# Chapter 2

# Oscillation of Impulsive Hyperbolic Differential Equations with Distributed Delay

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## 2.1 Introduction

In recent times, there has been rising interest in studying the oscillation and nonoscillation results of partial differential equations with continuous distributed deviating arguments [21, 22, 49, 51, 63, 68, 73, 75, 77]. The study of impulsive partial differential equations is motivated by having many applications in population models [9, 25], single species growth [23], quenching problems [13] and various scientific models [78, 80] with the boundary conditions of the type Dirichlet, Neumann and Robin. In this chapter, we focus on oscillation of impulsive second order partial differential equations with distributed deviating arguments

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} = a(t)\Delta u(x,t) + b(t)\Delta u(x,\tau(t)) - \sum\limits_{i=1}^{n} r_i(x,t)u(x,\sigma_i(t)) \\ + \int_c^d q(x,t,\xi) f(u(x,g(t,\xi)))d\eta(\xi), \quad t \neq t_k, \quad (x,t) \in \Omega \times (0,+\infty) \equiv G, \\ u(x,t_k^+) = (1+\alpha_k)u(x,t_k) \\ u_t(x,t_k^+) = (1+\beta_k)u_t(x,t_k), \quad k = 1,2,\cdots, \end{array}\right\} \tag{2.1.1}$$

with the boundary condition

$$u = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty), \tag{2.1.2}$$

Problem (2.1.1) is more generalization of the following equation

$$\frac{\partial^2 u}{\partial t^2} = a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,\rho_i(t)) - \sum_{j=1}^{k} p_j(x,t)u(x,\sigma_j(t)) + f(x,t),$$
$$(x,t) \in \Omega \times (0,\infty) \equiv G, \tag{2.1.3}$$

with impulse and distributed delay, which was studied by Baotong et al. [11] and also (2.1.1) includes the following equations

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = a(t)\Delta u(x,t) + b(t)\Delta u(x,\tau(t)) - \sum_{i=1}^{n} r_i(x,t)u(x,\sigma_i(t)) \\
&+ \sum_{j=1}^{l} q_j(x,t)f_j(u(x,g(t))), \quad t \neq t_k, \quad (x,t) \in \Omega \times (0,+\infty) \equiv G, \\
&u(x,t_k^+) = (1+\alpha_k)u(x,t_k) \\
&u_t(x,t_k^+) = (1+\beta_k)u_t(x,t_k), \quad k = 1,2,\cdots,
\end{aligned}$$

In this chapter, we assume that the following hypotheses hold:

$(H_1)$ $a(t), b(t) \in PC\left([0,+\infty),[0,+\infty)\right)$, where $PC$ represents the class of functions which are piecewise continuous in $t$ with discontinuities of first kind only at $t = t_k,\ k = 1,2,\cdots$, and left continuous at $t = t_k,\ k = 1,2,\cdots$.

$(H_2)$ $r_i(x,t) \in C\left(\bar{\Omega} \times [0,+\infty),[0,+\infty)\right)$, $\quad R_i(t) = \min_{x\in\bar{\Omega}} r_i(x,t)$, $q(x,t,\xi) \in C(\bar{\Omega} \times \mathbb{R}^+ \times [c,d], \mathbb{R}^+)$, $\quad Q(t,\xi) = \min_{x\in\bar{\Omega}} q(x,t,\xi), f(u) \in C(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}^+$, $uf(u) > 0$ and $\dfrac{f(u)}{u} \geq \epsilon > 0$ for $u \neq 0$.

$(H_3)$ $\tau(t) \in C([0,\infty),\mathbb{R}),\ \lim_{t\to\infty} \tau(t) = +\infty,\ \sigma_i(t) \in C([0,+\infty),\mathbb{R}),\ \lim_{t\to+\infty} \sigma_i(t) = \infty$, $\sigma(t) = \max_{1\leq i\leq n} \sigma_i(t),\ i = 1,2,\cdots,n,\ g(t,\xi) \in C(\mathbb{R}^+ \times [c,d],\mathbb{R}),\ g(t,\xi) \leq t$ for $\xi \in [c,d]$ and $g(t,\xi)$ is nondecreasing with respect to $t$ and $\xi$ respectively and $\liminf_{t\to\infty,\xi\in[c,d]} g(t,\xi) = +\infty, \eta(\xi) : [c,d] \to \mathbb{R}$ is nondecreasing and the integral is a stieltjes integral in (2.1.1).

