CHAPTER 3

MATHEMATICAL AND SIMULATION TOOLS FOR MANET ANALYSIS

3.1 INTRODUCTION

MANET analysis is a multidimensional affair. Many tools of mathematics are used in the analysis. Among them, the prime tools are probability theory and Random processes. This is due to the fact that most, if not all, of MANET parameters and characteristics function in a stochastic environment. Be it link reliability, link availability, link maintainability etc., measures of energy awareness and congestion control QoS metrics are all measured in terms of probability.

The generalization of the concept of random variable is the concept of random process and is used in a major way in MANET analysis. Simulation is another application of random numbers in probability theory. This is due to the fact that if the random variable, has density and distribution functions f(x) and F(x) respectively, then the random variable y = F(x), has uniform distribution over (0, 1). Simulation techniques have grown up with computers and are ideally suited for repetitive calculations.

In fact, mathematical and simulation tools in modern times, bring out the best of the interplay between computer capabilities and processes of mathematical analysis.
The rest of this chapter is organized as follows. In sections 3.2, 3.3, 3.4 and 3.5 the basics of probability, distributions, random processes and reliability are reviewed. In sections 3.6 and 3.7 simulation in general and elements of network simulator are discussed.

### 3.2 PROBABILITY THEORY

Probability is a concept associated with the outcomes of a random experiment as written by Walpole and Myers (1985). It allows one to quantify the variability in the outcome of any random experiment (E), whose exact outcome cannot be predicted with certainty.

Let S be the set of totality of outcomes of E, called the sample space of E. Let \( n(S) \) = Number of elements in S. Let \( A \subset S \) be any event of E. Then \( P(A) \), the probability of occurrence of the event is

\[
P(A) = \frac{n(A)}{n(S)}, \text{ where } n(A) = \text{Number of elements in S, favorable to } A
\]

Clearly, \( P(\emptyset) = 0 \)

\[
P(S) = 1
\]

\[
P(A^1) = 1 - P(A) \text{ where } A^1 \text{ is the complement of } A \text{ in } S
\]

Mutually exclusive events are a set of events, where the occurrence of any one, excludes the occurrence of all others. Independent events are a set of events where the occurrence of any one has no effect on the occurrence of any others. Conditional probability of event A with respect to event B, denoted by \( P(A/B) \) is the probability of A with B as sample space.

One can show \( P(A/B) = \frac{P(A\cap B)}{P(B)} \)  

\[(3.1)\]
As per the probability theory standards the theorems are,

Addition theorem

- For two general events $A$, $B$
  \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

- For three general events $A$, $B$, $C$
  \[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \]

- For two mutually exclusive events $A$, $B$
  \[ P(A \cup B) = P(A) + P(B) \]

- For three mutually exclusive events $A$, $B$, $C$
  \[ P(A \cup B \cup C) = P(A) + P(B) + P(C) \]

Multiplication theorem

- For two general events $A$, $B$
  \[ P(A \cap B) = P(A) \cdot P(B/A) \]

- For three general events $A$, $B$, $C$
  \[ P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B) \]

- For two independent events $A$, $B$
  \[ P(A \cap B) = P(A) \cdot P(B) \]

- For three pair-wise independent events $A$, $B$, $C$
  \[ P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) \]
If the events $B_i, i = 1, n$ forms a partition of the sample space $S$ and $A$ is any subset of $S$, then

The total probability theorem is

$$P(A) = \sum_i^n P(B_i) \cdot P(A/B_i) \quad (3.2)$$

The Baye’s theorem is

$$P(B_k/A) = \frac{P(B_k) \cdot P(A/B_k)}{\sum_i^n P(B_i) \cdot P(A/B_i)} \quad (3.3)$$

### 3.3 PROBABILITY DISTRIBUTIONS

#### 3.3.1 Concept of random variable

A random variable is a variable associated with a random experiment. It is called Discrete Random Variable (DRV) or Continuous Random Variable (CRV) according to whether it takes a discrete set of values or all values in an interval.

A random variable can take values, not with certainty, but only to a probability. Hence for a DRV $X$, there is an associated probability mass function $p_i$ or $p(x_i)$, defined by $p_i = P\{X = x_i\}$ satisfying the properties $p_i \geq 0$ and $\sum p_i = 1$. The values $\{x_1, x_2, \ldots, x_n\}$ which the DRV takes is said to be the domain of the DRV as written by Papoulls and Unnikrishnan Pillai (2002). The set of pairs $(x_i, p_i)$ for $i = 1$ to $n$ is called the probability distributions of the DRV.

