Chapter -3

Generalized Sasakian-Space-Forms

Admitting Quarter–Symmetric Metric Connection
Generalized Sasakian-Space-Forms Admitting
Quarter–Symmetric Metric Connection

3.1. Introduction:

In 1975, Golab [59] defined and studied quarter-symmetric connection in a differentiable manifold. A linear connection $\nabla$ on an $n$-dimensional Riemannian manifold $(M^n, g)$ is said to be a quarter-symmetric connection if its torsion tensor $\overline{T}$ given by

$$\overline{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

is of the form

$$\overline{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \quad (3.1.1)$$

where $\eta$ is 1-form and $\phi$ is a tensor of type $(1,1)$. In addition, a quarter-symmetric linear connection $\overline{\nabla}$ satisfying the condition

$$\overline{\nabla}_X g)(Y,Z) = 0, \quad (3.1.2)$$

for all $X, Y, Z \in T_p M^n$, where $T_p M^n$ is the Lie algebra of vector fields of the manifold $M^n$, $\overline{\nabla}$ is said to be quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric connection reduces to a semi-symmetric connection [59]. Quarter-symmetric metric
connections have also studied by Biswas and De [21], De and De [38], De and Mondal [40], Singh, Pandey and Tiwari [98], Yano and Imai [110] and many others.

On the other hand, a generalized Sasakian-space-form was defined by Alegre et al. [5] as the almost contact metric manifold \((M^n, \phi, \xi, \eta, g)\), whose curvature tensor is given by

\[
R = f_1 R_1 + f_2 R_2 + f_3 R_3, \tag{3.1.3}
\]

where \(f_1, f_2, f_3\) are some differentiable functions on \(M^n\) and

\[
R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \\
R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\
R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \tag{3.1.4}
\]

for any vector fields \(X, Y, Z\) on \(M^n\). In such a case, we denote the manifold as \(M^n(f_1, f_2, f_3)\). This kind of manifold appears as a generalization of the well-known Sasakian-space-forms by taking \(f_1 = (c - 1)/4\). It is known that any three-dimensional \((\alpha, \beta)\)-trans-Sasakian manifold with \(\alpha, \beta\) depending on \(\xi\) is a generalized Sasakian-space-form [7]. Alegre et al. [6] have studied about B. Y. Chen’s inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. Al-Ghefari et al. analyse the CR submanifolds of generalized Sasakian-space-forms ([10],[11]). Kim [66] has
studied conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms. De and Sarkar [45] have studied generalized Sasakian-space-forms regarding conharmonic curvature tensor. Conharmonic curvature tensor of generalized Sasakian-Space-forms have also been studied by De, Singh and Pandey [49]. Generalized Sasakian-space-forms have been studied by Singh and Pandey ([97],[99]) and many others.

### 3.2. Generalized Sasakian-Space-Forms

In an almost contact metric manifold, we have [23]

\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad (3.2.1)
\]

\[
\eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \quad (3.2.2)
\]

\[
g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (3.2.3)
\]

\[
g(\phi X,Y) = -g(X,\phi Y), \quad g(\phi X,X) = 0, \quad (3.2.4)
\]

\[
(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \quad (3.2.5)
\]

where \(\phi\) is a \((1, 1)\) tensor, \(\xi\) is a vector field, \(\eta\) is a 1-form and \(g\) is a Riemannian metric. The metric \(g\) induces an inner product on the tangent space of the manifold. Again, we have in a generalized Sasakian-space-forms [5]
\[ R(X,Y)Z = f_1 [g(Y,Z)X - g(X,Z)Y] + f_2 [g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X] + 2g(X,\phi Y)\phi Z + f_3 [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \]

(3.2.6)

for any vector fields \( X, Y, Z \) on \( M^n \), where \( R \) denotes the curvature tensor of \( M^n \) and \( f_1, f_2, f_3 \) are smooth functions on the manifold. The Ricci operator \( Q \), Ricci tensor \( S \) and the scalar curvature \( r \) of the manifold of dimension \((2n+1)\) are respectively given by [66]

\[ QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi, \]

(3.2.7)

\[ S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y), \]

(3.2.8)

\[ r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \]

(3.2.9)

In view of equations (3.2.6), (3.2.7) and (3.2.8), we have

\[ R(X,Y)\xi = (f_1 - f_3) [\eta(Y)X - \eta(X)Y], \]