$(H_4)$ $u(x,t)$ and $u_t(x,t)$ are piecewise continuous in $t$ with discontinuities of first kind only at $t = t_k,\ k = 1,2,\cdots$, and left continuous at $t = t_k,\ \alpha_k > -1$,

$\beta_k > -1$, $\alpha_k < \beta_k$, the sequence $t_k$ is a fixed strictly increasing sequence of positive real numbers with $t_k \to \infty$ as $k \to \infty$.

We begin with definitions, known results, notations and Lemma which are required to prove main results.

**Definition 2.1.1.** *A solution $u$ of the problem* (2.1.1) *is a function* $u \in C^2(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ *that satisfies (2.1.1), where*

$$t_{-1} := \min\left\{0, \min_{\xi\in[c,d]}\left\{\inf_{t\geq 0} g(t,\xi)\right\}\right\}, \quad \hat{t}_{-1} := \min\left\{0, \inf_{t\geq 0}\tau(t), \min_{1\leq s\leq n}\left\{\inf_{t\geq 0}\sigma_i(t)\right\}\right\}.$$

Now with this definition of solution, we can precisely define what we mean by oscillation.

**Definition 2.1.2.** *The solution $u$ of the problem (2.1.1)-(2.1.2) is said to be oscillatory in the domain $G$ if for any positive number $\ell$ there exists a point* $(x_0, t_0) \in \Omega \times [\ell, +\infty)$ *such that* $u(x_0, t_0) = 0$ *holds.*

**Definition 2.1.3.** *A function $U(t)$ is said to be eventually positive (negative), if there exists a $t_1 \geq t_0$ such that $U(t) > 0$ $(< 0)$ holds for all $t \geq t_1$.*

It is identified that [72] the least eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\begin{aligned} \Delta\omega(x) + \lambda\omega(x) &= 0, \text{ in } \Omega, \\ \omega(x) &= 0, \text{ on } \partial\Omega, \end{aligned}$$

and the consequent eigen function $\Phi(x) > 0$ in $\Omega$.

For each positive solution $u(x,t)$ of the problem (2.1.1)-(2.1.2) we define the functions

$$U(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx,$$

and

$$G(t) = \epsilon \int_c^d Q(t,\xi)d\eta(\xi).$$

**Lemma 2.1.1.** *[11]Suppose that $y(t) \in C^2([t_0,\infty),\mathbb{R})$ and that*

$$y(t) > 0,\ y'(t) > 0 \quad and \quad y''(t) \leq 0 \quad for \quad t \geq t_0 > 0.$$

*Then for any $\lambda_1 \in (0,1)$, there exists a number $t_1 > t_0$ such that*

$$y(t) \geq \lambda_1\ t\ y'(t) \quad for \quad t \geq t_1.$$

## 2.2 Main Results

In this section, we establish some sufficient conditions for the oscillation of all solutions of the problem (2.1.1)-(2.1.2).

**Theorem 2.2.1.** *Suppose that conditions $(H_1)$-$(H_4)$ hold and that every solution $u(x,t)$ of the problem (2.1.1)-(2.1.2) is oscillatory in $G$, if the impulsive delay differential inequality*

$$\left.\begin{array}{l} U''(t) + \sum\limits_{i=1}^{n} R_i(t)U(\sigma_i(t)) - G(t)U(g(t,\xi)) \leq 0, \qquad t \neq t_k, \qquad t \geq t_1, \\ U(t_k^+) = (1+\alpha_k)U(t_k) \\ U'(t_k^+) = (1+\beta_k)U'(t_k), \quad k = 1,2,\cdots, \end{array}\right\} \tag{2.2.1}$$

*has no eventually positive solutions.*

**Proof.** Let $u(x,t)$ be a nonoscillatory solution of the boundary value problem (2.1.1)-(2.1.2) and $u(x,t) > 0$. Then there exists a $t_1 > t_0 > 0$ such that $\tau(t) \geq 0$,

