Chapter 7

On the Oscillation of Higher order Impulsive Neutral Partial Differential Equations with Distributed Deviating Arguments

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7.1 Introduction

In this chapter, we begin oscillation criteria for even order impulsive neutral partial differential equation with damping which was not formerly studied. In the consequence, the main results of this chapter are the generalization of earlier results studied in [29, 42, 45, 46, 76] with additional force components along the system such as impulse, damping and distributed delay.

In this chapter, we consider the following impulsive system with damping of the form

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left[r(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\Big(u(x,t)+\textstyle\int_a^b g(t,\xi)u(x,\tau(t,\xi))d\eta(\xi)\Big)\right]\\
&+p(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\Big(u(x,t)+\textstyle\int_a^b g(t,\xi)u(x,\tau(t,\xi))d\eta(\xi)\Big)+\textstyle\int_a^b q(t,\xi)u(x,\sigma(t,\xi))d\eta(\xi)\\
&=a(t)\Delta u(x,t)+\sum_{j=1}^{n}b_j(t)\Delta u(x,\rho_j(t)),\quad t\neq t_k,\quad (x,t)\in\Omega\times[0,+\infty)\equiv G\\
&\frac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}} = I_k^{(i)}\left(x,t_k,\frac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right),\quad k=1,2,\cdots,\ i=0,1,2,\cdots,m-1
\end{aligned}\right\}\tag{7.1.1}$$

Equation (7.1.1) is supplemented by the following Robin boundary condition,

$$\alpha(x)\frac{\partial u(x,t)}{\partial\gamma}+\beta(x)u(x,t) = 0,\qquad (x,t)\in\partial\Omega\times[0,+\infty)\tag{7.1.2}$$

where γ is the outer surface normal vector to $\partial\Omega$ and $\alpha, \beta \in C(\partial\Omega, [0, +\infty))$, $\alpha^2(x) + \beta^2(x) \neq 0$.

In the sequel, we assume that the following hypotheses (H) hold:

(H_1) $r(t) \in C'([0, +\infty), (0, +\infty))$, $r'(t) \geq 0$, $p(t) \in C([0, +\infty), \mathbb{R})$,
$\int_{t_0}^{+\infty} \frac{1}{R(s)} ds = +\infty$, where $R(t) = \exp\left(\int_{t_0}^{t} \frac{r'(s) + p(s)}{r(s)} ds\right)$.

(H_2) $a(t), b_j(t) \in PC([0, +\infty), [0, +\infty))$, $j = 1, 2, \cdots, n$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, and left continuous at $t = t_k$, $k = 1, 2, \cdots$.

(H_3) $g(t, \xi) \in C^m([0, +\infty) \times [a, b], [0, +\infty))$, $q(t, \xi) \in C([0, +\infty) \times [a, b], [0, +\infty))$, $\rho_j(t) \in C([0, +\infty), \mathbb{R})$, $\rho_j(t) \leq t$, $\lim_{t \to +\infty} \rho_j(t) = +\infty$, $j = 1, 2, \cdots, n$, and a, b are non-positive constants with $a < b$.

(H_4) $\tau(t, \xi)$, $\sigma(t, \xi) \in C([0, +\infty) \times [a, b], \mathbb{R})$, $\tau(t, \xi) \leq t$, $\sigma(t, \xi) \leq t$ for $\xi \in [a, b]$, $\tau(t, \xi)$ and $\sigma(t, \xi)$ are nondecreasing with respect to t and ξ respectively and $\liminf_{t \to +\infty,\ \xi \in [a,b]} \tau(t, \xi) = \liminf_{t \to +\infty,\ \xi \in [a,b]} \sigma(t, \xi) = +\infty$.

(H_5) There exist a function $\theta(t) \in C([0, +\infty), [0, +\infty))$ satisfying $\theta(t) \leq \sigma(t, a)$, $\theta'(t) > 0$ and $\lim_{t \to +\infty} \theta(t) = +\infty$, $\eta(\xi) : [a, b] \to \mathbb{R}$ is nondecreasing and the integral is a Stieltjes integral in (7.1.1).

(H_6) $\frac{\partial^{(i)} u(x, t)}{\partial t^{(i)}}$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, and left continuous at $t = t_k$, $\frac{\partial^{(i)} u(x, t_k)}{\partial t^{(i)}} = \frac{\partial^{(i)} u(x, t_k^-)}{\partial t^{(i)}}$, $k = 1, 2, \cdots$, $i = 0, 1, 2, \cdots, m - 1$.