Associated with a CRV $X$, similarly the probability density function (PDF) $f(x)$ is defined as
\[ f(x)dx = P\{x < X < x + dx\} \quad (3.4) \]

satisfying the properties

\[ f(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} f(x)dx = 1 \] (Here -\(\infty\) and \(\infty\) are only the symbols for the lower/upper limits of the domain of the CRV)

Also associated with a DRV \(X\), one has the distribution function \(F(x_i)\) defined by \(F(x_i) = P(X \leq x_i)\) satisfying the properties

\[ F(x_i) \geq 0, F(-\infty) = 0, F(\infty) = 1 \]

\[ p_i = F(x_i) - F(x_{i-1}) \]

Similarly, for a CRV \(X\), the distribution function \(F(x)\) is defined as \(F(x) = P(X \leq x)\) having the properties

\[ F(x) \geq 0, F(-\infty) = 0, F(\infty) = 1 \]

\[ F(x) = \int_{-\infty}^{x} f(x)dx, F'(x) = f(x) \quad (3.5) \]

### 3.3.2 Expectation and moments of a random variable

The expectation of a random variable, \(E(X)\) is defined as

\[ E(X) = \sum x_i p_i \quad \text{for Discrete Random Variable} \]

\[ E(X) = \int_{-\infty}^{\infty} x f(x)dx \quad \text{for Continuous Random Variable} \]

If \(g(x)\) is any function of the random variable \(X\), then \(E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)dx\) If \(g(x) = x^r\), then \(E(x^r) = \mu_r\) the \(r^{th}\) order raw
moment of the random variable \( x \) and if \( g(x) = (x - \mu_1^r)^r \), then \( E(x - \mu_1^r)^r = \mu_r \) the \( r^{th} \) order central moment of the random variable.

In particular, \( \bar{x} = \text{mean} = \mu_1 \) and

\[
Var(X) = \sigma^2 = \mu_2
\]

Also \( \mu_r = \sum_{j=1}^{r} r C_j (-1)^j \mu_{r-j} (\mu_1')^j \) \hspace{1cm} (3.6)

and \( M_X(t) \) = moment generation function of the random variable \( X \) is defined as

\[
M_X(t) = E(e^{xt})
\]

For any moment generation function one can show that

\[
M_X(0) = 1
\]

If \( Y = aX + b \), then

\[
M_Y(t) = e^{bt} M_X(at) \hspace{1cm} (3.7)
\]

If \( X_i \) are independent random variables (for \( i = 1 \) to \( n \)) with moment generating function \( M_i(t) \), then for

\[
X = \sum X_i,
\]

moment generating function \( M_X(t) = \prod_{i=1}^{n} M_i(t) \)

### 3.3.3 Standard probability distributions used in MANET

Standard probability distributions used in MANET analysis are

- ‘Memoryless’ exponential distribution
- Gamma distribution which is a generalization of the exponential distribution
- Weibull distribution which is also a generalization of the exponential distribution.

- Poisson distribution which is used in Poisson random processes and Uniform distribution used in simulation.

1. The exponential random variable is a continuous random variable $x$, with density function

$$f(x) = \lambda e^{-\lambda x}, x > 0,$$

for which

the distribution function $F(x) = 1 - e^{-\lambda x}$,

moment generating function

$$M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

Mean $= \frac{1}{\lambda}$ and

Variance $\sigma^2 = \frac{1}{\lambda^2}$

$\lambda^{-1} (1/\lambda)$ specifies the exponentially distributed mean rate of epoch length. The memoryless property is due to the fact, that for the exponential distribution, the conditional probability

$$P(x > (t_0 + t)/x > t_0) = P(x > t)$$

Given that the life span is greater than $t_0$ in the conditional probability that the life span is greater than $t_0 + t$ is independent of $t_0$ and is equal to the probability that the life span is greater than $t$.

2. The gamma random variable is a continuous random variable (a generalization of the exponential random variable) with probability density function
\[ f(x) = \frac{\lambda^n}{\sqrt{n}} e^{-\lambda x} x^{n-1}; \lambda, n, x > 0, \]

with \( F(x) = \frac{\lambda^n}{\sqrt{n}} \int_0^x e^{-\lambda x} x^{n-1} dx \)

\[ M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-n} \]

\[ \bar{x} = \frac{n}{\lambda}, Var(x) = \frac{n}{\lambda^2} \]

For \( n=1 \), the gamma distribution becomes the exponential distribution given by \( f(x) = \lambda e^{-\lambda x}; x > 0 \)