$\sigma_i(t) \geq 0$ and $g(t,\xi) \geq 0$ for $(t,\xi) \in [t_1,+\infty) \times [c,d]$, we get that

$$\begin{aligned} u(x,\tau(t)) &> 0, \quad \text{for} \quad (x,t) \in \Omega \times [t_1,+\infty), \\ u(x,\sigma_i(t)) &> 0, \quad \text{for} \quad (x,t) \in \Omega \times [t_1,+\infty), \quad i = 1,2,\cdots,n, \\ and \;\; u(x,g(t,\xi)) &> 0, \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [c,d]. \end{aligned}$$

Multiplying both sides of (2.1.1) by $K_\Phi \Phi(x) > 0$ and integrating with respect to $x$ over $\Omega$, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\left[\int_\Omega u(x,t)K_\Phi\Phi(x)dx\right] &= a(t)\int_\Omega \Delta u(x,t)K_\Phi\Phi(x)dx + b(t)\int_\Omega \Delta u(x,\tau(t))K_\Phi\Phi(x)dx \\ -\sum_{i=1}^{n}\int_\Omega r_i(x,t)u(x,\sigma_i(t))K_\Phi\Phi(x)dx &+ \int_\Omega\int_c^d q(x,t,\xi)f(u(x,g(t,\xi)))K_\Phi\Phi(x)d\eta(\xi)dx \end{aligned} \tag{2.2.2}$$

From Green's formula and the boundary condition (2.1.2), we see that

$$\begin{aligned} K_\Phi\int_\Omega \Delta u(x,t)\Phi(x)dx &= K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u}{\partial\gamma} - u\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + K_\Phi\int_\Omega u(x,t)\Delta\Phi(x)dx \\ &= -\lambda_0 U(t). \end{aligned}$$

$$\therefore K_\Phi\int_\Omega \Delta u(x,t)\Phi(x)dx \leq 0 \tag{2.2.3}$$

and

$$\begin{aligned} K_\Phi\int_\Omega \Delta u(x,\tau(t))\Phi(x)dx &= K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,\tau(t))}{\partial\gamma} - u(x,\tau(t))\frac{\partial\Phi(x)}{\partial\gamma}\right]dS \\ &\quad + K_\Phi\int_\Omega u(x,\tau(t))\Delta\Phi(x)dx \\ &= -\lambda_0 U(\tau(t)) \end{aligned}$$

$$\therefore K_\Phi\int_\Omega \Delta u(x,\tau(t))\Phi(x)dx \leq 0. \tag{2.2.4}$$

Furthermore applying Jensen's inequality for convex functions and using the assumptions in $(H_2)$, we get that

$$\int_\Omega\int_c^d q(x,t,\xi)f(u(x,g(t,\xi)))K_\Phi\Phi(x)d\eta(\xi)dx$$

$$
\begin{aligned}
&\geq \int_c^d Q(t,\xi)\int_\Omega f(u(x,g(t,\xi)))K_\Phi\Phi(x)dxd\eta(\xi)\\
&\geq \int_c^d Q(t,\xi)\epsilon\int_\Omega u(x,g(t,\xi))K_\Phi\Phi(x)dxd\eta(\xi)\\
&\geq \epsilon\int_c^d Q(t,\xi)U(g(t,\xi))d\eta(\xi), \qquad (2.2.5)
\end{aligned}
$$

where $U(g(t,\xi)) = \int_\Omega u(x,g(t,\xi))K_\Phi\Phi(x)dx$.

Combining (2.2.2)-(2.2.5), we get that

$$
\begin{aligned}
&U''(t) \leq -\sum_{i=1}^n \int_\Omega R_i(t)u(x,\sigma_i(t))K_\Phi\Phi(x)dx + \epsilon\int_c^d Q(t,\xi)U(g(t,\xi))d\eta(\xi),\\
&U''(t) \leq -\sum_{i=1}^n R_i(t)U(\sigma_i(t)) + G(t)U(g(t,\xi)),\\
&U''(t) + \sum_{i=1}^n R_i(t)U(\sigma_i(t)) - G(t)U(g(t,\xi)) \leq 0, \quad t \neq t_k, \quad t \geq t_1,
\end{aligned}
$$

where $G(t) = \epsilon\int_c^d Q(t,\xi)d\eta(\xi)$. Also, multiplying both sides of (2.1.1) by $K_\Phi\Phi(x) > 0$, integrating with respect to $x$ over $\Omega$, and from $(H_4)$ we obtain

$$
\begin{aligned}
\int_\Omega u(x,t_k^+)K_\Phi\Phi(x)dx &= (1+\alpha_k)\int_\Omega u(x,t_k)K_\Phi\Phi(x)dx,\\
U(t_k^+) &= (1+\alpha_k)U(t_k),\\
U'(t_k^+) &= (1+\beta_k)U'(t_k), \quad t = t_k, k = 1,2,\cdots.
\end{aligned}
$$