(H_7) $I_k^{(i)}\left(x, t_k, \dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}\right) \in PC(\bar{\Omega} \times [0,+\infty) \times \mathbb{R}, \mathbb{R})$, and there exist positive constants $a_k^{(i)}, b_k^{(i)}$ such that for $i = 0, 1, 2, \cdots, m-1,\ k = 1, 2, \cdots,$

$$a_k^{(i)} \leq \frac{I_k^{(i)}\left(x, t_k, \dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}\right)}{\dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

This chapter is planned as follows: In Section 7.2, we discuss the oscillation of the problem (7.1.1)-(7.1.2). In Section 7.3, we present some examples to illustrate the main results.

For each positive solution $u(x,t)$ of the problem (7.1.1)-(7.1.2) we define the following functions

$$V(t) = \int_{\Omega} u(x,t)\Phi(x)dx, \qquad B(t) = \frac{A'(t)}{A(t)} - \frac{A(t)p(t)}{r(t)}$$
$$L(t) = \frac{M(\theta(t))^{m-2}\theta'(t)}{r(t)} \quad \text{and} \quad C(t) = g_1 \int_a^b q(t,\xi)d\eta(\xi),$$

where

$$g_1 = 1 - \int_a^b g(\sigma(t,\xi),\xi)d\eta(\xi).$$

7.2 Oscillation Theorems

In this section, we establish the oscillation criteria of the problem (7.1.1)-(7.1.2). The Lemma 6.1.1 which is very useful for establishing the main results.

Theorem 7.2.1. *If $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$, then the necessary and sufficient condition for all solutions of the problem* (7.1.1)-(7.1.2) *to oscillate is that all solutions of the* *impulsive differential equation*

$$\left.\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\right]\\
&+p(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\\
&+\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi)+\lambda_0 a(t)V(t)\\
&+\lambda_0\sum_{j=1}^{n} b_j(t)V(\rho_j(t))=0, \quad t\neq t_k\\
&a_k^{(i)}\leq\frac{\frac{\partial^{(i)}V(t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)}V(t_k)}{\partial t^{(i)}}}\leq b_k^{(i)}, \quad k=1,2,\cdots, \quad i=0,1,2,\cdots,m-1
\end{aligned}\right\} \tag{7.2.1}$$

to oscillate, where λ_0 *is the smallest eigenvalue of (7.1.1).*

Proof.(i) Sufficient part: Suppose that there is a non-oscillatory solution $u(x,t)$ of the problem (7.1.1)-(7.1.2) which has no zero in $\Omega\times[t_0,+\infty)$ for some $t_0\geq 0$. Without loss of generality, we assume that $u(x,t)>0$, $(x,t)\in\Omega\times[t_0,+\infty)$, $t_0\geq 0$. By the conditions (H_3) and (H_4), there exists a $t_1>t_0>0$ such that $\tau(t,\xi)\geq t_0$, $\sigma(t,\xi)\geq t_0$ for $(t,\xi)\in[t_1,+\infty)\times[a,b]$ and $\rho_j(t)\geq t_0$, $j=1,2,\cdots,n$ for $t\geq t_1$, then

$$u(x,\tau(t,\xi))>0 \quad \text{for} \quad (x,t,\xi)\in\Omega\times[t_1,+\infty)\times[a,b],$$

$$u(x,\sigma(t,\xi))>0 \quad \text{for} \quad (x,t,\xi)\in\Omega\times[t_1,+\infty)\times[a,b]$$

$$\text{and} \quad u(x,\rho_j(t))>0 \quad \text{for} \quad (x,t)\in\Omega\times[t_1,+\infty), \quad j=1,2,\cdots,n.$$

For $t \geq t_0,\ t \neq t_k,\ k = 1, 2, \cdots$, multiplying both sides of (7.1.1) by $\Phi(x) > 0$ and integrating with respect to x over the domain Ω, we attain

$$\left.\begin{array}{l} \dfrac{d}{dt}\left[r(t)\dfrac{d^{m-1}}{dt^{m-1}}\left(\int_\Omega u(x,t)\Phi(x)dx + \int_\Omega \int_a^b g(t,\xi)u(x,\tau(t,\xi))\Phi(x)d\eta(\xi)dx\right)\right] \\ +p(t)\dfrac{d^{m-1}}{dt^{m-1}}\left(\int_\Omega u(x,t)\Phi(x)dx + \int_\Omega \int_a^b g(t,\xi)u(x,\tau(t,\xi))\Phi(x)d\eta(\xi)dx\right) \\ +\int_\Omega \int_a^b q(t,\xi)u(x,\sigma(t,\xi))\Phi(x)d\eta(\xi)dx \\ = a(t)\int_\Omega \Delta u(x,t)\Phi(x)dx + \sum\limits_{j=1}^{n} b_j(t)\int_\Omega \Delta u(x,\rho_j(t))\Phi(x)dx. \end{array}\right\} \tag{7.2.2}$$

