From 3.3.58, one gets $N(t+k_1) \geq -(L_{k_1}^{[3]}N(t))^{1/\lambda}B(t+k_1)$ and using in (3.3.57), we have

$$\left(L^{[3]}N(t) + \mu_1^\lambda\, L_{-m_1}^{[3]}N(t) + \mu_2^\lambda\, L_{m_2}^{[3]}N(t)\right)' - \frac{P_2(t)}{3^{\lambda-1}}\, L_{k_1}^{[3]}N(t)B^\lambda(t+k_1) \leq 0. \quad (3.3.59)$$

Now, set

$$\psi(t) = L^{[3]}N(t) + \mu_1^\lambda\, L_{-m_1}^{[3]}N(t) + \mu_2^\lambda\, L_{m_2}^{[3]}N(t).$$

Then $\psi(t) > 0$, $\psi'(t) \geq 0$ and the fact that $L^{[3]}N(t)$ is nondecreasing, we have

$$\psi(t) \leq L_{m_2}^{[3]}N(t)\left(1 + \mu_1^\lambda + \mu_2^\lambda\right). \quad (3.3.60)$$

Using (3.3.60) and (3.3.59), we see that $\psi(t)$ is a nonincreasing positive solution of the first order differential inequality

$$\psi'(t) - \frac{P_2(t)}{3^{\lambda-1}}\frac{B^\lambda(t+k_1)}{1+\mu_1^\lambda+\mu_2^\lambda}\psi(t-m_2+k_1) \geq 0, \quad (3.3.61)$$

which is contradiction to (3.3.52). □

Corollary 3.3.6. *Let* (3.1.4), (3.2.6) *hold and* $k_1 > m_1$, $k_1 > m_2$. *If*

$$\liminf_{t\to\infty} \int_{t-k_1+m_1}^{t} P_1(s)A^\lambda(s-k_1)\,ds > \frac{3^{\lambda-1}}{e\,(1+\mu_1^\lambda+\mu_2^\lambda)} \quad (3.3.62)$$

and

$$\liminf_{t\to\infty} \int_{t-m_2+k_1}^{t} P_2(s)B^\lambda(s+k_1)\,ds > \frac{3^{\lambda-1}}{e\,(1+\mu_1^\lambda+\mu_2^\lambda)} \quad (3.3.63)$$

holds, then Eq. (3.1.1) *oscillatory.*

Proof. The proof follows from Theorem 3.3.5 and [[66], Theorem 2.1.1], and the details are omitted. □

Example 3.3.7. *Consider the third order differential equation*

$$\left(\left(\left(\left(y(t) + (e^{-2}/4)\,y(t-2) + (e^{1}/4)y(t+1)\right)'\right)'\right)^{5/3}\right)'$$
$$+\frac{5e^{-5}}{6}\left(\frac{3}{2}\right)^{5/3} y^{5/3}(t-3) + \frac{5e^{5}}{6}\left(\frac{3}{2}\right)^{5/3} y^{5/3}(t+3) = 0. \quad (3.3.64)$$

Compared with (3.1.1), *we can see that* $a_1(t) = a_2(t) = 1$, $p_1(t) = \frac{e^{-2}}{4}$, $p_2(t) = \frac{e^{1}}{4}$, $q_1(t) = \frac{5e^{-5}}{6}\left(\frac{3}{2}\right)^{5/3}$, $q_2(t) = \frac{5e^{5}}{6}\left(\frac{3}{2}\right)^{5/3}$, $\lambda = 5/3$, $m_1 = 2$, $m_2 = 1$ *and* $k_1 = 3$. *By taking* $m(t) = 1$, $H(t,s) = (t-s)^2$, *it is easy to verify that all conditions of Theorem 3.3.3 are satisfied. Therefore, all the solutions of* (3.3.64) *is either oscillates or tends to* 0 *and* $y(t) = e^{-t}$ *is a such solution of* (3.3.64).

Example 3.3.8. *Consider the third order differential equation*

$$\Big[t^2\big(y(t)+k_1y(t-m_1)+k_2y(t+m_2)\big)''\Big]'+k_3t\,y(t-k_1)+k_4y(t+k_1)=0. \quad (3.3.65)$$

Compared with (3.1.1)*, we can see that* $a_1(t)=1$*,* $a_2(t)=t^2$*,* $p_1(t)=k_1$*,* $p_2(t)=k_2$*,* $q_1(t)=k_3t$*,* $q_2(t)=k_4$*,* $\lambda=1$ *and* k_1*,* k_2*,* k_3*,* k_4 *are nonnegative constants. It is easy to verify that all conditions of Corollary 3.3.6 are satisfied and hence all solutions of equation* (3.3.65) *are oscillatory.*

