Chapter 6

Nonlinear Oscillation of Certain Third-Order Neutral Differential Equation with Distributed Delay

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6.1 Introduction

[1] In the present chapter, we concerned with the nonlinear oscillation of certain NDE's with continuously distributed delay of third order

$$\left[r_1(t)\left[(r_2(t)N'(t))'\right]^{\gamma}\right]' + \int_c^d q(t,\mu)x^{\gamma}(\sigma(t,\mu))\,d\mu = 0, \tag{6.1.1}$$

and

$$\left[r_1(t)\left(r_2(t)N'(t)\right)'\right]' + \int_c^d q(t,\mu)f(x(\sigma(t,\mu)))\,d\mu = 0, \tag{6.1.2}$$

where

$$N(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu$$

and $a < b$, $c < d$. Throughout this chapter we following hypotheses are tacitly supposed to hold:

(H₁) $\gamma \geq 1$ is a ratio of two odd positive integers, $r_1(t), r_2(t) \in C^1([t_0,+\infty))$, $r_1(t), r_2(t) > 0$, $r_1'(t) \geq 0$ and

$$\int_{t_0}^{\infty} \frac{1}{r_1^{1/\gamma}(t)}dt = \infty, \quad \int_{t_0}^{\infty} \frac{1}{r_2(t)}dt = \infty. \tag{6.1.3}$$

(H₂) $p(t,\mu) \in C\big([t_0,+\infty) \times [a,b], R^+\big)$, $q(t,\mu) \in C\big([t_0,+\infty) \times [c,d], R^+\big)$, $0 \leq \int_a^b p(t,\mu)d\mu \leq P < 1$ and $q(t,\mu)$ is not identically zero for $[t_*,+\infty) \times [c,d]$, $t_* \geq t$.

[1] Nonlinear oscillation of certain third order neutral differential equation with distributed delay (a part of this Chapter is published in the *Journal of Mahani Mathematical Research Center*, **7** (1-2) (2018), 1-12).

(H₃) $\tau(t,\mu) \in C\big([t_0,+\infty)\times[a,b],R^+\big)$, $\tau(t,\mu)\le t$, $\tau(t,\mu)$ is nondecreasing in μ, $\tau(t,\mu)\to\infty$ as $t\to+\infty$ for $\mu\in[a,b]$ and $\sigma(t,\mu)\in C\big([t_0,+\infty)\times[c,d],R^+\big)$, $\sigma(t,\mu)\le t$, $\sigma(t,\mu)$ is nondecreasing in μ, $\sigma(t,\mu)\to\infty$ as $t\to+\infty$ for $\mu\in[c,d]$.

By a solution of equation (6.1.1) we mean a function $x(t)\in C([T_x,\infty))$, $T_x\ge t_0$, which has the property $N'(t)$, $r_2(t)N'(t)$ and $r_1(t)[(r_2(t)N'(t))']^\gamma$ are continuously differentiable and satisfies (6.1.1) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (6.1.1) which satisfy $\sup\{|x(t)|: t\ge T\}>0$ for all $T\ge T_x$.

Very recently, Candan [17], Zhang et al. [104], Bartuek et al. [12], Tian et al. [93], Elabbasy et al. [25], Fu et al. [27], Jiang et al. [58], and Wang et al. [95] are investigated oscillation and asymptotic behavior of solutions of NDE's with distributed delay of third order.

Till necessarily very few results has been initiated with regard to oscillation and asymptotic behavior of equation (6.1.1) and (6.1.2) with distributed delay. By using generalized Riccati transformation and integral averaging technique, this chapter presents some sufficient conditions which guarantees that every solution of (6.1.1) and (6.1.2) oscillates or converges to zero.

6.2 Nonlinear Neutral Equation with Continuously Distributed Delay I

In this section, we establish several sufficient conditions for oscillation of solutions to equation (6.1.1). For convenience, we use the following notations

$$q_*(t)=(1-P)^\gamma\int_c^d q(t,\mu)\,d\mu,\quad \phi'_+(t)=\max\{0,\phi'(t)\},$$
$$\sigma_1(t)=\sigma(t,c),\quad \Phi(t)=\left(\frac{\int_{t_2}^{\sigma_1(t)}\left(\int_{t_1}^{s}r_1^{-1/\gamma}(u)du/r_2(s)\right)ds}{\int_{t_1}^{\sigma_1(t)}r_1^{-1/\gamma}(u)du}\right)^\gamma$$
$$\psi(t)=\phi(s)q_*(s)\Phi(s)-\frac{1}{(1+\gamma)^{1+\gamma}}\frac{r_1(s)(\phi'_+(s))^{1+\gamma}}{\phi^\gamma(s)}. \tag{6.2.1}$$