From Green's formula and boundary condition (7.1.2), it follows that

$$\begin{aligned} \int_\Omega \Delta u(x,t)\Phi(x)dx = & \int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + \int_\Omega u(x,t)\Delta\Phi(x)dx \\ = & \int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS - \lambda_0\int_\Omega u(x,t)\Phi(x)dx. \end{aligned}$$

If $\alpha(x) \equiv 0,\ x \in \partial\Omega$, then from (7.1.2) we have $\beta(x) \not\equiv 0,\ u(x,t) = 0,\ (x,t) \in \partial\Omega \times [0,+\infty)$. Hence, we obtain

$$\int_{\partial\Omega}\left(\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\Omega}\right)dS \equiv 0, \quad t \geq t_1, \quad t \neq t_k.$$

If $\alpha(x) \not\equiv 0,\ x \in \partial\Omega$. Noting that $\partial\Omega$ is piecewise smooth, $\alpha, \beta \in C(\partial\Omega, [0,+\infty))$, $\alpha^2(x) + \beta^2(x) \neq 0$, without loss of generality, we can assume that $\alpha(x) > 0$, $x \in \partial\Omega$. Then by (7.1.2) and (7.2.1), we have

$$\begin{aligned} & \int_{\partial\Omega}\left(\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right)dS \\ = & \int_{\partial\Omega}\left(-\Phi(x)\frac{\beta(x)}{\alpha(x)}u(x,t) + \frac{\beta(x)}{\alpha(x)}\Phi(x)u(x,t)\right)dS = 0, \quad t \geq t_1. \end{aligned}$$

Therefore, using Lemma 6.1.1, we obtain

$$\begin{aligned} \int_\Omega \Delta u(x,t)\Phi(x)dx = -\lambda_0\int_\Omega u(x,t)\Phi(x)dx, \quad t \geq t_1 \\ = -\lambda_0 V(t) \end{aligned} \tag{7.2.3}$$

and for $j = 1, 2 \cdots, n$,

$$\int_\Omega \Delta u(x,\rho_j(t))\Phi(x)dx = -\lambda_0 \int_\Omega u(x,\rho_j(t))\Phi(x)dx, \quad t \geq t_1$$
$$= -\lambda_0 V(\rho_i(t)). \tag{7.2.4}$$

It is easy to see that

$$\int_\Omega \int_a^b q(t,\xi)u(x,\sigma(t,\xi))\Phi(x)d\eta(\xi)dx$$
$$= \int_a^b q(t,\xi)\int_\Omega u(x,\sigma(t,\xi))\Phi(x)dxd\eta(\xi)$$
$$= \int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi). \tag{7.2.5}$$

In consideration of (7.2.2)-(7.2.5), we acquire

$$\left.\begin{array}{l} \frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\right] \\ +p(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right) \\ +\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi)+\lambda_0 a(t)V(t) \\ +\lambda_0\sum_{j=1}^{n} b_j(t)V(\rho_j(t))=0, \quad t\geq t_1,\ t\neq t_k. \end{array}\right\} \tag{7.2.6}$$

For $t \geq t_0,\ t = t_k,\ k = 1, 2, \cdots,\ i = 0, 1, 2, \cdots, m-1$, multiplying both sides of (7.1.1) by $\Phi(x) > 0$, integrating with respect to x over the domain Ω, and from (H_7), we obtain

$$a_k^{(i)} \leq \frac{\frac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

According to $V(t) = \int_\Omega u(x,t)\Phi(x)dx$, we have

$$a_k^{(i)} \leq \frac{\frac{\partial^{(i)}V(t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)}V(t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

Therefore $V(t)$ is a positive solution of (7.2.1), which contradicts the fact that all solutions of equation (7.2.1) are oscillatory.