(3.2.10)

\[ R(\xi,X)Y = (f_1 - f_3) [g(X,Y)\xi - \eta(Y)X], \]

(3.2.11)

\[ \eta(R(X,Y)Z) = (f_1 - f_3) [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \]

(3.2.12)

\[ S(X,\xi) = 2n (f_1 - f_3)\eta(X). \]

(3.2.13)
3.3. Quarter-Symmetric Metric Connection

Let $\bar{\nabla}$ be the linear connection and $\nabla$ be the Levi-Civita connection of a generalized Sasakian space-form $M^n$ such that

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y), \quad (3.3.1)$$

where $H$ is a tensor field of type (1,2). The linear connection $\bar{\nabla}$ is said to be a quarter-symmetric metric connection if it satisfies

$$H(X,Y) = \frac{1}{2} [\bar{T}(X,Y) + \bar{T}'(X,Y) + \bar{T}'(Y,X)] \quad (3.3.2)$$

where $\bar{T}'$ is a tensor of type (1,2) defined on $M^n$ as

$$g(\bar{T}'(Z,X),Y) = g(\bar{T}'(X,Y),Z). \quad (3.3.3)$$

In view of equations (3.1.1) and (3.3.3), we have

$$\bar{T}'(X,Y) = \eta(X)\phi Y - g(\phi X,Y)\xi. \quad (3.3.4)$$

Now, using equations (3.1.1) and (3.3.4) in equation (3.3.2), we get

$$H(X,Y) = \eta(Y)\phi X - g(\phi X,Y)\xi. \quad (3.3.5)$$

Hence a quarter-symmetric metric connection $\bar{\nabla}$ in a generalized Sasakian space form $M^n$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi. \quad (3.3.6)$$
Thus, the above equation is the relation between quarter-symmetric metric connection and the Levi–Civita connection. The curvature tensor $\overline{R}$ in $M^n$ of quarter–symmetric metric connection $\overline{\nabla}$ is defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_YZ - \overline{\nabla}_Y \overline{\nabla}_XZ - \overline{\nabla}_{[X,Y]}Z.$$  \hspace{1cm} (3.3.7)

Using equation (3.3.6) in above equation, we get

$$\overline{R}(X,Y)Z = R(X,Y)Z + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X$$

$$+ (f_1 - f_3)[\{\eta(X)Y - \eta(Y)X\} \eta(Z)$$

$$+ \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\} \xi],$$ \hspace{1cm} (3.3.8)

where $\overline{R}$ and $R$ are the curvature tensors of $M^n$ with respect to $\overline{\nabla}$ and $\nabla$ respectively. Taking the inner product of equation (3.3.8) with $U$, we get

$$\overline{R}(X,Y,Z,U) = R(X,Y,Z,U) + g(\phi X,Z)g(\phi Y,U) - g(\phi Y,Z)g(\phi X,U)$$

$$+ (f_1 - f_3)[\{\eta(X)g(U,Y) - \eta(Y)g(U,X)\} \eta(Z)$$

$$+ \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\} \eta(U).$$ \hspace{1cm} (3.3.9)

Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $X = U = e_i$ in equation (3.3.9) and summing over $i$, $1 \leq i \leq n$, we get

$$\overline{S}(Y,Z) = S(Y,Z) - (1 + f_1 - f_3)g(Y,Z).$$
\[ + (1 - f_1 + f_3) \eta(Y) \eta(Z), \]  

which gives

\[ \bar{Q}Y = QY - (1 + f_1 - f_3) Y + (1 - f_1 + f_3) \eta(Y) \xi, \]  

where \( \bar{Q} \) and \( Q \) are the Ricci operators of type \((1,1)\), i.e. \( S(Y, Z) = g(\bar{Q}Y, Z) \) and \( S(Y, Z) = g(QY, Z) \) with respect to \( \bar{\nabla} \) and \( \nabla \) respectively.

Again, putting \( Y = Z = e_i \) in equation (3.3.10), we get

\[ \bar{r} = r - (n-1) - (n+1)(f_1 - f_3), \]  

where \( \bar{r} \) and \( r \) are the scalar curvatures with respect to \( \bar{\nabla} \) and \( \nabla \) respectively.