Therefore $U(t)$ is an eventually positive solution of (2.2.1). This disagree with the hypothesis and completes the proof of the theorem. □

**Theorem 2.2.2.** *Suppose that conditions $(H_1)$-$(H_4)$ hold. If for every number $\lambda_1, \lambda_2 \in (0,1)$,*

$$
\limsup_{t\to\infty}\int_{\sigma(t)}^t \prod_{t_0\leq t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)\left[\sum_{i=1}^n \lambda_1\sigma_i(s)R_i(s) - \lambda_2 G(s)g(s,\xi)\right]ds > 1, \quad (2.2.6)
$$

*then every solution $u(x,t)$ of the problem (2.1.1)-(2.1.2) is oscillatory in $G$.*

**Proof.** On the contrary, let $u(x,t)$ be a nonoscillatory solution of the boundary value problem (2.1.1)-(2.1.2) which we assume to be positive. Now we can use

$$U''(t) + \sum_{i=1}^{n} R_i(t)U(\sigma_i(t)) - G(t)U(g(t,\xi)) \leq 0, \quad t \geq t_1.$$

By Lemma 2.1.1

$$U(\sigma_i(t)) \geq \lambda_1 \sigma_i(t) U'(\sigma_i(t)) \quad and \quad U(g(t,\xi)) \geq \lambda_2 g(t,\xi) U'(g(t,\xi)), \quad t \geq t_1,$$

which implies that

$$U''(t) + \sum_{i=1}^{n} R_i(t)\lambda_1 \sigma_i(t) U^{'}(\sigma_i(t)) - G(t)\lambda_2 g(t,\xi) U^{'}(g(t,\xi)) \leq 0, \quad t \geq t_1. \quad (2.2.7)$$

Define

$$W(t) := \prod_{t_0 \leq t_k < t} \left( \frac{1+\beta_k}{1+\alpha_k} \right)^{-1} U(t).$$

In fact, $W(t)$ is continuous on each interval $[t_k, t_{k+1}]$, and in consideration of $W(t_k^+) \leq \left( \frac{1+\beta_k}{1+\alpha_k} \right) W(t_k)$, it follows for $t \geq t_0$ that, inequality (2.2.7) has no eventually positive solution, if the following inequality has no eventually positive solution.

$$W''(t) + \sum_{i=1}^{n} \lambda_1 \sigma_i(t) R_i(t) W^{'}(\sigma_i(t)) - \lambda_2 G(t) g(t,\xi) W^{'}(g(t,\xi)) \leq 0, \quad t \geq t_1.$$

Where

$$W(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left( \frac{1+\beta_k}{1+\alpha_k} \right)^{-1} U(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left( \frac{1+\beta_k}{1+\alpha_k} \right)^{-1} U(t_k) = W(t_k),$$

$$W(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left( \frac{1+\beta_k}{1+\alpha_k} \right)^{-1} U(t_k^-) = \prod_{t_0 \leq t_j < t_k} \left( \frac{1+\beta_k}{1+\alpha_k} \right)^{-1} U(t_k) = W(t_k).$$

Which implies that, $W(t)$ is continuous on $[t_0, +\infty)$. Then we get that

$$\prod_{t_0 \le t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-2} U''(t) \;+\; \sum_{i=1}^{n} \lambda_1 \sigma_i(t) R_i(t) \prod_{t_0 \le t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} U'(\sigma_i(t))$$
$$-\;\lambda_2 G(t) g(t,\xi) \prod_{t_0 \le t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} U'(g(t,\xi)) \le 0.$$

Integrating the above inequality from $\sigma(t)$ to $t$, we have

$$\int_{\sigma(t)}^{t} \prod_{t_0 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-2} U''(s)ds + \int_{\sigma(t)}^{t} \sum_{i=1}^{n} \lambda_1 \sigma_i(s) R_i(s) \prod_{t_0 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} U'(\sigma_i(s))ds$$
$$-\int_{\sigma(t)}^{t} \lambda_2 G(s) g(s,\xi) \prod_{t_0 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} U'(g(s,\xi))ds \le 0$$