(ii) Necessary part: Suppose that (7.2.1) has a non-oscillatory solution $\tilde{V}(t) > 0$. Without loss of generality we assume $\tilde{V}(t) > 0$ for $t \geq t_* \geq 0$, where t_* is some large number. From (7.2.1), we have

$$\left.\begin{array}{l} \frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}\left(\tilde{V}(t)+\int_a^b g(t,\xi)\tilde{V}(\tau(t,\xi))d\eta(\xi)\right)\right] \\ +p(t)\frac{d^{m-1}}{dt^{m-1}}\left(\tilde{V}(t)+\int_a^b g(t,\xi)\tilde{V}(\tau(t,\xi))d\eta(\xi)\right) \\ +\int_a^b q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi)+\lambda_0 a(t)\tilde{V}(t) \\ +\lambda_0\sum_{j=1}^n b_j(t)\tilde{V}(\rho_j(t))=0, \quad t\geq t_*,\ t\neq t_k,\ x\in\Omega \end{array}\right\} \tag{7.2.7}$$

For $t \geq t_0,\ t \neq t_k,\ k = 1, 2, \cdots$, multiplying both sides of (7.2.7) by $\Phi(x) > 0$ we obtain

$$\left.\begin{array}{l} \frac{\partial}{\partial t}\left[r(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{V}(t)\Phi(x)+\int_a^b g(t,\xi)\tilde{V}(\tau(t,\xi))\Phi(x)d\eta(\xi)\right)\right] \\ +p(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{V}(t)\Phi(x)+\int_a^b g(t,\xi)\tilde{V}(\tau(t,\xi))\Phi(x)d\eta(\xi)\right) \\ +\int_a^b q(t,\xi)\tilde{V}(\sigma(t,\xi))\Phi(x)d\eta(\xi)+\lambda_0 a(t)\tilde{V}(t)\Phi(x) \\ +\lambda_0\sum_{j=1}^n b_j(t)\Delta\tilde{V}(\rho_j(t))\Phi(x)=0, \quad t\geq t_*,\ x\in\Omega. \end{array}\right\} \tag{7.2.8}$$

Let $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [0,+\infty)$. By Lemma 6.1.1, we have $\Delta w(x) = -\lambda_0 w(x)$, $x \in \Omega$. Then (7.2.8) implies

$$\left.\begin{array}{l} \frac{\partial}{\partial t}\left[r(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{u}(x,t)+\int_a^b g(t,\xi)\tilde{u}(x,\tau(t,\xi))d\eta(\xi)\right)\right] \\ +p(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{u}(x,t)+\int_a^b g(t,\xi)\tilde{u}(x,\tau(t,\xi))d\eta(\xi)\right) \\ +\int_a^b q(t,\xi)\tilde{u}(x,\sigma(t,\xi))d\eta(\xi) \\ =a(t)\Delta\tilde{u}(x,t)+\sum_{j=1}^n b_j(t)\Delta\tilde{u}(x,\rho_j(t)), \quad t\geq t_*,\ x\in\Omega. \end{array}\right\} \tag{7.2.9}$$

For $t \geq t_0,\ t = t_k, k = 1, 2, \cdots$, multiplying both sides of (7.2.7) by $\Phi(x) > 0$, we have

$$a_k^{(i)}\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{V}(t_k)\Phi(x) \leq \frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{V}(t_k^+)\Phi(x) \leq b_k^{(i)}\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{V}(t_k)\Phi(x)$$

since $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [0,+\infty)$

$$a_k^{(i)} \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x,t_k) \leq \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x,t_k^+) \leq b_k^{(i)} \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x,t_k)$$
$$\frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x,t_k^+) = I_k^{(i)} \left(x, t_k, \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x,t_k) \right)$$

which shows that $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [t_*,+\infty)$, satisfies (7.1.1). From Lemma 6.1.1, we get

$$\alpha(x) \frac{\partial w(x)}{\partial \gamma} + \beta(x) w(x) = 0, \quad x \in \partial\Omega$$

which implies

$$\alpha(x) \frac{\partial \tilde{u}(x,t)}{\partial \gamma} + \beta(x) \tilde{u}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,+\infty). \tag{7.2.10}$$

Hence $\tilde{u}(x,t) = \tilde{v}(t)\Phi(x) > 0$ is a nonoscillatory solution of the problem (7.1.1)-(7.1.2) and which is a contradiction. □

Remark 7.2.1. *Theorem 7.2.1 shows that the oscillation of the problem (7.1.1)-(7.1.2) is equivalent to the oscillation of the impulsive differential equation* (7.2.1).