3. Another generalization of the exponential distribution is the weibull continuous random variable having density function

\[ f(x) = \alpha \beta t^{\beta - 1} e^{-\alpha t^\beta}; \alpha, \beta > 0, x > 0 \]

with \( F(x) = 1 - e^{-\alpha t^\beta} \)

\[ M_x(t) = \alpha \beta \int_0^x t^{\beta - 1} e^{-\alpha t^\beta} dx \]

Mean \( \bar{x} = \alpha^{-1/\beta} \sqrt{1 + \frac{1}{\beta}} \)

and \( Var(x) = \alpha^{-2/\beta} \left[ \sqrt{1 + \frac{2}{\beta}} - \left( \sqrt{1 + \frac{1}{\beta}} \right)^2 \right] \)

\( \alpha \) and \( \beta \) determines the mean and variance of epoch length.

For \( \beta = 1 \), one gets the exponential random variable and for \( \beta = 2 \), one gets the Rayleigh random variable.

4. The Poisson random variable is a Discrete random variable \( X \) with domain \( x = 0, 1, 2, \ldots \), with probability mass function
\[ f(x) = e^{-\lambda} \frac{\lambda^x}{x!} \]

\[ F(x) = e^{-\lambda} \sum_{0}^{x} \frac{\lambda^x}{x!} \]

\[ M_X(t) = e^{\lambda(e^t - 1)} \]

Also here Mean = \lambda = variance

Poisson variate can also be considered as a limiting case of the binomial variate and the limiting case of the Poisson variate is the normal distribution.

5. The uniform CRV \( X \) in the interval \((a,b)\) has density function given by

\[ f(x) = \frac{1}{b-a}; \quad b > a, a < x < b \]

and the distribution function

\[ F(x) = \frac{x-a}{b-a} \]

and the moment generating function

\[ M_X(t) = \frac{e^{bt} - e^{at}}{t(b-a)} \]

Also \( \bar{x} = (b + a)/2 \)

and \( \sigma^2 = (b - a)^2/12 \)

when \( b = 1, a = 0 \) we get the uniform variate over \((0,1)\) which is used in simulation studies.

### 3.4 RANDOM PROCESSES

#### 3.4.1 Introduction

Many MANET processes are stochastic processes. In prediction based link reliability estimation, the number of times the nodes change their
velocities (ie, number of epochs) within prediction period \( t_p \) is a Poisson process which is a discrete random process satisfying Poisson postulates.

The MANET route discovery process is “Renewal Process” where within a specific time period, link disruption and restoration takes place with respect to communication between two nodes.

3.4.2 Definition of a Random process

A random process \( X(\lambda, t) \) (mostly written as \( X(t) \)), where \( \lambda \) is the random variable over sample space \( S \) and \( t \) is the time variable over time space \( T \), is an ensemble of random variables each of which is a time function.

When \( t \) is fixed, \( X(t) \) is a random variable. When \( \lambda \) is fixed, \( X(t) \) is a time function and when \( \lambda \& t \) are fixed, \( X(t) \) is a real number.

3.4.3 Four way classification of random processes

(D: Discrete, C: Continuous   S: Sample space,   T: Time space)

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Note: Poisson process and Renewal process used in MANET analysis are discrete random processes.
3.4.4  Stationarity of a random process

Let X(t) be a random process. For $t_1, t_2, \ldots, t_k$ let the random variable $x_1, x_2, \ldots, x_k$ (where $x_k = X(t_k)$) have the joint distribution function $F(x_1, x_2, \ldots, x_k)$.

If these distribution functions $F(x_1, x_2, \ldots, x_k)$ are invariant under any time shift for all orders up to and including $k$ then $X(t)$ is said to be $k^{th}$ order stationary. If $k=1$, $X(t)$ is first order stationary. If $k=2$, $X(t)$ is Wide Sense Stationary. If $k=\infty$, $X(t)$ is Strict Sense Stationary. Poisson process used in MANET analysis is not a stationary process of any order.

3.4.5  Statistical averages of a Random Process

Mean $= E(X(t)) = \mu(t)$

For a fixed $t$, $X(t)$ is a random variable, having density $f(x)$ (say)

Then $E(X(t)) = \int_{-\infty}^{\infty} X(t) f(x) dx$

$Var(X) = E(X^2) - (E(X))^2$

is the variance.

Autocorrelation function $R_{XX}(t_1, t_2)$ is defined as (also written as $R(t_1, t_2)$)

$R(t_1, t_2) = E(X(t_1)X(t_2))$

Auto covariance function $C_{XX}(t_1, t_2) = R(t_1, t_2) - \overline{X}(t_1)\overline{X}(t_2)$
3.4.6 Poisson random process

Poisson random process is a discrete random process (where the state space is discrete and time space is continuous). It is an event counting process \( N(t) \) where \( N(t) \) denotes the number of occurrences of the event in \((0,t)\) satisfying Poisson postulates.