Theorem 6.2.1. *Assume $(H_1)-(H_3)$ holds. If there exists a positive function $\phi\in C^1([t_0,\infty),\mathbb{R})$, such that for all sufficiently large $t_i>t_1\ge t_0$ $(i=2,3,4)$, we have*

$$\limsup_{t\to\infty}\int_{t_3}^{t}\psi(s)\,ds=\infty, \tag{6.2.2}$$

and

$$\int_{t_4}^{\infty}\frac{1}{r_2(v)}\int_{v}^{\infty}\left(\frac{1}{r_1(u)}\int_{u}^{\infty}\int_{c}^{d}q(s,\mu)d\mu\,ds\right)^{1/\gamma}du\,dv=\infty, \qquad (6.2.3)$$

where $\psi(t)$ *is defined in* (6.2.1)*, then every solution* $x(t)$ *of* (6.1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$*.*

Proof. Assume, for sake of contradiction, that equation (6.1.1) has an eventually positive solution $x(t)$. That is $x(t)>0$, $x(\tau(t,\mu))>0$ and $x(\sigma(t,\mu))>0$ for $t\geq t_1$ some $t_1\geq t_0$, by definition of $N(t)$. By condition (6.1.3), there exist two possible cases:

(i) $N(t)>0$, $N'(t)>0$, $(r_2(t)N'(t))'>0$, $\left(r_1(t)[(r_2(t)N'(t))']^{\gamma}\right)'<0$,

(ii) $N(t)>0$, $N'(t)<0$, $(r_2(t)N'(t))'>0$, $\left(r_1(t)[(r_2(t)N'(t))']^{\gamma}\right)'<0$, for $t\geq t_1$, t_1 is large enough.

Suppose, Case (i) holds for $t\geq t_2$. From the definition of $N(t)$, $N(t)\geq x(t)$ for $t\geq t_2$ and

$$\begin{aligned}
x(t) &= N(t)-\int_a^b p(t,\mu)x(\tau(t,\mu))d\mu\\
&\geq N(t)-\int_a^b p(t,\mu)N(\tau(t,\mu))d\mu\\
&\geq N(t)-N(\tau(t,b))\int_a^b p(t,\mu)d\mu\\
&\geq N(t)\left(1-\int_a^b p(t,\mu)d\mu\right)=N(t)(1-P). \qquad (6.2.4)
\end{aligned}$$

Setting (6.2.4) into (6.1.1), we get

$$\begin{aligned}
\left(r_1(t)\left[\left(r_2(t)N'(t)\right)'\right]^{\gamma}\right)' &= -\int_c^d q(t,\mu)x^{\gamma}(\sigma(t,\mu))\,d\mu\\
&\leq -(1-p_0)^{\gamma}\int_c^d q(t,\mu)N^{\gamma}(\sigma(t,\mu))\,d\mu\\
&\leq -(1-p_0)^{\gamma}N^{\gamma}(\sigma(t,c))\int_c^d q(t,\mu)\,d\mu\\
&= -q_*(t)N^{\gamma}(\sigma_1(t)). \qquad (6.2.5)
\end{aligned}$$

Using the fact that $N'(t)>0$, we have

$$r_2(t)N'(t)\geq\int_{t_1}^{t}\frac{r_1^{1/\gamma}(s)(r_2(s)N'(s))'}{r_1^{1/\gamma}(s)}ds\geq r_1^{1/\gamma}(t)(r_2(t)N'(t))'\int_{t_1}^{t}\frac{1}{r_1^{1/\gamma}(s)}ds.$$

Thus

$$N(t) = N(t_2) + \int_{t_2}^{t} \frac{r_2(s)N'(s)}{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du} \frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)} ds$$
$$\geq \frac{r_2(t)N'(t)}{\int_{t_1}^{t} r_1^{-1/\gamma}(u)du} \int_{t_2}^{t} \frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)} ds.$$