Theorem 7.2.2. *If $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$ and the impulsive differential inequality*

$$\left.\begin{aligned} &\left(r(t)Z^{(m-1)}(t)\right)' + p(t)Z^{(m-1)}(t) + g_0 \int_a^b q(t,\xi) Z(\theta(t)) d\eta(\xi) \leq 0 \\ &a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)} Z(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)} Z(t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}, \quad t = t_k, \quad k = 1,2,\cdots, \quad i = 0,1,2,\cdots,m-1 \end{aligned}\right\} \tag{7.2.11}$$

has no eventually positive solution, then every solution of the problem (7.1.1)-(7.1.2) is oscillatory in G.

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of the problem (7.1.1)-(7.1.2) which has no zero in $\Omega \times [t_0,+\infty)$ for some $t_0 \geq 0$.

Without loss of generality, we assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0, +\infty)$, $t_0 \geq 0$. By assumption that there exists a $t_1 > t_0$ such that $\tau(t,\xi) \geq t_0$, $\sigma(t,\xi) \geq t_0$ for $(t,\xi) \in [t_1, +\infty) \times [a,b]$ and $\rho_j(t) \geq t_0$, $j = 1, 2, \cdots, n$ for $t \geq t_1$, then

$$\begin{aligned} & u(x,\tau(t,\xi)) > 0 \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b], \\ & u(x,\sigma(t,\xi)) > 0 \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b] \\ \text{and} \quad & u(x,\rho_j(t)) > 0 \quad \text{for} \quad (x,t) \in \Omega \times [t_1,+\infty), \quad j = 1,2,\cdots,n. \end{aligned}$$

Proceeding as in the proof of Theorem 7.2.1, by Lemma 6.1.1 and from (7.2.6), we have that

$$\left.\begin{aligned} & \frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t) + \textstyle\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\right] \\ & + p(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t) + \textstyle\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right) \\ & + \textstyle\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \\ & = -\lambda_0 a(t)V(t) - \lambda_0 \sum_{j=1}^{n} b_j(t)V(\rho_j(t)) \\ & \leq 0, \quad t \geq t_1, \ t \neq t_k. \end{aligned}\right\} \tag{7.2.12}$$

Set $Z(t) = V(t) + \int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)$. Equation (7.2.12) can be written as

$$\left(r(t)Z^{(m-1)}(t)\right)' + p(t)Z^{(m-1)} + \int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \leq 0, \qquad t \neq t_k. \tag{7.2.13}$$

From Theorem 6.2.2, we have $Z(\sigma(t,\xi)) \geq Z(\sigma(t,\xi) - \tau(t,\xi)) \geq x(\sigma(t,\xi) - \tau(t,\xi))$, and thus

$$\left(r(t)Z^{(m-1)}(t)\right)' + p(t)Z^{(m-1)}(t) + \int_a^b q(t,\xi)Z(\sigma(t,\xi))\left(1 - \int_a^b g(\sigma(t,\xi),\xi)d\eta(\xi)\right)d\eta(\xi).$$

From (7.2.13), we get

$$\left(r(t)Z^{(m-1)}(t)\right)' + p(t)Z^{(m-1)}(t) + g_1\int_a^b q(t,\xi)Z(\sigma(t,\xi))d\eta(\xi) \leq 0. \tag{7.2.14}$$

From (H_4) and (H_5), we obtain

$$Z(\sigma(t,\xi)) \geq Z(\sigma(t,a)) > 0, \quad \xi \in [a,b] \text{ and } \theta(t) \leq \sigma(t,\xi) \leq t.$$

Thus $Z(\theta(t)) \leq Z(\sigma(t,a))$ for $t \geq t_2$. Then (7.2.14) can be written as

$$\left(r(t)Z^{(m-1)}(t)\right)' + p(t)Z^{(m-1)}(t) + g_0 \int_a^b q(t,\xi)Z(\theta(t))d\eta(\xi) \leq 0. \qquad (7.2.15)$$

For $t \geq t_0,\ t = t_k,\ k = 1, 2, \cdots$ from (7.2.1) we have

$$a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)} Z(x, t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)} Z(x, t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

Therefore $Z(t)$ is an eventually positive solution of (7.2.11). This contradicts the hypothesis and completes the proof. □

Theorem 7.2.3. *Assume that $\frac{d}{dt}\sigma(t,a)$ exists and $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$. If for $t_0 > 0$ and there exists a function $A(t) \in C'([0,+\infty),(0,+\infty))$ which is nondecreasing with respect to t, such that*

$$\int_{t_0}^{+\infty} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[A(s)C(s) - \frac{A(s)B^2(s)}{4L(s)}\right] ds = +\infty, \qquad (7.2.16)$$

then every solution of the boundary value problem (7.1.1)-(7.1.2) is oscillatory in G.