Now, writing two more equations by the cyclic permutations of \( X, Y \) and \( Z \) from equation (3.3.8), we get

\[ \bar{R}(Y, Z)X = R(Y, Z)X + g(\phi Y, X) \phi Z - g(\phi Z, X) \phi Y \]

\[ + (f_1 - f_3) \{ \eta(Y) Z - \eta(Z) Y \} \eta(X) \]

\[ + \{ g(Y, X) \eta(Z) - g(Z, X) \eta(Y) \} \xi, \]  

(3.3.13)
and

\[
\bar{R}(Z,X)Y = R(Z,X)Y + g(\phi Z,Y)\phi X - g(\phi X,Y)\phi Z \\
+ (f_1 - f_3)\left\{ [\eta(Z)X - \eta(X)Z] \eta(Y) \\
+ \{ g(Z,Y)\eta(X) - g(X,Y)\eta(Z) \} \xi \right\}. \quad (3.3.14)
\]

Adding equations (3.3.8), (3.3.13) and (3.3.14) and using the fact that

\[ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \]

we get

\[
\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 2 \left[ g(\phi X,Z)\phi Y \\
+ g(\phi Y,X)\phi Z + g(\phi Z,Y)\phi X \right]. \quad (3.3.15)
\]

Thus we can state as follows-

**Theorem (3.3.1)**: A generalized Sasakian-space-form admitting quarter symmetric metric connection satisfies equation (3.3.15).

Now, interchanging \( X \) and \( Y \) in equation (3.3.9), we get

\[
\bar{R}(Y,X,Z,U) = R(Y,X,Z,U) + g(\phi Y,Z)g(\phi X,U) - g(\phi X,Z)g(\phi Y,U) \\
+ (f_1 - f_3)\left\{ [\eta(Y)g(U,X) - \eta(X)g(U,Y)] \eta(Z) \\
+ \{ g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \} \eta(U) \right\}. \quad (3.3.16)
\]
Adding equations (3.3.9) and (3.3.16) and using the fact that $R(X,Y,Z,U) + R(Y,X,Z,U) = 0,$ we get

$$\overline{R}(X,Y,Z,U) + \overline{R}(Y,X,Z,U) = 0.
$$ (3.3.17)

Again interchanging $Z$ and $U$ in equation (3.3.9) and adding it to equation (3.3.9) with fact that $R(X,Y,Z,U) + R(X,Y,U,Z) = 0,$ we get

$$\overline{R}(X,Y,Z,U) + \overline{R}(X,Y,U,Z) = 0.
$$ (3.3.18)

Now, interchanging pair of slots in equation (3.3.9), we get

$$\overline{R}(Z,U,X,Y) = R(Z,U,X,Y)

+ g(\phi Z,X)g(\phi U,Y) - g(\phi U,X)g(\phi Z,Y)

+ (f_1 - f_2)\left[ \eta(Z)g(Y,U) - \eta(U)g(Y,Z) \right] \eta(X)

+ \left[ g(Z,X)\eta(U) - g(U,X)\eta(Z) \right] \eta(Y)].
$$ (3.3.19)

Subtracting this equation from equation (3.3.9) and using the fact that $R(X,Y,Z,U) = R(Z,U,X,Y),$ we get

$$\overline{R}(X,Y,Z,U) = \overline{R}(Z,U,X,Y).
$$ (3.3.20)

In view of equations (3.3.17), (3.3.18) and (3.3.20), we can state as follows-
**Theorem (3.3.2):** A generalized Sasakian-space-form admitting quarter-symmetric metric connection satisfies

(i) \( \bar{R}(X,Y,Z,U)+\bar{R}(Y,X,Z,U) = 0. \)

(ii) \( \bar{R}(X,Y,Z,U)+\bar{R}(X,Y,U,Z) = 0. \)

(iii) \( \bar{R}(X,Y,Z,U)-\bar{R}(Z,U,X,Y) = 0. \)

Now, putting \( Z = \xi \) in equation (3.3.8) and using equations (3.2.2) and (3.2.10), we get

\[
\bar{R}(X,Y)\xi = 0. \tag{3.3.21}
\]

Taking the inner product of equation (3.3.8) with \( \xi \) and using equations (3.2.2) and (3.2.12), we get

\[
\eta(\bar{R}(X,Y)Z) = 0. \tag{3.3.22}
\]

Putting \( X = \xi \) in equation (3.3.8) and using equations (3.2.1) and (3.2.11), we get

\[
\bar{R}(\xi,Y)Z = 0. \tag{3.3.23}
\]