Then, we get

$$\frac{N(\sigma_1(t))}{r_2(\sigma_1(t))N'(\sigma_1(t))} \geq \frac{\int_{t_2}^{\sigma_1(t)} \left(\frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)} \right) ds}{\int_{t_1}^{\sigma_1(t)} r_1^{-1/\gamma}(u)du}, \tag{6.2.6}$$

and

$$\frac{r_2(\sigma_1(t))N'(\sigma_1(t))}{r_2(t)N'(t)} \geq \frac{\int_{t_1}^{\sigma_1(t)} r_1^{-1/\gamma}(u)du}{\int_{t_1}^{t} r_1^{-1/\gamma}(u)du}. \tag{6.2.7}$$

Define

$$W(t) := \phi(t)r_1(t) \left[\frac{(r_2(t)N'(t))'}{r_2(t)N'(t)} \right]^{\gamma}, \tag{6.2.8}$$

and $W(t) > 0$ for $t \geq t_1$. Differentiating (6.2.8), we obtain

$$W'(t) = \frac{\phi'(t)}{\phi(t)} W(t) + \phi(t) \frac{\left(r_1(t)[(r_2(t)N'(t))']^{\gamma} \right)'}{[r_2(t)N'(t)]^{\gamma}}$$
$$-\gamma\phi(t)r_1(t) \left[\frac{(r_2(t)N'(t))'}{r_2(t)N'(t)} \right]^{\gamma+1}. \tag{6.2.9}$$

By (6.2.8), we get

$$\left[\frac{W(t)}{\phi(t)r_1(t)} \right]^{(\gamma+1)/\gamma} = \left[\frac{(r_2(t)N'(t))'}{r_2(t)N'(t)} \right]^{\gamma+1}, \tag{6.2.10}$$

By (6.2.5), (6.2.10), and (6.2.9) that

$$W'(t) \leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t)q_*(t) \left(\frac{N(\sigma(t))}{r_2(t)N'(t)} \right)^{\gamma} - \gamma \frac{W^{\frac{(\gamma+1)}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}}$$
$$= \frac{\phi'(t)}{\phi(t)} W(t) - \gamma \frac{W^{\frac{(\gamma+1)}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}}$$
$$-\phi(t)q_*(t) \left(\frac{N(\sigma_1(t))}{r_2(\sigma_1(t))N'(\sigma_1(t))} \frac{r_2(\sigma_1(t))N'(\sigma_1(t))}{r_2(t)N'(t)} \right)^{\gamma} \tag{6.2.11}$$

Taking (6.2.6), (6.2.7) and (6.2.11), into account

$$\begin{aligned} W'(t) &\le \frac{\phi'_+(t)}{\phi(t)}W(t) - \gamma\frac{W^{\frac{(\gamma+1)}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}} \\ &\qquad -\phi(t)q_*(t)\left(\frac{\int_{t_2}^{\sigma_1(t)}\left(\frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)}\right)ds}{\int_{t_1}^{\sigma_1(t)} r_1^{-1/\gamma}(u)du}\right)^{\gamma} \\ &\le -\phi(t)q_*(t)\Phi(t) + \frac{\phi'_+(t)}{\phi(t)}W(t) - \gamma\frac{W^{\frac{\gamma+1}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}}. \end{aligned} \tag{6.2.12}$$

Then, using (6.2.12) and inequality

$$Bu - Au^{(\gamma+1)/\gamma} \le \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}, \tag{6.2.13}$$

where $u = W(t)$, $A = \frac{\gamma}{[\phi(t)r_1(t)]^{1/\gamma}}$, $B = \frac{\phi'_+(t)}{\phi(t)}$. We find that

$$W'(t) \le -\phi(t)q_*(t)\Phi(t) + \frac{1}{(1+\gamma)^{1+\gamma}}\frac{r_1(t)(\phi'_+(t))^{1+\gamma}}{\phi^{\gamma}(t)}. \tag{6.2.14}$$

Integrating (6.2.14) from t_3 $(> t_2)$ to t gives

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(\phi(s)q_*(s)\Phi(s) - \frac{1}{(1+\gamma)^{1+\gamma}}\frac{r_1(s)(\phi'_+(s))^{1+\gamma}}{\phi^{\gamma}(s)}\right)ds \le W(t_3), \tag{6.2.15}$$

which contradicts (6.2.2).