Proof. To prove the solutions of (7.1.1)-(7.1.2) are oscillatory in G, from Theorem 7.2.2, it is enough to prove that the impulsive differential inequality (7.2.11) has no eventually positive solution. Suppose that $Z(t) > 0$ is a solution of the inequality (7.2.11). Define

$$W(t) = A(t)\frac{r(t)Z^{(m-1)}(t)}{Z(\theta(t))}, \quad t \geq t_0, \qquad (7.2.17)$$

then $W(t) \geq 0$ for $t \geq t_0$, and

$$W'(t) \leq \frac{A'(t)}{A(t)}W(t) + \frac{A(t)\left[-p(t)Z^{(m-1)}(t) - g_1\int_a^b q(t,\xi)Z(\theta(t))d\eta(\xi)\right]}{Z(\theta(t))} - \frac{A(t)\left(r(t)Z^{(m-1)}(t)\right)Z'(\theta(t))\theta'(t)}{Z^2(\theta(t))}.$$

From $Z^{(m)}(t) \leq 0$, according to Lemma 6.1.3, we obtain

$$Z'(\theta(t)) \geq M(\theta(t))^{m-2}Z^{(m-1)}(t). \tag{7.2.18}$$

Thus

$$W'(t) \leq B(t)W(t) - A(t)C(t) - \frac{L(t)}{A(t)}W^2(t).$$

Proceeding as in the proof of Theorem 6.2.3, we have

$$U(t) \leq U(t_0) - \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[A(s)C(s) - \frac{A(s)B^2(s)}{4L(s)}\right] ds.$$

Letting $t \to +\infty$, from (7.2.16), we have $\lim_{t\to+\infty} U(t) = -\infty$, which leads to a contradiction with $U(t) \geq 0$. □

Similar to Theorem 6.2.4, we can obtain the following Thereom.

Theorem 7.2.4. *Assume that* $\frac{d}{dt}\sigma(t,a)$ *exists and* $\beta(x) \not\equiv 0$ *for* $x \in \partial\Omega$. *Moreover, suppose that there exist functions* $A(t)$ *and* $\rho(s) \in C'([0,+\infty),(0,+\infty))$ *in which* $A(t)$ *is nondecreasing with respect to* t, *and the functions* $H(t,s), h(t,s) \in C'(D,\mathbb{R})$, *in which* $D = \{(t,s)|t \geq s \geq t_0 > 0\}$, *such that*

(H_8) $H(t,t) = 0, \quad t \geq t_0; \quad H(t,s) > 0, \quad t > s \geq t_0,$

(H_9) $H_t'(t,s) \geq 0, \quad H_s'(t,s) \leq 0,$

(H_{10}) $-\dfrac{\partial}{\partial s}[H(t,s)\rho(s)] - H(t,s)\rho(s)B(s) = h(t,s).$

If

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[A(s)C(s)H(t,s)\rho(s)-\frac{1}{4}\frac{|h(t,s)|^2A(s)}{L(s)H(t,s)\rho(s)}\right]ds = +\infty, \qquad (7.2.19)$$

then every solution of the boundary value problem (7.1.1)-(7.1.2) is oscillatory in G.

Remark 7.2.2. *In Theorem 7.2.4, by choosing $\rho(s) = A(s) \equiv 1$, we have the following corollary.*

Corollary 7.2.1. *Assume that the conditions of Theorem 7.2.4 hold, and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[C(s)H(t,s)-\frac{1}{4}\frac{|h(t,s)|^2}{L(s)H(t,s)}\right]ds = +\infty,$$

then every solution of the boundary value problem (7.1.1)-(7.1.2) is oscillatory in G.