Again

\[
\bar{R}(\xi,Y)Z = -\bar{R}(Y,\xi)Z = 0. \tag{3.3.24}
\]

Putting \( Y = \xi \) in equation (3.3.10) and using equation (3.2.13), we obtain
\[ \bar{S}(\xi, Z) = (2n-2)(f_1 - f_3)\eta(Z). \] \hfill (3.3.25)

By virtue of equations (3.3.21), (3.3.22), (3.3.23), (3.3.24) and (3.3.25), we can state as follows-

**Theorem (3.3.3):** A generalized Sasakian–space-form admitting quarter-symmetric metric connection satisfies

(i) \[ R(X, Y)\xi = 0, \]

(ii) \[ \eta(\bar{R}(X, Y)Z) = 0, \]

(iii) \[ \bar{R}(\xi, Y)Z = 0, \]

(iv) \[ \bar{R}(\xi, Y)Z = -\bar{R}(Y, \xi)Z = 0, \]

(v) \[ \bar{S}(\xi, Z) = (2n-2)(f_1 - f_3)\eta(Z). \]

Now, consider

\[ (R(\xi, X)\bar{R})(Y, Z)U = 0, \] \hfill (3.3.26)

which gives

\[ R(\xi, X)\bar{R}(Y, Z)U - \bar{R}(R(\xi, X)Y, Z)U - \bar{R}(Y, R(\xi, X)Z)U \]

\[ -\bar{R}(Y, Z)R(\xi, X)U = 0. \] \hfill (3.3.27)

Using equations (3.2.11), (3.3.21) and (3.3.22) in above equation, we obtain
\[(f_1 - f_3)\{g(X, R(Y, Z)U) + \eta(Y)R(X, Z)U + \eta(Z)R(Y, X)U

- g(X, U) R(Y, Z)\xi + \eta(U)R(Y, Z)X = 0. \quad (3.3.28)\]

By virtue of equation (3.3.8), above equation reduces to

\[(f_1 - f_3)\{g(X, R(Y, Z, U)\xi + \eta(Y)R(X, Z)U + \eta(Z)R(Y, X)U

- g(X, U)R(Y, Z)\xi + \eta(U)R(Y, Z)X + g(\phi Y, U)g(X, \phi Z)\xi

- g(\phi Z, U)g(X, \phi Y)\xi + g(\phi X, U)\eta(Y)\phi Z - g(\phi Z, U)\eta(Y)\phi X

+ g(\phi Y, U)\eta(Z)\phi X - g(\phi X, U)\eta(Z)\phi Y - g(X, U)\eta(\phi Y)\phi Z

+ g(X, U)\eta(\phi Z)\phi Y + g(\phi Y, X)\eta(U)\phi Z - g(\phi Z, X)\eta(U)\phi Y

+ (f_1 - f_3)\{2g(Y, U)\eta(X)\eta(Z)\xi - g(Z, U)\eta(X)\eta(Y)\xi

+ 2\eta(X)\eta(Y)\eta(U)Z - 2\eta(X)\eta(U)\eta(Z)Y - 2g(X, U)\eta(Y)\eta(Z)\xi

- g(X, U)\eta(Y)Z + g(X, U)\eta(Z)Y - g(X, U)\eta(Y)\eta(Z)\xi\} = 0. \quad (3.3.29)\]
Taking the inner product of above equation with $\xi$ and using equations (3.2.2) and (3.2.12), we get

\[
(f_1 - f_3)[R(Y,Z,U)X + (f_1 - f_3)\{g(Z,U)\eta(X)\eta(Y) - g(Y,U)\eta(X)\eta(Z) \\
+ g(Z,X)\eta(U)\eta(Y) - g(Y,X)\eta(Z)\eta(U)\} + (f_1 - f_3)[2g(Y,U)\eta(X)\eta(Z) \\
- 3g(X,U)\eta(Y)\eta(Z) - g(Z,U)\eta(X)\eta(Y)] \\
+ g(\phi Y,U)g(X,\phi Z) - g(\phi Z,U)g(X,\phi Y)] = 0. \quad (3.3.30)
\]

Putting $Y = X = e_i$ in above equation and taking summation over $i, \ 1 \leq i \leq n,$ we get

\[
S(Z,U) = g(U,Z) + [(f_1 - f_3)(n+1) - 1]\eta(U)\eta(Z). \quad (3.3.31)
\]

Thus we can state as follows-

**Theorem (3.3.4):** A generalized Sasakian-space-form with quarter-symmetric metric connection satisfying $(R(\xi,X),\tilde{R})(Y,Z)U = 0,$ is an $\eta$-Einstein manifold.