Suppose Case (ii) holds. Since $N(t) > 0$ and $N'(t) < 0$, we have $N(t) \to L \ge 0$. If $L > 0$, then for $\epsilon = \frac{L(1-P)}{2P} > 0$, there exists $t_4 \ge t_1$ such that $L < N(t) < L + \epsilon$ for $t \ge t_4$. Then for $t \ge t_4$, we have

$$\begin{aligned} x(t) &= N(t) - \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu > L - \int_a^b p(t,\mu)N(\tau(t,\mu))d\mu \\ &\ge L - PN(\tau(t,a)) \ge L - P(L+\epsilon) > kN(t) \end{aligned} \tag{6.2.16}$$

where $k = \frac{L-P(L+\epsilon)}{L+\epsilon}$ and $\sigma_0(t) = \sigma(t,d)$. Using (6.2.16), we obtained from (6.1.1) , we have

$$\Big(r_1(t)\big(r_2(t)N'(t)\big)'\Big)' = -k^{\gamma}\int_c^d q(t,\mu)N(\sigma(t,\mu))\,d\mu = -q^*(t)N(\sigma_0(t)),$$

where $q^*(t) = k^{\gamma}\int_c^d q(t,\mu)d\mu$. Integrating from $t(\ge t_4)$ to ∞ and $r_1(t)\big(r_2(t)N'(t)\big)' \ge 0$ is decreasing, we get

$$\big(r_2(t)N'(t)\big)' \ge \left(\frac{1}{r_1(t)}\int_t^{\infty} q^*(s)N^{\gamma}(\sigma_0(s))ds\right)^{1/\gamma}.$$

Using $N(\sigma_0(t)) \geq L$, we obtain

$$\left(r_2(t)N'(t)\right)' \geq L\left(\frac{1}{r_1(t)}\int_t^\infty q^*(s)ds\right)^{1/\gamma}.$$

Again integrating

$$r_2(t)N'(t) \geq -L\int_v^\infty \left(\frac{1}{r_1(u)}\int_u^\infty q^*(s)\,ds\right)^{1/\gamma} du,$$

and finally integration from t_4 to ∞, we get

$$N(t_4) \geq L\int_{t_4}^\infty \frac{1}{r_2(v)}\int_v^\infty \left(\frac{1}{r_1(u)}\int_u^\infty q^*(s)ds\right)^{1/\gamma} du\,dv.$$

This contradicts to (6.2.3) and hence $L = 0$. □

Remark 6.2.2. *Suitable choice of $\phi(t)$, by Theorem 6.2.1 gives a various asymptotic criteria for* (6.1.1).

Corollary 6.2.3. *Assume $(H_1) - (H_3)$ and* (6.2.3) *holds. If $\phi = 1$, such that for all sufficiently large $t_3 > t_1 \geq t_0$, we have*

$$\limsup_{t\to\infty}\int_{t_3}^t q_*(s)\Phi(s)\,ds = \infty, \tag{6.2.17}$$

then all solution $x(t)$ of (6.1.1) *is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Next, examine the oscillation results of solutions of (6.1.1) by Philos-type.
Let $\mathbb{S}_0 = \{(t,s) : a \leq s < t < +\infty\}$, $\mathbb{S} = \{(t,s) : a \leq s \leq t < +\infty\}$ the continuous function $E(t,s)$, $E : \mathbb{S} \to \mathbb{R}$ belongs to the class function $\Re$,

(i) $E(t,t) = 0$ for $t \geq t_0$ and $E(t,s) > 0$ for $(t,s) \in \mathbb{S}_0$,

(ii) $\frac{\partial E(t,s)}{\partial s} \leq 0$, $(t,s) \in \mathbb{S}_0$ and some locally integrable function e such that

$$\frac{\partial E(t,s)}{\partial s} + \frac{\phi'(t)}{\phi(t)}E(t,s) = -e(t,s)(E(t,s))^{\frac{\gamma}{1+\gamma}} \quad \text{for all } (t,s) \in \mathbb{S}_0.$$