Remark 7.2.3. *From Theorem 7.2.4 and Corollary 7.2.1, we can attain various oscillatory criteria by different choices of the weighted function $H(t,s)$. For example, choosing $H(t,s) = (t-s)^{\mu-1}$, $t \ge s \ge t_0$, in which $\mu > 2$ is an integer, then* $h(t,s) = (t-s)^{\mu-1}$, $t \ge s \ge t_0$. *From Corollary 7.2.1, we have*

Corollary 7.2.2. *If there exists a $\frac{d}{dt}\sigma(t,a)$ and an integer $\mu > 2$ such that*

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{\mu-1}}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}(t-s)^{\mu-1}\left[C(s)-\frac{1}{4L(s)}\left[\frac{\mu-1}{t-s}-B(s)\right]^2\right]ds = +\infty,$$

then every solution of the problem (7.1.1)-(7.1.2) is oscillatory in G.

Now we consider $H(t,s) = [R(t)-R(s)]^{\mu}$, $t \ge s \ge t_0$, where $R(t) = \int_{t_0}^{t}\frac{1}{r(s)}ds$ and $\lim_{t\to+\infty} R(t) = +\infty$, then $h(t,s) = [R(t)-R(s)]^{\mu}\left[\frac{\mu}{R(t)-R(s)} - B(s)\right]$.

Corollary 7.2.3. *If there exists $\frac{d}{dt}\sigma(t,a)$ and an integer $\mu > 2$ such that*

$$\limsup_{t\to+\infty} \frac{1}{[R(t)-R(t_0)]^{\mu}} \int_{t_0}^{t} \prod_{t_0\le t_k<s} [R(t)-R(s)]^{\mu} \left[C(s) - \frac{1}{4L(s)} \left[\frac{\mu}{R(t)-R(s)} - B(s) \right]^2 \right] ds = +\infty,$$

then every solution of the boundary value problem (7.1.1)-(7.1.2) is oscillatory in G.

7.3 Examples

In this section, we present couple of examples to point up our results established in Section 7.2.

Example 7.3.1. *Consider the following equation of the form*

$$\left.\begin{array}{l}
\dfrac{\partial}{\partial t}\left(\dfrac{3}{4}\dfrac{\partial^5}{\partial t^5}\left(u(x,t)+\dfrac{2}{3}\int_{-\pi/2}^{-\pi/4} u(x,t+2\xi)d\xi\right)\right) \\
-\dfrac{1}{2}\dfrac{\partial^5}{\partial t^5}\left(u(x,t)+\dfrac{2}{3}\int_{-\pi/2}^{-\pi/4} u(x,t+2\xi)d\xi\right) \\
+\dfrac{4}{3}\int_{-\pi/2}^{-\pi/4} u(x,t+2\xi)d\xi = \dfrac{4}{3}\Delta u(x,t) + \dfrac{3}{4}\Delta u(x,t-\frac{3\pi}{2}), \quad t\neq t_k,\ k=1,2,\cdots, \\
u(x,t_k^+) = \dfrac{k}{k+1}u(x,t_k), \\
\dfrac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k^+) = \dfrac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k), \quad i=1,2,3,4,5, \quad k=1,2,\cdots
\end{array}\right\} \tag{7.3.1}$$

for $(x,t) \in (0,\pi)\times[0,+\infty)$, with the boundary condition

$$u(0,t) = u(\pi,t) = 0, \quad t\neq t_k. \tag{7.3.2}$$

Here $\Omega = (0,\pi)$, $m = 6$, $a_k^{(0)} = b_k^{(0)} = \dfrac{k}{k+1}$, $a_k^{(i)} = b_k^{(i)} = 1$, $i = 1,2,3,4,5$, $r(t) = \dfrac{3}{4}$, $g(t,\xi) = \dfrac{2}{3}$, $p(t) = -\dfrac{1}{2}$, $q(t,\xi) = \dfrac{4}{3}$, $\tau(t,\xi) = \sigma(t,\xi) = t+2\xi$, $a(t) = \dfrac{4}{3}$, $b_1(t) = \dfrac{3}{4}$, $\rho_1(t) = t-\frac{3\pi}{2}$, $j = 1$, $[a,b] = \left[-\dfrac{\pi}{2}, -\dfrac{\pi}{4}\right]$, $\eta(\xi) = \xi$, $M = 1$, $\theta(t) = t$,

$\theta'(t) = 1,\ \mu = 3.$ *Since* $t_0 = 1,\ t_k = 2^k,\ g_1 = 1 - \frac{\pi}{6},\ C(s) = \frac{\pi}{3} - \frac{\pi^2}{18},\ L(s) = \frac{4s^4}{3}.$