Let \( \lambda \) be the mean occurrence rate of the event. Then the Poisson postulates are

\[
P \{ \text{Event occurs only once in (t, t + dt)} \} = \lambda \ dt
\]

\[
P \{ \text{No event occurs in (t, t + dt)} \} = 1 - \lambda \ dt
\]

\[
P \{ \text{More than one occurs in (t, t + dt)} \} = 0
\]

Events are independent of past and future occurrences

The number of occurrences in \((t_0, t_0 + t)\) depends only on the span \( t \) of the interval and not on \( t_0 \)

Using these postulates, one can show that,

\[
P_n(t) = P\{N(t) = n\}
\]

\[
e^{-\lambda t} \frac{\lambda^t}{n!} \text{ for } n=0,1,2 \ldots \ldots \quad (3.8)
\]

Also for the Poisson process \( N(t) \),

\[
E(N(t)) = \lambda t
\]

\[
Var(N(t)) = \lambda t
\]

\[
E(N^2(t)) = \lambda t + \lambda^2 t^2
\]
Standard properties of the Poisson random process are

- Poisson process is a Markov process

- The sum of two independent Poisson processes with parameters $\lambda_1, \lambda_2$ is a Poisson process with parameter $(\lambda_1 + \lambda_2)$.

- The difference of two independent Poisson process is not a Poisson process.

- Poisson process is not a stationary process. Its autocorrelation function

$$R(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

$$C(t_1, t_2) = \lambda \min(t_1, t_2)$$

- The inter occurrence times of events (which are epoch lengths in MANET analysis) of Poisson process with parameter $\lambda$, has exponential distribution with mean $1/\lambda$.

Poisson random process is of great use in MANET analysis.

3.5 RELIABILITY

3.5.1 Introduction

Link reliability is a parameter of prominence in MANET studies. So are link availability and link maintainability. These are measured using the tools of probability theory. They are briefly reviewed in this section.

Let $T$ be the CRV, representing the life time (of satisfactory performance) of a system (MANET) or component (e.g. Link in a MANET).
Let $f(t)$ be the density function of $T$, $F(t)$ the distribution function, $R(t)$ the reliability function and $Z(t)$, the hazard rate function. Then, in terms of probability

$$f(t)dt = P(t < T < t + dt)$$

$$F(t) = P(T \leq t), R(t) = P(T > t) \text{ and}$$

$$Z(t)dt = P((t < T < (t + dt))/T > t)$$

one can show that

$$R(t) = 1 - F(t) = 1 - \int_{0}^{t} f(t)dt = e^{-\int_{0}^{t} Z(t)dt} \quad (3.9)$$

$$F(t) = 1 - R(t) = \int_{0}^{t} F(t)dt = 1 - e^{-\int_{0}^{t} Z(t)dt} \quad (3.10)$$

$$Z(t) = \frac{f(t)}{1-\int_{0}^{t} f(t)dt} = \frac{F'(t)}{1-F(t)} = -\frac{R'(t)}{R(t)} \quad (3.11)$$

Also Mean Time to Failure (MTTF) = $E(T) = \int_{0}^{\infty} tf(t)dt = \int_{0}^{\infty} R(t)dt$

For the ‘memoryless’ exponential variate $T$, and for parameter $\lambda$

$$f(t) = \lambda e^{-\lambda t}$$

$$F(t) = 1 - e^{-\lambda t}$$

$$R(t) = e^{-\lambda t} \text{ and}$$

$$Z(t) = \lambda (\text{constant hazard rate})$$

and $\text{MTTF} = \frac{1}{\lambda}$
If the life time variable $T$ has the Weibull distribution, then

$$f(t) = \alpha \beta t^{\beta-1}e^{-\alpha t^\beta}$$

$$F(t) = 1 - e^{-\alpha t^\beta}$$

$$R(t) = e^{-\alpha t^\beta}$$

$$Z(t) = \alpha \beta t^{\beta-1} \text{ (Here hazard rate is a power of } t)$$

and $MTTF = \alpha^{-1/\beta} \sqrt{(1 + 1/\beta)}$

### 3.5.2 Reliability of systems

If a system consists of $n$ components connected in series, with $R_j(t)$ being the reliability of the $j^{th}$ component, then system reliability $R = \prod_1^n R_j(t)$

If these $n$ components are connected in parallel, then

$$R = 1 - \prod_{j=1}^n (1 - R_j)$$

Availability is the probability that a component or system is performing well at time $t$, when it is operated and maintained at prescribed conditions. $A(t)$, called point availability differs from $R(t)$, the reliability in the sense that the system might have been restored to life (due to failure and repair) in the time interval for $A(t)$ while it is not so in the case of $R(t)$.