Theorem 6.2.4. *Assume that* (6.2.3) *holds. If there exists a positive function $E \in \Re$ and $\phi \in C^1([t_0,\infty),\mathbb{R})$, such that for all sufficiently large $t_5 > t_1 \geq t_0$, we have*

$$\limsup_{t\to\infty}\frac{1}{E(t,t_5)}\int_{t_5}^t \left[E(t,s)\phi(s)q_*(s)\Phi(s) - \frac{r_1(s)\phi(s)|e(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}E^\gamma(t,s)}\right] ds = \infty, \tag{6.2.18}$$

then all solution $x(t)$ of (6.1.1) *is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. Suppose that $x(t)$ is a positive solution of (6.1.1). Then by the proof of Theorem 6.2.1, we have Cases (i) and (ii). Let Case (i) hold; $W(t)$ is defined as in (6.2.8). Then, we get

$$W'(t) \leq -\phi(t)q_*(t)\Phi(t) + \frac{\phi'(t)}{\phi(t)}W(t) - \gamma\frac{W^{\frac{\gamma+1}{\gamma}}(t)}{[\phi(t)r_1(t)]}. \tag{6.2.19}$$

Take $X(t) = \frac{\phi'(t)}{\phi(t)}$, $Y(t) = \gamma\left(\frac{1}{\phi(t)r_1(t)}\right)^{1/\gamma}$, we have

$$\phi(t)q_*(t)\Phi(t) \leq -W'(t) + X(t)W(t) - Y(t)W^{\frac{\gamma+1}{\gamma}}. \tag{6.2.20}$$

Multiplying (6.2.20) by $E(t,s)$ and integrating from t_5 to t, with $T \geq t_1$, we have

$$\begin{aligned}
&\int_{t_5}^{t} E(t,s)\phi(s)q_*(s)\Phi(s)ds \\
&\quad\leq \int_{t_5}^{t} E(t,s)\left[-W'(s) + X(s)w(s) - B(s)W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\quad= E(t,t_5)W(t_5) + \int_{t_5}^{t}\left[\frac{\partial E(t,s)}{\partial s} + E(t,s)X(s)\right]W(s)ds \\
&\qquad\quad - \int_{t_5}^{t}\left[E(t,s)Y(s)W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\quad\leq E(t,t_5)W(t_5) + \int_{t_5}^{t}\left[|e(t,s)|W(s) - E(t,s)Y(s)W^{\frac{\gamma+1}{\gamma}}(s)\right]ds.
\end{aligned}$$

Then, using (6.2.12) and inequality, we obtain

$$\int_{t_5}^{t} E(t,s)\phi(s)q_*(s)\Phi(s)ds \leq$$

$$E(t,t_5)W(t_5) + \int_{t_5}^{t}\frac{r_1(s)\phi(s)|e(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}E^{\gamma}(t,s)}ds. \tag{6.2.21}$$

Therefore, we have

$$\frac{1}{E(t,t_5)}\int_{t_5}^{t}\left[E(t,s)\phi(s)q_*(s)\Phi(s) - \frac{r_1(s)\phi(s)|e(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}E^{\gamma}(t,s)}\right]ds \leq W(t_5), \tag{6.2.22}$$

which contradicts (6.2.18).

Next, Assume that Case (ii) holds, we get $\lim_{t\to\infty} x(t) = 0$. □

Next, Based on Theorem 6.2.1, we present a Kamenev-type criterion for (6.1.1).

Theorem 6.2.5. *Assume that* (6.2.3) *holds. If there exists a positive function* $\phi \in C^1([t_0,\infty),\mathbb{R})$*, such that for sufficiently large* $t_6 > t_0$*, we have*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_6}^{t}(t-s)^n\psi(s)ds = \infty, \tag{6.2.23}$$

where $\psi(t)$ *is defined in* (6.2.1)*, then all solution* $x(t)$ *of* (6.1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume, for sake of contradiction, that equation (6.1.1) has an eventually positive solution $x(t)$. Then by the proof of Theorem 6.2.1, we have Cases (i) and (ii). Let Case (i) hold. By using the same arguments as in the proof of Theorem 6.2.1, we obtain (6.2.14), then

$$W'(t) \le -\phi(t)q_*(t)\Phi(t) + \frac{1}{(1+\gamma)^{1+\gamma}}\frac{r_1(t)(\phi'_+(t))^{1+\gamma}}{\phi^\gamma(t)}. \tag{6.2.24}$$

Multiplying by $(t-s)^n$ and integrating (6.2.24) from t_6 to t gives

$$\int_{t_6}^t (t-s)^n\,\psi(s)ds \;=\; -\int_{t_1}^t (t-s)^n W'(s)ds. \tag{6.2.25}$$