3.4 Mixed Neutral Equations with Continuous Delay

In this section, we will establish several oscillation criteria for (3.1.2). For convenience, we define,

$$\begin{aligned}
Q_1(t,\xi)&=\min\{\tilde{q}_1(t,\xi),\,\tilde{q}_1(t-m_1,\xi),\,\tilde{q}_1(t+m_2,\xi)\},\\
Q_2(t,\xi)&=\min\{\tilde{q}_2(t,\xi),\,\tilde{q}_2(t-m_1,\xi),\,\tilde{q}_2(t+m_2,\xi)\},\\
Q(t,\xi)&=Q_1(t,\xi)+Q_2(t,\xi).
\end{aligned}$$

Theorem 3.4.1. *Let* (3.1.3) *hold and* $c+d\geq 0$*,* $d\geq m_1$*. If there exists an* $m(t)\in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.7) *and*

$$\int_{t_3}^{\infty}\Bigg[m(s)\frac{\int_c^d Q(s,\xi)d\xi}{3^{\lambda-1}} -\Big(1+\mu_1^\lambda+\mu_2^\lambda\Big)\left(\frac{(m'(s))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(s-d)}{m(s)\pi_1[t_0,s-d]}\right)^{\lambda}\Bigg]ds=\infty, \quad (3.4.1)$$

then every solution $y(t)$ *of* (3.1.2) *is either oscillatory or tends to* 0*.*

Proof. Suppose that (3.1.2) has a nonoscillatory solution y. Without loss of generality, we may take $y(t)>0$, $y(t-m_1)>0$, $y(t+m_2)>0$, $y(t-\xi)>0$ and $y(t+\xi)>0$ for $t\geq t_1\geq t_0$ and $\xi\in[c,d]$. Since $y(t)>0$ for all $t\geq t_1$, in view of (3.1.2), we have

$$L^{[4]}N(t)=-\int_c^d\tilde{q}_1(t,\xi)y^\lambda(t-\xi)\,d\xi-\int_c^d\tilde{q}_2(t,\xi)y^\lambda(t+\xi)\,d\xi\leq 0. \quad (3.4.2)$$

Assumption of (3.1.3), by Lemma 3.2.4 there exists two cases (C_1) and (C_2). If (C_2)

holds, then by Lemma 3.2.7, $\lim_{t\to\infty} N(t) = 0$. If (C_1) holds.

$$\begin{aligned}
L^{[4]}N(t) &+ \int_c^d \tilde{q}_1(t,\xi)y^\lambda(t-\xi)\,d\xi + \int_c^d \tilde{q}_2(t,\xi)y^\lambda(t+\xi)\,d\xi \\
&+\mu_1^\lambda L_{-m_1}^{[4]}N(t) + \mu_1^\lambda \int_c^d \tilde{q}_1(t-m_1,\xi)y^\lambda(t-m_1-\xi)\,d\xi \\
&+\mu_1^\lambda \int_c^d \tilde{q}_2(t-m_1,\xi)y^\lambda(t-m_1+\xi)\,d\xi + \mu_2^\lambda L_{m_2}^{[4]}N(t) \\
&+\mu_2^\lambda \int_c^d \tilde{q}_1(t+m_2,\xi)y^\lambda(t+m_2-\xi)\,d\xi \\
&+\mu_2^\lambda \int_c^d \tilde{q}_2(t+m_2,\xi)y^\lambda(t+m_2+\xi)\,d\xi = 0. \qquad (3.4.3)
\end{aligned}$$

Furthermore, from Lemma 3.2.1, we have

$$\begin{aligned}
\tilde{q}_1(t,\xi)\,y^\lambda(t-\xi) &+ \mu_1^\lambda\,\tilde{q}_1(t-m_1,\xi)\,y^\lambda(t-m_1-\xi) \\
&+\mu_1^\lambda\,\tilde{q}_1(t+m_2,\xi)\,y^\lambda(t+m_2-\xi) \geq \frac{Q_1(t,\xi)}{3^{\lambda-1}}\,N^\lambda(t-\xi). \qquad (3.4.4)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\tilde{q}_2(t,\xi)\,y^\lambda(t+\xi) &+ \mu_2^\lambda\,\tilde{q}_2(t-m_1,\xi)\,y^\lambda(t-m_1+\xi) \\
&+\mu_2^\lambda\,\tilde{q}_2(t+m_2,\xi)\,y^\lambda(t+m_2+\xi) \geq \frac{Q_2(t,\xi)}{3^{\lambda-1}}\,N^\lambda(t+\xi). \qquad (3.4.5)
\end{aligned}$$