If 1, 0 denote the states of good performance, downstate due to repair of a system then the point availability

$$A(t) = P_1(t)$$
the probability that at time $t$, the system is in upstate 1.

Interval availability $A(\tau)$ over $(0,\tau) = \frac{1}{\tau} \int_0^\tau A(t) \, dt$ and inherent availability is the $\lim_{\tau \to \infty} A(\tau)$ and is $A(\infty)$.

$$A(\infty) = \frac{MTTF}{MTTF + MTTR}$$

(3.12)

where MTTR - Mean Time to Repair

If a system has constant hazard rate $\lambda$ and has constant repair rate $\mu$, then $A(\infty) = \frac{\mu}{\mu + \lambda}$ since $MTTF = \frac{1}{\lambda}$ and $= \frac{1}{\mu}$.

System availability $A(t)$ of a series system having $n$ components

i) Connected in series, with $i^{th}$ components having $A_i(t)$ as its point availability is

$$A(t) = \prod_1^n A_i(t) \text{ and }$$

(3.13)

ii) If they are connected in parallel then

$$A(t) = 1 - \prod_1^n \left(1 - A_i(t)\right)$$

(3.14)

3.6 SIMULATION

In MANET analysis, simulation has a significant role to play. Due to the versatility of computers, the repetitive calculations of simulation are easily done. To simulate the observation of continuous random variables, one usually starts with uniform random numbers, and relates them to the distribution function of interest. This is due to the fact that if $X$ is any CRV with distribution function $F(X)$, then the CRV $Y=F(X)$ has uniform distribution over $(0,1)$. 
Suppose to simulate an observation from the exponential distribution \( F(x) = 1 - e^{-\lambda x}, x > 0 \). The computer will first generate a value i.e., a random number \( y \) in \((0,1)\). Now \( y = F(x) \) gives \( x = F^{-1}(y) \).

Here \( 1 - e^{-\lambda x} = r \eta \) (random number in \((0,1)\))

\[
\therefore x = -\lambda^{-1}\ln(1 - r \eta) \text{ is the exponential variate value corresponding to random number.}
\]

Similarly, to simulate observations from the weibull distribution with parameter \( \alpha, \beta \) for which, the distribution function \( F(x) \) is

\[
F(x) = 1 - e^{-\alpha x^\beta}
\]

So for random number \( rn \) in \((0,1)\) the weibull variable

\[
x = \left( -\alpha^{-1}\ln(1 - rn) \right)^{\frac{1}{\beta}}
\]

Similarly, one can simulate observations of any random variable \( x \) with distribution function \( F(x) \) as

\[
x = F^{-1}(rn)
\]

where \( rn \) is the random number in \((0,1)\).

Nowadays, simulation software packages are available with various capabilities. The NS simulator is one such simulation package used widely and many updated versions are also available.

- **NS Simulator**

  The Network Simulator 2.32 (ns-2) simulator covers a very large number of applications, of protocols, of network types, of network elements and traffic models. They are called ‘simulated objects” using network
simulator. The analysis of the behavior of the simulated objects is focused here. The network simulator is based on two languages: an Object oriented simulator, written in C++ and an Object oriented Tool Command Language (OTcl) interpreter, used to execute user’s command scripts.

ns-2 has a rich library of network and protocol objects. There are two class hierarchies: the compiled C++ hierarchy and the interpreted OTcl hierarchy with one-to-one correspondence between them. ns-2 is a discrete event simulator, where the advance of time depends on the timing of events which are maintained by a scheduler. The scheduler keeps an ordered data structure (a simple linked list) with the events to be executed and tries them one by one, invoking the handler of the event.

ns-2 simulator can be used to run many applications involving problems of

- File Transfer Protocol
- Constant Bit Rate using User Datagram Protocol (UDP)
- Transport Control Protocol (TCP)
- TCP over DSR
- TCP over AODV
- TCP over Temporally Ordered Routing Algorithm (TORA)
- TCP with Medium Access Control (MAC) protocol

The ns-2 has thus become a powerful tool in the hand of the researcher, offering the type of support which only a computer can provide in manifold calculations.
3.7 CONCLUSION

MANET parameters and characteristics function in a stochastic environment. Be it link reliability or link availability are measured in terms of probability. In this chapter, mathematical tools namely probability theory and random process which are essential to estimate the link reliability is discussed.