From this,

$$\prod_{t_0 \le t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-2} \left[U'(t) - U'(\sigma(t))\right] + \int_{\sigma(t)}^{t} \sum_{i=1}^{n} \lambda_1 \sigma_i(s) R_i(s) \prod_{t_0 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}$$

$$U'(\sigma_i(s))ds - \int_{\sigma(t)}^{t} \lambda_2 G(s) g(s,\xi) \prod_{t_0 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} U'(g(s,\xi))ds \le 0, \quad t \ge t_1.$$

Therefore,

$$\int_{\sigma(t)}^{t} \prod_{t_0 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right) \left[\sum_{i=1}^{n} \lambda_1 \sigma_i(s) R_i(s) - \lambda_2 G(s) g(s,\xi)\right] ds \le 1 - \frac{U'(t)}{U'(\sigma(t))} < 1,$$

and hence

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \prod_{t_0 \le t_k < s} \left(\frac{1+\alpha_k}{1+\alpha_k}\right) \left[\sum_{i=1}^{n} \lambda_1 \sigma_i(s) R_i(s) - \lambda_2 G(s) g(s,\xi)\right] ds \le 1, \quad (2.2.8)$$

which is a contradiction with (2.2.6). The proof of the case $u(x,t) < 0$ is similar and is omitted. □

## 2.3 Example

In this section, we present an example to point up the main results established in Section 2.2.

**Example 2.3.1.** *Consider the following equation*

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{5}{2}\Delta u(x,t) + \Delta u(x,t-3\pi/2) - \dfrac{5}{2}u(x,t-\pi) \\ \qquad\qquad - \int_{-\pi/2}^{0} u(x,t+\xi)d\xi, \qquad t > 1, t \neq t_k, k = 1,2,\cdots, \\ u(x,t_k^+) = (1+\alpha_k)u(x,t_k) \\ u_t(x,t_k^+) = (1+\beta_k)u_t(x,t_k), \quad k = 1,2,\cdots, \end{array}\right\} \tag{2.3.1}$$

*for* $(x,t) \in (0,\pi) \times [0,+\infty)$*, with the boundary condition*

$$u(0,t) = u(\pi,t) = 0. \tag{2.3.2}$$

*Here* $\Omega = (0,\pi)$*,* $a(t) = \frac{5}{2}$*,* $b(t) = 1$*,* $\tau(t) = t - \frac{3\pi}{2}$*,* $n = 1$*,* $R(t) = \frac{5}{2}$*,* $\sigma(t) = t - \pi$*,* $Q(t,\xi) = 1$*,* $f(u) = u$*,* $g(t,\xi) = t + \xi$*,* $\alpha_k = \frac{1}{2^k}$*,* $\beta_k = 2^k$*,* $\epsilon = 1$*, and* $G(s) = \frac{\pi}{2}$*.*

*Also, we see from the above assumption that the hypotheses* $(H_1)$*-*$(H_4)$ *hold, moreover*

$$\begin{aligned} \lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right) ds &= \int_{1}^{+\infty} \prod_{1<t_k<s} \left(\frac{1+2^k}{1+(1/2^k)}\right) ds \\ &= \int_{1}^{t_1} \prod_{1<t_k<s} 2^k ds + \int_{t_1^+}^{t_2} \prod_{1<t_k<s} 2^k ds + \int_{t_2^+}^{t_3} \prod_{1<t_k<s} 2^k ds + \cdots \\ &= +\infty. \end{aligned}$$

*Now, the condition (2.2.6) reads,*

$$\limsup_{t\to\infty} \int_{t-\pi}^{t} \left(\prod_{t_0 \le t_k < s} 2^k\right) \left[\frac{5}{6}(s-\pi) + \frac{\pi}{8}(s+\xi)\right] ds > 1. \tag{2.3.3}$$

*Therefore all the conditions of the Theorem 2.2.2 are satisfied. So, every solution of the problem (2.3.1)-(2.3.2) is oscillatory in G. In fact* $u(x,t) = \sin x \cos t$ *is such a solution.*