Substituting (3.4.4), (3.4.5) into (3.4.3), we have

$$\begin{aligned}
L^{[4]}N(t) + \mu_1^\lambda\,L_{-m_1}^{[4]}N(t) + \mu_2^\lambda\,L_{m_2}^{[4]}N(t) &+ \frac{\int_c^d Q_1(t,\xi)d\xi}{3^{\lambda-1}}\,N^\lambda(t-\xi) \\
&+\frac{\int_c^d Q_2(t,\xi)d\xi}{3^{\lambda-1}}\,N^\lambda(t+\xi) \leq 0. \qquad (3.4.6)
\end{aligned}$$

Using the fact of $L^{[1]}N(t) > 0$ and $c + d \geq 0$, we have

$$L^{[4]}N(t) + \mu_1^\lambda\,L_{-m_1}^{[4]}N(t) + \mu_2^\lambda\,L_{m_2}^{[4]}N(t) + \frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}}\,N^\lambda(t-d) \leq 0. \qquad (3.4.7)$$

Define a function

$$w_1(t) = m(t)\frac{L^{[3]}N(t)}{N^\lambda(t-d)}. \qquad (3.4.8)$$

We obtain $w_1(t) > 0$, then

$$w_1'(t) = m'(t)\frac{L^{[3]}N(t)}{N^\lambda(t-d)} + m(t)\frac{L^{[4]}N(t)}{N^\lambda(t-d)} - \lambda m(t)\frac{L^{[3]}N(t)N'(t-d)}{N^{\lambda+1}(t-d)}. \qquad (3.4.9)$$

By Lemma 3.2.5, one gets $N'(t-d) \geq \frac{a_2^{1/\lambda}(t)}{a_1(t-d)}\pi_1[t_0,t-d]\, L^{[2]}N(t)$. Therefore

$$w_1'(t) \leq m'(t)\frac{L^{[3]}N(t)}{N^\lambda(t-d)}+m(t)\frac{L^{[4]}N(t)}{N^\lambda(t-d)}-\lambda m(t)\frac{a_2^{\frac{\lambda+1}{\lambda}}(t)\pi_1[t_0,t-d]L^{[2]}N(t)N'(t-d)}{N^{\lambda+1}(t-d)a_1(t-d)}. \tag{3.4.10}$$

Using (3.4.8) in (3.4.10), we have

$$w_1'(t) \leq \frac{(m'(t))_+}{m(t)}w_1(t)+m(t)\frac{L^{[4]}N(t)}{N^\lambda(t-d)}-\lambda\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}. \tag{3.4.11}$$

Next, define

$$w_2(t) = m(t)\frac{L^{[3]}_{-m_1}N(t)}{N^\lambda(t-d)}. \tag{3.4.12}$$

We obtain $w_2(t)>0$, then

$$w_2'(t) = m'(t)\frac{L^{[3]}_{-m_1}N(t)}{N^\lambda(t-d)}+m(t)\frac{L^{[4]}_{-m_1}N(t)}{N^\lambda(t-d)}-\lambda m(t)\frac{L^{[3]}_{-m_1}N(t)N'(t-d)}{N^{\lambda+1}(t-d)}. \tag{3.4.13}$$

By Lemma 3.2.5, one gets $N'(t-d) \geq \frac{a_2^{1/\lambda}(t-m_1)}{a_1(t-d)}\pi_1[t_0,t-d]\, L^{[2]}_{-m_1}N(t)$ and using (3.4.12) in (3.4.13), we have

$$w_2'(t) \leq \frac{(m'(t))_+}{m(t)}w_2(t)+m(t)\frac{L^{[4]}_{-m_1}N(t)}{N^\lambda(t-d)}-\lambda\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}. \tag{3.4.14}$$

Finally, define

$$w_3(t) = m(t)\frac{L^{[3]}_{m_2}N(t)}{N^\lambda(t-d)}. \tag{3.4.15}$$

We obtain $w_3(t)>0$, then

$$w_3'(t) = m'(t)\frac{L^{[3]}_{m_2}N(t)}{N^\lambda(t-d)}+m(t)\frac{L^{[4]}_{m_2}N(t)}{N^\lambda(t-d)}-\lambda m(t)\frac{L^{[3]}_{m_2}N(t)N'(t-d)}{N^{\lambda+1}(t-d)}. \tag{3.4.16}$$

By Lemma 3.2.5, one gets $N'(t-d) \geq \frac{a_2^{1/\lambda}(t+m_2)}{a_1(t-d)}\pi_1[t_0,t-d]\, L^{[2]}_{m_2}N(t)$ and using (3.4.15) in (3.4.16), we have

$$w_3'(t) \leq \frac{(m'(t))_+}{m(t)}w_3(t)+m(t)\frac{L^{[4]}_{m_2}N(t)}{N^\lambda(t-d)}-\lambda\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}. \tag{3.4.17}$$