Then hypotheses $(H_1) - (H_7)$ *hold, moreover*

$$\begin{aligned}
\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \frac{a_k^{(0)}}{b_k^{(i)}} ds &= \int_1^{+\infty} \prod_{1<t_k<s} \frac{k}{k+1} ds \\
&= \int_1^{t_1} \prod_{1<t_k<s} \frac{k}{k+1} ds + \int_{t_1^+}^{t_2} \prod_{1<t_k<s} \frac{k}{k+1} ds + \cdots \\
&= 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times 2^3 + \cdots \\
&= \sum_{n=0}^{+\infty} \frac{2^n}{n+1} = +\infty.
\end{aligned}$$

Thus,

$$\limsup_{t\to+\infty} \frac{1}{(t-1)^2} \left\{ \int_1^t \prod_{1<t_k<s} \frac{k+1}{k} (t-s)^2 \left[\left(\frac{\pi}{3} - \frac{\pi^2}{18} \right) - \frac{3}{4s^4} \times \left(\frac{1}{(t-s)^2} + \frac{1}{9} - \frac{2}{3(t-s)} \right) \right] ds \right\} = +\infty.$$

Therefore all the conditions of the Corollary 7.2.2 are satisfied. Therefore, every solution of the problem (7.3.1)-(7.3.2) *is oscillatory in* G. *In fact* $u(x,t) = \sin x \cos t$ *is such a solution.*

Example 7.3.2. *Consider the following equation of the form*

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left(t^2 \frac{\partial^3}{\partial t^3} \left(u(x,t) + \frac{1}{2} \int_{-\pi}^{0} u(x,t+\xi) d\xi \right) \right) \\
&+ (-2t) \frac{\partial^3}{\partial t^3} \left(u(x,t) + \frac{1}{2} \int_{-\pi}^{0} u(x,t+\xi) d\xi \right) \\
&+ \frac{3}{2} \int_{-\pi}^{0} u(x,t+\xi) d\xi = t^2 \Delta u(x,t) + (t^2+3) \Delta u(x, t - \tfrac{\pi}{2}), \quad t \ne t_k,\ k = 1,2,\cdots, \\
&u(x,t_k^+) = \frac{k}{k+1} u(x,t_k), \\
&\frac{\partial^{(i)}}{\partial t^{(i)}} u(x,t_k^+) = \frac{\partial^{(i)}}{\partial t^{(i)}} u(x,t_k), \quad i = 1,2,3, \quad k = 1,2,\cdots
\end{aligned}\right\} \tag{7.3.3}$$

for $(x,t) \in (0,\pi) \times [0,+\infty)$, *with the boundary condition*

$$u(0,t) = u(\pi,t) = 0, \quad t \neq t_k. \tag{7.3.4}$$

Here $\Omega = (0,\pi)$, $m = 4$, $a_k^{(0)} = b_k^{(0)} = \dfrac{k}{k+1}$, $a_k^{(i)} = b_k^{(i)} = 1$, $i = 1,2,3$, $r(t) = t^2$, $g(t,\xi) = \dfrac{1}{2}$, $p(t) = -2t$, $q(t,\xi) = \dfrac{3}{2}$, $\tau(t,\xi) = \sigma(t,\xi) = t+\xi$, $a(t) = t^2$, $b_1(t) = t^2+3$, $\rho_1(t) = t - \frac{\pi}{2}$, $j = 1$, $[a,b] = [-\pi,0]$, $\eta(\xi) = \xi$, $M = 1$, $\theta(t) = t$, $\theta'(t) = 1$, $\mu = 3$.
Since $t_0 = 1$, $t_k = 2^k$, $g_1 = 1 - \dfrac{\pi}{2}$, $C(s) = \dfrac{3\pi}{2} - \dfrac{3\pi^2}{4}$, $L(s) = 1$.
Then hypotheses $(H_1) - (H_7)$ *hold, thus,*

$$\limsup_{t\to+\infty} \frac{1}{(t-1)^2}\left\{ \int_1^t \prod_{1<t_k<s} \frac{k+1}{k}(t-s)^2 \left[\left(\frac{3\pi}{2} - \frac{3\pi^2}{4}\right) - \left(\frac{1}{s^2} + \frac{1}{(t-s)^2} - \frac{2}{s(t-s)}\right)\right] ds \right\} = +\infty.$$

Therefore all the conditions of the Corollary 7.2.2 are satisfied. Therefore, every solution of the problem (7.3.3)-(7.3.4) *is oscillatory in* G. *In fact* $u(x,t) = e^{-x}\sin t$ *is such a solution.*