We get

$$\begin{aligned}\frac{1}{t^n}\int_{t_6}^t (t-s)^n\,\psi(s)ds &= -\frac{1}{t^n}\int_{t_6}^t (t-s)^n W'(s)ds\\ &\le -\frac{n}{t^n}\int_{t_6}^t (t-s)^{n-1}W(s)ds + \big(1-\frac{t_6}{t}\big)^n W(t_1)\\ &< \big(1-\frac{t_6}{t}\big)^n W(t_1) < \infty, \end{aligned}\tag{6.2.26}$$

which contradicts (6.2.24).
Next, Assume that Case (ii) holds, we get $\lim_{t\to\infty} x(t) = 0$. □

6.3 Nonlinear Neutral Equation with Continuously Distributed Delay II

This section is concerned with the oscillatory and asymptotic behavior of third-order NDE with continuously distributed delay when $\gamma = 1$. Additionally, we assume that,

(H$_4$) $f \in C(\mathbb{R},\mathbb{R})$ satisfies $uf(u) > 0$ for $u \ne 0$ and there exists a constant $\delta > 0$ such that $\frac{f(u)}{u} \ge \delta$.

Further references, we introduce the notations

$$q_*(t) = \delta(1-P)\int_c^d q(t,\mu)\,d\mu, \quad \rho'_+(t) = \min\{0,\rho'(t)\},$$

$$\Phi(t) = \frac{\int_{t_2}^{\sigma_1(s)} \left(\frac{\int_{t_1}^{v} \frac{1}{r_1(u)} du}{r_2(v)} \right) dv}{\int_{t_1}^{\sigma_1(t)} \frac{1}{r_1(u)} du}, \quad \sigma_1(t) = \sigma(t, c).$$

Theorem 6.3.1. *Let $\gamma = 1$ and $(H_1) - (H_4)$ holds. If there exists a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$, such that for all sufficiently large $t_i > t_1 \geq t_0$ $(i = 2, 3, 4)$, we have*

$$\limsup_{t\to\infty} \int_{t_3}^{t} \left(\rho(s) q_*(s) \Phi(s) - \frac{r_1(s)(\rho'(s))_+^2}{4\rho(s)} \right) ds = \infty, \tag{6.3.1}$$

and

$$\int_{t_4}^{\infty} \frac{1}{r_2(v)} \int_{v}^{\infty} \left[\frac{1}{r_1(u)} \int_{u}^{\infty} \int_{c}^{d} q(s, \mu) d\mu \, ds \right] du \, dv = \infty, \tag{6.3.2}$$

then every solution $x(t)$ of (6.1.2) is either oscillatory or converges to zero asymptotically.

Corollary 6.3.2. *Let $\gamma = 1$, $(H_1) - (H_4)$ holds.and (6.3.2) holds. If $\rho = 1$, such that for all sufficiently large $t_3 > t_2 > t_1 \geq t_0$, we have*

$$\limsup_{t\to\infty} \int_{t_3}^{t} q_*(s) \Phi(s) ds = \infty, \tag{6.3.3}$$

then any solution $x(t)$ of (6.1.2) is either oscillatory or satisfies $x(t) \to 0$ as $t \to \infty$.

Theorem 6.3.3. *Let $\gamma = 1$, $(H_1) - (H_4)$ and (6.3.2) holds. If there exists a positive function $H \in \Re$ and $\rho \in C^1([t_0, \infty), \mathbb{R})$, such that for all sufficiently large $T \geq t_1 \geq t_0$, we have*

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[H(t, s) \rho(s) q_*(s) \Phi(s) - \frac{h_*^2(t, s)}{4 r_1(t) \rho(t)} \right] ds = \infty, \tag{6.3.4}$$

where

$$h_*(t, s) = h(t, s) - \frac{\rho'(t)}{\rho(t)} \sqrt{H(t, s)},$$

then every solution $x(t)$ of (6.1.2) is either oscillatory or converges to zero asymptotically.

Theorem 6.3.4. *Let $\gamma = 1$, $(H_1) - (H_4)$ and (6.3.2) holds. If there exists a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$, such that for all sufficiently large $t_1 \geq t_0$, we have*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_1}^{t} (t - s)^n \left(\rho(s) q_*(t) \Phi(s) - \frac{r_1(s)(\rho'(s))^2}{4\rho(s)} \right) ds = \infty, \tag{6.3.5}$$

then every solution $x(t)$ of (6.1.2) is either oscillatory or converges to zero asymptotically.

6.4 Examples

We provide the following examples to illustrate the main results.