From (3.4.8), (3.4.10) and (3.4.15), we have

$$\begin{aligned} w_1'(t)+\mu_1^\lambda w_2'(t)+\mu_2^\lambda w_3'(t) \leq{}& m(t)\left[\frac{L^{[4]}N(t)+\mu_1^\lambda L^{[4]}_{-m_1}N(t)+\mu_2^\lambda L^{[4]}_{m_2}N(t)}{N^\lambda(t-d)}\right] \\ &+\left[\frac{(m'(t))_+}{m(t)}w_1(t)-\lambda\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}\right] \\ &+\mu_1^\lambda\left[\frac{(m'(t))_+}{m(t)}w_2(t)-\lambda\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}\right] \\ &+\mu_2^\lambda\left[\frac{(m'(t))_+}{m(t)}w_3(t)-\lambda\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}\right]. \end{aligned} \tag{3.4.18}$$

Using (3.4.7) in (3.4.18), we have

$$
\begin{aligned}
w_1'(t) + \mu_1^\lambda w_2'(t) + \mu_2^\lambda w_3'(t) \le & -m(t)\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}} \\
& + \left[\frac{(m'(t))_+}{m(t)}w_1(t) - \lambda\,\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}\right] \\
& + \mu_1^\lambda\left[\frac{(m'(t))_+}{m(t)}w_2(t) - \lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}\right] \\
& + \mu_2^\lambda\left[\frac{(m'(t))_+}{m(t)}w_3(t) - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}\right]. \quad (3.4.19)
\end{aligned}
$$

Applying Lemma 3.2.3, we conclude that

$$
\frac{(m'(t))_+}{m(t)}w_i(t) - \frac{\lambda\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}(w_i(t))^{\frac{\lambda+1}{\lambda}} = \left(\frac{(m'(t))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(t-d)}{m(t)\pi_1[t_0,t-d]}\right)^{\lambda},
$$

for $i = 1,2,3$ and we conclude from (3.4.19), we have

$$
\begin{aligned}
w_1'(t) + \mu_1^\lambda w_2'(t) + \mu_2^\lambda w_3'(t) \le & -m(t)\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}} \\
& + \left(1+\mu_1^\lambda+\mu_2^\lambda\right)\left(\frac{(m'(t))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(t-d)}{m(t)\pi_1[t_0,t-d]}\right)^{\lambda}. \quad (3.4.20)
\end{aligned}
$$

Integrating (3.4.20) from t_3 to t, we obtain

$$
\begin{aligned}
\int_{t_3}^{t}\left[m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}} - \left(1+\mu_1^\lambda+\mu_2^\lambda\right)\left(\frac{(m'(s))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(s-d)}{m(s)\pi_1[t_0,s-d]}\right)^{\lambda}\right]ds \\
\le w_1'(t_3) + \mu_1^\lambda w_2'(t_3) + \mu_2^\lambda w_3'(t_3),
\end{aligned}
$$

we get a contradiction to (3.4.1). ☐

Theorem 3.4.2. *Let* (3.1.3) *holds and* $c + d \ge 0$, $-c \ge m_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.7) *and*

$$
\begin{aligned}
\int_{t_3}^{\infty}\Bigg[m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}} \\
-\left(1+\mu_1^\lambda+\mu_2^\lambda\right)\left(\frac{(m'(s))_+}{\lambda+1}\right)^{\lambda+1}\left(\frac{a_1(s+c)}{m(s)\pi_1[t_0,s+c]}\right)^{\lambda}\Bigg]ds = \infty,
\end{aligned}
\quad (3.4.21)
$$

then every solution $y(t)$ *of* (3.1.2) *is either oscillatory or tends to* 0.

Proof. Suppose that (3.1.2) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-m_1) > 0$, $y(t+m_2) > 0$, $y(t-\xi) > 0$ and $y(t+\xi) > 0$ for $t \geq t_1 \geq t_0$ and $\xi \in [c,d]$. Assumption of (3.1.3), by Lemma 3.2.4 there exists two cases (C_1) and (C_2). If (C_2) holds, then by Lemma 3.2.7, $\lim_{t\to\infty} N(t) = 0$. We only consider (C_1), by using the fact that $N'(t) > 0$ and $-c \geq m_1$, using the fact of $L^{[1]}N(t) > 0$, we obtain

$$L^{[4]}N(t) + \mu_1^\lambda L^{[4]}_{-m_1}N(t) + \mu_2^\lambda L^{[4]}_{m_2}N(t) + \frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}} N^\lambda(t+c) \leq 0. \quad (3.4.22)$$

Next, we categorize the functions as $w_1(t) = m(t)\frac{L^{[3]}N(t)}{N^\lambda(t+c)}$, $w_2(t) = m(t)\frac{L^{[3]}_{-m_1}N(t)}{N^\lambda(t+c)}$ and $w_3(t) = m(t)\frac{L^{[3]}_{m_2}N(t)}{N^\lambda(t+c)}$ respectively. The rest of the proof is similar to that of Theorem 3.4.2, therefore, it is omitted.

