n\) defined in (4.3.23) can be written as

\[ H_n = \sum_{i=1}^{n} [2\beta_i (X_{i-1} - \bar{X}_{n-1})^3 (b - \hat{b}_n) + 2\varepsilon_i (X_{i-1} - \bar{X}_{n-1}) (b - \hat{b}_n) \]

\[ + (X_{i-1} - \bar{X}_{n-1})^3 (b - \hat{b}_n)^2 - 2\bar{\varepsilon}_i (X_{i-1} - \bar{X}_{n-1})^3 (b - \hat{b}_n) \]

\[ - 2\bar{\varepsilon}_i (X_{i-1} - \bar{X}_{n-1}) (b - \hat{b}_n) - \bar{\varepsilon}_i (X_{i-1} - \bar{X}_{n-1})^3 (b - \hat{b}_n)^2]. \]

Now repeating arguments in Lemma 4.3.2 here also we can show that

\[ P[|n^{-1} H_n - 0| > \varepsilon] = O(n^{p/2}). \] (4.3.29)

Now (4.3.27), (4.3.29), (4.3.22) and Result 1.6.12 leads to

\[ P[|\tilde{\gamma}_n - \gamma| > \varepsilon_2] = O(n^{p/2}). \] (4.3.30)

Application of (4.3.28) and (4.3.30) in (4.3.24) gives

\[ P[|\hat{\gamma}_n - \gamma| > \varepsilon_2] = O(n^{p/2}). \]

This completes the proof.

**Lemma 4.3.4:** Suppose that the conditions of Lemma 4.3.2 hold, then for \(\varepsilon > 0\)

\[ P[|\hat{\sigma}^2_n - \sigma^2| > \varepsilon] = O(n^{p/2}). \]
**Proof:** Let $\tilde{\sigma}_n^2$ be the expression for $\hat{\sigma}_n^2$ obtained by replacing $U_i$ in the place of $\hat{U}_i$ in equation (4.3.3).

Consider

$$\tilde{\sigma}_n^2 - \sigma^2 = n^{-1} \sum_{i=1}^{n} U_i^2 - \bar{\gamma}_n \bar{Z} - \sigma^2$$

and

$$\hat{\sigma}_n^2 - \tilde{\sigma}_n^2 = n^{-1} \sum_{i=1}^{n} (\hat{U}_i^2 - \bar{U}_i^2) - (\hat{\gamma}_n - \bar{\gamma}_n)\bar{Z}.$$  

Now using lemma 4.3.3 and using similar arguments as before we can show that

$$P[|\tilde{\sigma}_n^2 - \sigma^2| > \varepsilon/2] = O(n^{p/2}) \quad (4.3.31)$$

and

$$P[|\hat{\sigma}_n^2 - \tilde{\sigma}_n^2| > \varepsilon/2] = O(n^{p/2}). \quad (4.3.32)$$

But

$$P[|\hat{\sigma}_n^2 - \sigma^2| > \varepsilon] \leq P[|\tilde{\sigma}_n^2 - \sigma^2| > \varepsilon/2] + P[|\hat{\sigma}_n^2 - \tilde{\sigma}_n^2| > \varepsilon/2]. \quad (4.3.33)$$

Now Lemma 4.3.4 follows from (4.3.31), (4.3.32) and (4.3.33).

This completes the proof of Lemma 4.3.4.

The results of this chapter will be used in the following chapters for studying the sequential estimation.