Example 6.4.1. *For $t \geq 1$, consider the 3rd order differential equation*

$$\Big[t\Big[t^{-1}\Big[x(t)+\frac{1}{2}\int_{-1}^{0}x(\frac{t+\xi}{3})d\xi\Big]''\Big]^3\Big]'+\int_0^1\frac{\xi}{t^3}x^3(\frac{t+\xi}{2})d\xi=0, \quad (6.4.1)$$

where $\gamma=3$, $r_1(t)=t$, $r_2(t)=t^{-1}$, $\tau(t,\mu)=(t+\mu)/3$, $\sigma(t,\mu)=(t+\mu)/2$, $a=-1$, $b=0$, $c=0$, $d=1$, then we obtain $\int_c^d q(t,\mu)\,d\mu=1/2=P$, $\sigma_1(t)=\sigma(t,c)=t/2$. Choose $\phi(t)=t$, easily verified that the conditions (6.2.2), (6.2.3) *of Theorem 6.2.1 are satisfied. Since all solutions of* (6.4.1) *is oscillatory or $\lim_{t\to\infty}x(t)=0$.*

Example 6.4.2. *Consider the 3rd order differential equation*

$$\Big[x(t)+\int_{1/2}^{1}\frac{1}{2}x(\frac{t+\xi}{3})\Big]'''+\int_1^2\frac{\xi}{t^2}x(\frac{t+\xi}{2})d\xi=0, \quad t\geq 1, \quad (6.4.2)$$

where $\gamma=1$, $r_1(t)=r_2(t)=1$, $\tau(t,\mu)=(t+\mu)/3$, $\sigma(t,\mu)=(t+\mu)/2$, $a=1/2$, $b=1$, $c=1$, $d=2$, then we obtain $\int_c^d q(t,\mu)\,d\mu=1/2=P$, $\sigma_1(t)=\sigma(t,c)=(t+1)/2$. Pick $\phi(t)=1$, it is not difficult to verified that the conditions of Theorem 6.2.5. Since all solutions of (6.4.2) *is oscillatory or $\lim_{t\to\infty}x(t)=0$.*

Example 6.4.3. *For $t\geq 1$, consider the 3rd order differential equation*

$$\left(x(t)+\frac{1}{2}\int_{-1}^{0}x\left(\frac{t+\mu}{3}\right)d\mu\right)'''+\frac{1}{4}\int_0^1 x\left(\frac{t+\mu}{2}\right)d\mu=0, \quad (6.4.3)$$

where $r_1(t)=r_2(t)=1$, $\tau(t,\mu)=(t+\mu)/3$, $\sigma(t,\mu)=(t+\mu)/2$, $a=-1$, $b=0$, $c=0$, $d=1$, then we obtain $\int_c^d q(t,\mu)\,d\mu=1/2=P$, $\sigma_1(t)=\sigma(t,c)=t/2$, $q_*(t)=(1-P)\int_0^1 1/4\,d\mu=1/2$. Choose $\rho(t)=1$, then

$$\Phi(t)=\frac{\int_{t_2}^{t/2}\left(\frac{\int_{t_1}^{v}\frac{1}{r_1(u)}du}{r_2(v)}\right)dv}{\int_{t_1}^{t/2}\frac{1}{r_1(u)}du}=\frac{(t^2-4tt_1-4t_2^2+8t_1t_2)}{4(t-t_1)}.$$

Therefore,

$$\limsup_{t\to\infty}\int_{t_3}^{t}q_*(s)\Phi(s)ds=\limsup_{t\to\infty}\frac{\delta}{8}\int_{t_3}^{t}\frac{(s^2-4st_1-4t_2^2+8t_1t_2)}{(s-t_1)}ds=\infty,$$

and

$$\int_{t_4}^{\infty}\int_{v}^{\infty}\int_{u}^{\infty}1/2\,du\,dv=\infty.$$

Thus, by Theorem 6.3.1, any solution of (6.4.3) is oscillatory, or satisfies $x(t)\to 0$ as $t\to\infty$.

6.5 Conclusion

In this chapter, we are using generalized Riccati transformation, Philos and Kamenev-type criteria to established three new oscillation and asymptotic theorems for (6.1.1) and (6.1.2) in the case of (6.1.3). Our result improves and complements results in the cited papers. This results easily extended to the corresponding dynamic equations on time scales.