□

Theorem 3.4.3. *Let* (3.1.3) *holds and* $c + d \geq 0$, $b \geq m_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.7) *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_3)} \int_{t_3}^t \Bigg[H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}} - \frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}} \left(\frac{|h(t,s)|a_1(s-d)}{m(s)\pi_1[t_0,s-d]} \right)^\lambda \Bigg] ds = \infty, \quad (3.4.23)$$

then every solution $y(t)$ *of* (3.1.2) *is either oscillatory or tends to* 0.

Proof. Proceeding by the similar argument as in proof of Theorem 3.4.1, we obtain the inequality (3.4.19),

$$\begin{aligned} m(t)\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}} &\leq -w_1'(t) - \mu_1^\lambda w_2'(t) - \mu_2^\lambda w_3'(t) + \frac{(m'(t))_+}{m(t)} w_1(t) \\ &\quad -\lambda \frac{\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)} (w_1(t))^{\frac{\lambda+1}{\lambda}} \\ &\quad +\mu_1^\lambda \left[\frac{(m'(t))_+}{m(t)} w_2(t) - \lambda \frac{\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)} (w_2(t))^{\frac{\lambda+1}{\lambda}} \right] \\ &\quad +\mu_2^\lambda \left[\frac{(m'(t))_+}{m(t)} w_3(t) - \lambda \frac{\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)} (w_3(t))^{\frac{\lambda+1}{\lambda}} \right] \end{aligned} \quad (3.4.24)$$

Multiply both sides $H(t,s)$ and integrate (3.4.24) from t_3 to t, one can get that

$$\begin{aligned}\int_{t_3}^{t} H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}ds \leq &-\int_{t_3}^{t} H(t,s)w_1'(s)ds-\mu_1^{\lambda}\int_{t_3}^{t} H(t,s)w_2'(s)ds\\ &-\mu_2^{\lambda}\int_{t_3}^{t} H(t,s)w_3'(s)ds+\int_{t_3}^{t} H(t,s)\frac{(m'(s))_+}{m(s)}w_1(s)\,ds\\ &-\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\ &+\mu_1^{\lambda}\int_{t_3}^{t} H(t,s)\frac{(m'(s))_+}{m(s)}w_2(s)ds-\mu_1^{\lambda}\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\ &+\mu_2^{\lambda}\int_{t_3}^{t} H(t,s)\frac{(m'(s))_+}{m(s)}w_3(s)ds-\mu_2^{\lambda}\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.\end{aligned} \tag{3.4.25}$$

Thus, we obtain

$$\begin{aligned}\int_{t_3}^{t} H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}ds \leq &H(t,t_3)w_1(t_3)+\mu_1^{\lambda}H(t,t_3)w_2(t_3)+\mu_2^{\lambda}H(t,t_3)w_3(t_3)\\ &-\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-H(t,s)\frac{m'(s)}{m(s)}\right]w_1(s)\,ds\\ &-\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\ &-\mu_1^{\lambda}\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-H(t,s)\frac{m'(s)}{m(s)}\right]w_2(s)ds\\ &-\mu_1^{\lambda}\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\ &-\mu_2^{\lambda}\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-H(t,s)\frac{m'(s)}{m(s)}\right]w_3(s)ds\\ &-\mu_2^{\lambda}\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.\end{aligned} \tag{3.4.26}$$

Then

$$\int_{t_3}^{t} H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}ds \le H(t,t_3)w_1(t_3) + \mu_1^\lambda H(t,t_3)w_2(t_3) + \mu_2^\lambda H(t,t_3)w_3(t_3)$$
$$+\int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_1(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_1(s))^{\frac{\lambda+1}{\lambda}}\right]ds$$
$$+\mu_1^\lambda\int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_2(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_2(s))^{\frac{\lambda+1}{\lambda}}\right]ds$$
$$+\mu_2^\lambda\int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_3(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_3(s))^{\frac{\lambda+1}{\lambda}}\right]ds. \tag{3.4.27}$$