### 3.4. Projective Curvature Tensor of Generalized Sasakian-Space-Forms Admitting Quarter-Symmetric Metric Connection:

Projective curvature tensor of generalized Sasakian space-forms admitting quarter-symmetric metric connection $\nabla$ is defined as

\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \quad (3.4.1)
\]
which on using equations (3.3.8) and (3.3.10), gives

\[
\bar{P}(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)} [S(Y,Z)X - S(X,Z)Y] + g(\phi X,Z)\phi Y
\]

\[- g(\phi Y,Z)\phi X - \left(\frac{(n-1)(f_1-f_3)+1}{(n-1)}\right)\eta(X)Y - \eta(Y)X]\eta(Z)
\]

\[+ \left(\frac{1+3f_1-f_3}{(n-1)}\right) \eta(Y)Z - \eta(Y)X]\eta(Z),
\]

which gives

\[
\bar{P}(X,Y)Z = P(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X
\]

\[- \left(\frac{(n-1)(f_1-f_3)+1}{(n-1)}\right)\eta(X)Y - \eta(Y)X]\eta(Z)
\]

\[+ \left(\frac{1+3f_1-f_3}{(n-1)}\right) \eta(Y)Z - \eta(Y)X]\eta(Z),
\]

where \(P(X,Y)Z\) is the projective curvature tensor \([72]\) of connection \(\nabla\) defined as

\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1} [S(Y,Z)X - S(X,Z)Y],
\]

(3.4.4)
which on putting \( X = \xi \) and by use of equations (3.2.1), (3.2.2), (3.2.6), (3.2.11) and (3.2.13), gives

\[
\overline{P}(\xi, Y) \xi = \left[ \frac{(n-2)(f_2-f_3)-1}{n-1} \right] g(Y, Z) \xi - \eta(Z) Y + \left[ \frac{(3n-2)(f_2-f_3)+1}{n-1} \right] \eta(Z) Y
\]

\[
+ \left[ \frac{(3f_2+(3n-3)f_3-(n-2)f_2)-1}{(n-1)} \right] \eta(Y) \eta(\xi) Y - \left[ \frac{2nf_2+3f_2-f_2}{n-1} \right] g(Y, Z) \xi. \tag{3.4.5}
\]

Again putting \( Z = \xi \) in equation (3.4.2) and using equations (3.2.1), (3.2.2), (3.2.10) and (3.2.13), we get

\[
\overline{P}(X, Y) \xi = -\left[ \frac{2n(f_2-f_3)+1}{n-1} \right] [\eta(Y) X - \eta(X) Y]. \tag{3.4.6}
\]

Taking the inner product of equation (3.4.1) with \( \xi \) and using equations (3.2.1), (3.2.2), (3.2.4), (3.2.8) and (3.3.21), we get

\[
\eta(\overline{P}(X, Y) Z) = -\frac{1}{n-1} [\overline{S}(Y, Z) \eta(X) - \overline{S}(X, Z) \eta(Y)]
\]

\[
- \left( \frac{1-f_3+f_2}{n-1} \right) g(X, Z) \eta(Y) - g(Y, Z) \eta(X),
\]

which reduces, due to equation (3.3.10) as

\[
\eta(\overline{P}(X, Y) Z) = -\frac{1}{(n-1)} [S(Y, Z) \eta(X) - S(X, Z) \eta(Y)]
\]

\[
- \left[ \frac{(1-f_3+f_2)}{n-1} \right] [g(X, Z) \eta(Y) - g(Y, Z) \eta(X)], \tag{3.4.7}
\]
which gives

\[ \eta(\bar{P}(X,Y)Z) = \left[ \frac{(2n+1)f_1 - 3f_2 - 2f_3 - 1}{(n-1)} \right] [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \quad (3.4.8) \]