Setting $Y = \frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}$, $X = \frac{H(t,s)\,\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}$ and $u = w_i(t)$ for $i = 1,2,3$. By using the Lemma 3.2.3, we conclude that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t}\left[H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}} - \frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}}\left(\frac{|h(t,s)|a_1(s-d)}{m(s)\pi_1[t_0,s-d]}\right)^\lambda\right]ds$$
$$\le w_1(t_3) + \mu_1^\lambda w_2(t_3) + \mu_2^\lambda w_3(t_3) \tag{3.4.28}$$

which contradicts condition (3.4.23). □

Theorem 3.4.4. *Let* (3.1.4) *holds and* $b \ge m_1$ *(or* $b \le m_1$*). If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.7), (3.4.1) *and*

$$\int_{t_3}^{\infty}\left[\pi_*^\lambda(s+m_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^\lambda - \left(\frac{\lambda}{1+\lambda}\right)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+m_2+d)}{a_2^{1+\frac{1}{\lambda}}(s)\beta^\lambda(s+m_2)}\right]ds = \infty, \tag{3.4.29}$$

where $\beta(t) = \int_{t+d}^\infty a_2^{-1/\lambda}(s)ds$*, then every solution* $y(t)$ *of* (3.1.2) *is either oscillatory or tends to* 0*.*

Proof. Suppose that (3.1.1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-m_1) > 0$, $y(t+m_2) > 0$, $y(t-\xi) > 0$ and $y(t+\xi) > 0$ for $t \ge t_1 \ge t_0$ and $\xi \in [c,d]$. Since $y(t) > 0$ for all $t \ge t_1$. Assumption of (3.1.4), by Lemma 3.2.4 there exists three cases (C_1), (C_2) and (C_3). If case (C_1) and (C_2) holds, the proof is follows from Theorem 3.4.1.

If case (C_3) holds, $N'(t-d) < 0$ for $t \geq t_1$. The facts that $N'(t) < 0$, $c+d \geq 0$ and (3.4.6), we obtain

$$L^{[4]}N(t) + \mu_1^{\lambda}\, L^{[4]}_{-m_1}N(t) + \mu_2^{\lambda}\, L^{[4]}_{m_2}N(t) + \frac{\int_c^d Q(t,\xi)\, d\xi}{3^{\lambda-1}}\, N^{\lambda}(t+d) \leq 0. \quad (3.4.30)$$

Define

$$w_*(t) = \frac{L^{[3]}N(t)}{(a_1(t+d)N'(t+d))^{\lambda}}. \quad (3.4.31)$$

We obtain $w_*(t) < 0$ for $t \geq t_2$. Noting that $L^{[3]}N(t)$ is decreasing, we obtain

$$a_2(s)\Big[L^{[2]}N(s)\Big]^{\lambda} \leq a_2(t)\Big[L^{[2]}N(t)\Big]^{\lambda} \quad (3.4.32)$$

for $s \geq t \geq t_2$. Dividing (3.4.32) by $a_2(s)$ and integrating from $t+d$ to $l(l \geq t)$, we get

$$a_1(l)N'(l) \leq a_1(t+d)N'(t+d) + a_2^{1/\lambda}(t)\Big[L^{[2]}N(t)\Big]\int_{t+d}^{l} a_2^{-1/\lambda}(s)ds.$$

Letting $l \to \infty$, we get

$$-1 \leq \frac{a_2^{1/\lambda}(t)\Big[L^{[2]}N(t)\Big]}{a_1(t+d)N'(t+d)}\pi_*(t), \quad t \geq t_2. \quad (3.4.33)$$

From (3.4.31), we have

$$-1 \leq w_*(t)\beta^{\lambda}(t) \leq 0. \quad (3.4.34)$$

By (3.4.2) we have $a_1(t+d)N'(t+d) \leq a_1(t)N'(t)$. Differentiating (3.4.31) gives,

$$w_*'(t) \leq \frac{(L^{[3]}N(t))'}{(a_1(t+d)N'(t+d))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}N(t)}{a_1(t+d)N'(t+d)}\Big]^{\lambda+1}. \quad (3.4.35)$$

Using (3.4.31) in (3.4.35), we have

$$w_*'(t) \leq \frac{L^{[4]}N(t)}{(a_1(t+d)N'(t+d))^{\lambda}} - \lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.4.36)$$

Next, we define

$$w_{**}(t) = \frac{L^{[3]}_{-m_1}N(t)}{(a_1(t+d)N'(t+d))^{\lambda}}. \quad (3.4.37)$$