Thus in view of equations (3.4.5), (3.4.6) and (3.4.8), we can state as follows-

**Theorem (3.4.1):** A generalized Sasakian-space–form with quarter–symmetric metric connection, we obtain

(i) \[ \bar{P}(\xi,Y)Z = \left[ \frac{(n-2)(f_1 - f_3) + 1}{n-1} \right] [g(Y,Z)\xi - \eta(Z)Y] + \left[ \frac{(3n-2)(f_1 - f_3) + 1}{n-1} \right] \eta(Z)Y \]

\[ + \left[ \frac{(3f_2 + (3n-3)f_2 - (n-2)f_3 - 1)}{(n-1)} \right] \eta(Y)\eta(Z)\xi - \left[ \frac{(2nf_1 + 3f_2 - f_3)}{n-1} \right] g(Y,Z)\xi. \]

(iii) \[ \bar{P}(X,Y)\xi = -\left[ \frac{2n(f_1 - f_3) + 2}{n-1} \right] [\eta(Y)X - \eta(X)Y]. \]

(iii) \[ \eta(\bar{P}(X,Y)Z) = -\frac{1}{(n-1)} [S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] - \left[ \frac{(1-f_1 + f_3)}{n-1} \right] [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \]

Now, interchanging \( X \) and \( Y \) in equation (3.4.2) and adding the resulting equation to equation (3.4.2) with the fact that \( R(X,Y)Z + R(Y,Z)X = 0 \), we get

\[ \bar{P}(X,Y)Z + \bar{P}(Y,X)Z = 0. \quad (3.4.9) \]
Again from equation (3.4.2), writing two more equations by the cyclic permutations of \( X, Y \) and \( Z \), adding these two equations with equation (3.4.2) and using the fact that \( R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \), we get

\[
\bar{P}(X,Y)Z + \bar{P}(Y,Z)X + \bar{P}(Z,X)Y = 0. \tag{3.4.10}
\]

Thus we can state as follows-

**Theorem (3.4.2):** A generalized Sasakian-space-form admitting quarter-symmetric metric connection satisfies

(i) \( \bar{P}(X,Y)Z + \bar{P}(Y,X)Z = 0. \)

(ii) \( \bar{P}(X,Y)Z + \bar{P}(Y,Z)X + \bar{P}(Z,X)Y = 0. \)

Now, suppose

\[
(R(\xi,X)\bar{P})(Y,Z)U = 0, \tag{3.4.11}
\]

which gives

\[
R(\xi,X)\bar{P}(Y,Z)U - \bar{P}(R(\xi,X)Y,Z)U - \bar{P}(Y,R(\xi,X)Z)U - \bar{P}(Y,Z)R(\xi,X)U = 0. \tag{3.4.12}
\]
By virtue of equation (3.2.11), the above equation reduces to

\[(f_1 - f_3)\{g(X,\bar{P}(Y,\xi)U)\xi - \eta(\bar{P}(Y,Z)U)X - g(X,Y)\bar{P}(\xi,Z)U
\]

\[+ \eta(Y)\bar{P}(X,Z)U - g(X,Z)\bar{P}(Y,\xi)U + \eta(Z)\bar{P}(Y,X)U
\]

\[- g(X,U)\bar{P}(Y,Z)\xi + \eta(U)\bar{P}(Y,Z)X\} = 0. \quad (3.4.13)\]

Taking the inner product of above equation with \(\xi\) and using equation (3.2.2), we get

\[(f_1 - f_3)\{g(X,\bar{P}(Y,\xi)U)\eta(Y)\bar{P}(X,Z)U - g(Y)\eta(X)\bar{P}(\xi,Z)U
\]

\[+ \eta(Y)\eta(\bar{P}(X,Z)U) - g(X,Z)\eta(\bar{P}(Y,\xi)U) + \eta(Z)\eta(\bar{P}(Y,X)U
\]

\[+ g(X,U)\eta(\bar{P}(Y,Z)\xi) + \eta(U)\eta(\bar{P}(Y,Z)X)\} = 0. \quad (3.4.14)\]
In view of equations (3.4.2), (3.4.5), (3.4.6) and (3.4.7), above equation reduces to

\[
(f_1 - f_3) \left[ g(X,R(Y,Z,U)) - \frac{1}{n-1} \{S(Z,U)g(X,Y) - S(Y,U)g(X,Z) \}ight] \\
+ g(\phi Y,U)g(X,\phi Z) - g(\phi Z,U)g(X,\phi Y) + \left[ \frac{(n-2)(f_1-f_3)-1}{n-1} \right] g(X,Z)\eta(Y) \\
- g(X,Y)\eta(Z) - \left[ \frac{1+ f_1 - f_3}{n-1} \right] \left[ g(Z,U)g(X,Y) - g(Y,U)g(X,Z) \right] \\
+ \left[ \frac{(2n+1)f_1 + 3f_2 - 2f_3 - 1}{n-1} \right] g(Z,U)g(X,Y) + g(X,Z)g(Y,U) \\
+ g(X,U)\eta(Y)\eta(Z) - g(X,Y)\eta(U)\eta(Z) \right] = 0. \tag{3.4.15}
\]