We obtain $w_{**}(t) < 0$ and $w_{**}(t) \geq w_*(t)$ for $t \geq t_2$. By (3.4.34), we obtain

$$-1 \leq w_{**}(t)\beta^{\lambda}(t) \leq 0. \quad (3.4.38)$$

By (3.3.2) we have $a_1(t+d)N'(t+d) \leq a_1(t-m_1)N'(t-m_1)$. Differentiating (3.4.37) gives,

$$w'_{**}(t) \leq \frac{(L^{[3]}_{-m_1}N(t))'}{(a_1(t+d)N'(t+d))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}_{-m_1}N(t)}{a_1(t+d)N'(t+d)}\Big]^{\lambda+1}. \quad (3.4.39)$$

Using (3.4.37) in (3.4.39), we have

$$w'_{**}(t) \leq \frac{L^{[4]}_{-m_1}N(t)}{(a_1(t+d)N'(t+d))^{\lambda}} - \lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.4.40)$$

Finally, We define a function

$$w_{***}(t) = \frac{L^{[3]}_{m_2}N(t)}{(a_1(t+m_2+d)N'(t+m_2+d))^{\lambda}}. \quad (3.4.41)$$

We obtain $w_{***}(t) < 0$ and $w_{***}(t) = w_*(t+m_2)$ for $t \geq t_2$. By (3.4.34), we obtain

$$-1 \leq w_{***}(t)\beta^{\lambda}(t+m_2) \leq 0. \quad (3.4.42)$$

By (3.4.2) we have $a_1(t+m_2+d)N'(t+m_2+d) \leq a_1(t+m_2)N'(t+m_2)$. Differentiating (3.4.41) gives,

$$w'_{***}(t) \leq \frac{(L^{[3]}_{m_2}N(t))'}{(a_1(t+d)N'(t+d))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}_{m_2}N(t)}{a_1(t+m_2+d)N'(t+m_2+d)}\Big]^{\lambda+1} \quad (3.4.43)$$

Using (3.4.41) in (3.4.43), we have

$$w'_{***}(t) \leq \frac{L^{[4]}_{m_2}N(t)}{(a_1(t+d)N'(t+d))^{\lambda}} - \lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.4.44)$$

From (3.4.36), (3.4.40), (3.4.44) and (3.4.30) which implies

$$\begin{aligned} w'_*(t) + \mu_1^{\lambda}w'_{**}(t) + \mu_2^{\lambda}w'_{***}(t) \leq & -\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}}\frac{N^{\lambda}(t+d)}{(a_1(t+d)N'(t+d))^{\lambda}} \\ & -\lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_1^{\lambda}\lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_2^{\lambda}\lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.4.45) \end{aligned}$$

In case (C_3), $(a_1(t)N'(t))' < 0$ we seen that

$$N(t) \geq a_1(t)N'(t)\int_{t_2}^{t}\frac{ds}{a_1(s)}. \quad (3.4.46)$$

Using (3.4.46) in (3.4.45), we get

$$\begin{aligned} w'_*(t) + \mu_1^{\lambda}w'_{**}(t) + \mu_2^{\lambda}w'_{***}(t) \quad \leq \quad & -\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{t+d}\frac{ds}{a_1(s)}\Big)^{\lambda} - \lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} \\ & -\mu_1^{\lambda}\lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_2^{\lambda}\lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.4.47) \end{aligned}$$

Multiplying $\beta^\lambda(t+m_2)$ and integrating from t_3 ($t_3 > t_2$) to t, yields

$$\begin{aligned}
\beta^\lambda(t+m_2)w_*(t) - \beta^\lambda(t_3+m_2)w_*(t_3) + \beta^\lambda(t+m_2)\,\mu_1^\lambda\,w_{**}(t)\\
-\beta^\lambda(t_3+m_2)\,\mu_1^\lambda\,w_{**}(t_3) + \beta^\lambda(t+m_2)\,\mu_2^\lambda\,w_{***}(t) - \beta^\lambda(t_3+m_2)\,\mu_2^\lambda\,w_{***}(t_3)\\
-\lambda\int_{t_3}^t\left[\frac{\beta^{\lambda-1}(s+m_2)(-w_*(s))}{a_2^{1/\lambda}(s+m_2)} - \frac{\beta^\lambda(s+m_2)(-w_*(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\right]ds\\
-\lambda\,\mu_1^\lambda\int_{t_3}^t\left[\frac{\beta^{\lambda-1}(s+m_2)(-w_{**}(s))}{a_2^{1/\lambda}(s+m_2)} - \frac{\beta^\lambda(s+m_2)(-w_{**}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\right]ds\\
-\lambda\,\mu_2^\lambda\int_{t_3}^t\left[\frac{\beta^{\lambda-1}(s+m_2)(-w_{***}(s))}{a_2^{1/\lambda}(s+m_2)} - \frac{\beta^\lambda(s+m_2)(-w_{***}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\right]ds\\
+\int_{t_3}^t\beta^\lambda(s+m_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^\lambda ds \le 0. \qquad (3.4.48)
\end{aligned}$$