Putting \( Y = X = e_i \) in above equation and taking summation over \( i, 1 \leq i \leq n \), we get

\[
S(Z,U) = \left[ \frac{2(n-1) + (4n-2n^2)f_1 + 3(n+1)f_2 - (3n+1)f_3}{n-1} \right] g(U,Z) \\
- [(2n+1)f_1 + 3f_2 - 2f_3 - 1)(n + 1) - 2] \eta(U)\eta(Z). \tag{3.4.16}
\]

Thus we can state as follows-

**Theorem (3.4.3):** A generalized Sasakian-space-form with quarter-symmetric metric connection satisfying \( (R(\xi,X).R)(Y,Z)U = 0 \) is an \( \eta \)-Einstein manifold.
3.5. Conformal Curvature Tensor of Generalized Sasakian-Space–Form Admitting Quarter-Symmetric Metric Connection:

Conformal curvature tensor $\tilde{C}$ of generalized Sasakian-space-forms admitting quarter-symmetric metric connection $\tilde{\nabla}$ is defined as

$$
\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)(QX) \\
- g(X,Z)(QY)] + \frac{r}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y]. 
$$

(3.5.1)

Using equations (3.3.8), (3.3.10), (3.3.11) and (3.3.12) in above equation, we get

$$
\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\
+ \frac{r}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y] + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\
+ \left[ - (f_1 - f_3) + \frac{(1-f_1+f_3)}{(n-2)} \right] \{\eta(Y)X - \eta(X)Y\} \eta(Z) \\
+ \left[ (f_1 - f_3) + \frac{(1-f_1-f_3)}{(n-2)} \right] [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\
+ \left[ \frac{(1-f_1+f_3) - n(1+f_1-f_3)}{(n-1)(n-2)} \right] [g(Y,Z)X - g(X,Z)Y].
$$

(3.5.2)
which yields

\[
\mathcal{C}(X,Y)Z = C(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X
\]

\[
+ \left[ \frac{(1-f_1+f_3)-(n-2)(f_1-f_3)}{(n-2)} \right] \left[ \eta(Y)X - \eta(X)Y \right] \eta(Z)
\]

\[
+ \left[ \frac{(f_1-f_3)(n-2)-(1-f_1+f_3)}{(n-2)} \right] [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]
\]

\[
+ \left[ \frac{(1-f_1+f_3)-(n+1)(f_1-f_3)}{(n-1)(n-2)} \right] [g(Y,Z)X - g(X,Z)Y],
\]

(3.5.3)

where \( C(X,Y)Z \) is the conformal curvature tensor of connection \( \nabla \) in \( M^n \) [34] defined as

\[
C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX
\]

\[-g(X,Z)QY] + \frac{r}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y].
\]

(3.5.4)

Now, interchanging \( X \) and \( Y \) in equation (3.5.2) and adding this equation to equation (3.5.2) with the fact that \( R(X,Y)Z + R(Y,Z)X = 0 \), we get

\[
\mathcal{C}(X,Y)Z + \mathcal{C}(Y,X)Z = 0.
\]

(3.5.5)

Again from equation (3.5.2), writing two more equations by the cyclic permutations of \( X, Y \) and \( Z \) and adding these two equations with the equation (3.5.2) and using the fact that \( R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \), we get

\[
\mathcal{C}(X,Y)Z + \mathcal{C}(Y,Z)X + \mathcal{C}(Z,X)Y = 0.
\]

(3.5.6)
Thus we can state as follows-

**Theorem (3.5.1):** A generalized Sasakian-space-form admitting quarter-symmetric metric connection, we obtain

(i) \(\bar{\nabla}(X,Y)Z + \bar{\nabla}(Y,X)Z = 0.\)

(ii) \(\bar{\nabla}(X,Y)Z + \bar{\nabla}(Y,Z)X + \bar{\nabla}(Z,X)Y = 0.\)