Applying Lemma 3.2.3, we conclude that

$$\begin{aligned}
\int_{t_3}^t\Bigg[\beta^\lambda(s+m_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^\lambda\\
-\left(\frac{\lambda}{1+\lambda}\right)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+m_2+d)}{a_2^{1+\frac{1}{\lambda}}(s)\;\beta^\lambda(s+m_2)}\Bigg]ds\\
\le -\Big[\beta^\lambda(t+m_2)w_*(t) + \mu_1^\lambda\beta^\lambda(t+m_2)w_{**}(t) + \mu_2^\lambda\beta^\lambda(t+m_2)w_{***}(t)\Big].
\end{aligned}$$
(3.4.49)

Using the fact of $\beta^\lambda(t+m_2) \le \beta^\lambda(t)$ in (3.4.34), (3.4.38), (3.4.42) and (3.4.49) imply that

$$\begin{aligned}
\int_{t_3}^t\Bigg[\beta^\lambda(s+m_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^\lambda\\
-\left(\frac{\lambda}{1+\lambda}\right)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+m_2+d)}{a_2^{1+\frac{1}{\lambda}}(s)\beta^\lambda(s+m_2)}\Bigg]ds \le 1+\mu_1^\lambda+\mu_2^\lambda,
\end{aligned}$$
(3.4.50)

a contradiction to (3.4.29). □

Example 3.4.5. *Consider a third-order differential equation*

$$\begin{aligned}
\left(\frac{1}{2}\Big(y(t)+(1/3)y(t-\pi/4)+(2/3)y(t+\pi/2)\Big)''\right)'\\
+\int_0^\pi y(t-3)d\xi + \frac{3}{2}\int_0^\pi y(t+3)d\xi = 0. \qquad (3.4.51)
\end{aligned}$$

Compared with (3.1.2)*, we can see that* $c = 0$*,* $d = \pi$*,* $a_1(t) = 1/2$*,* $a_2(t) = 1$*,* $p_1(t) = \frac{1}{3}$*,* $p_2(t) = \frac{2}{3}$*,* $\tilde{q}_1(t,\xi) = \tilde{q}_2(t,\xi) = 1$*,* $\lambda = 1$*,* $m_1 = \pi/4$*,* $m_2 = \pi/2$ *and* $k_1 = 3$*. By taking* $m(t) = 1$*, we obtain*

$$\frac{1}{2}\int_{t_4}^{\infty}\int_{v}^{\infty}\int_{u}^{\infty} 2\pi \, ds \, du \, dv = \infty$$

and

$$\int_{t_3}^{\infty}\left[2\pi - \left(1 + \mu_1^{\lambda} + \mu_2^{\lambda}\right)\left(\frac{(m'(s))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(s-d)}{m(s)\pi_1[t_0, s-d]}\right)^{\lambda}\right] ds = \int_{t_3}^{\infty} 2\pi \, ds = \infty.$$

Hence all conditions of Theorem 3.4.1 are satisfied. Therefore, all the solutions of (3.4.51) *is either oscillates or tends to* 0*.*

Example 3.4.6. *Consider the third order differential equation*

$$\begin{aligned}&\left(e^{-t/2}\left(e^{-t/2}\left[y(t) + \frac{6}{10}\, y(t-\pi) + \frac{1}{10} y(t + (3\pi/2))\right)'\right)'\right)' \\ &+ \frac{9}{20}\int_{\pi}^{3\pi/2} e^{-t} y(t-3)\, d\xi + \frac{2}{20}\int_{\pi}^{3\pi/2} e^{-t} y(t+3)\, d\xi = 0. \qquad (3.4.52)\end{aligned}$$

Compared with (3.1.2)*, we can see that* $c = \pi$*,* $d = 3\pi/2$*,* $a_1(t) = e^{-t/2}$*,* $a_2(t) = e^{-t/2}$*,* $p_1(t) = \frac{6}{10}$*,* $p_2(t) = \frac{1}{10}$*,* $\tilde{q}_1(t,\xi) = \frac{9}{20}e^{-t}$*,* $\tilde{q}_2(t,\xi) = \frac{2}{20}e^{-t}$*,* $\lambda = 1$*,* $m_1 = \pi$*,* $m_2 = 3\pi/2$ *and* $k_1 = 3$*. By taking* $m(t) = 1$*,* $H(t,s) = (t-s)^2$*, it is easy to verify that all conditions of Theorem 3.4.3 are satisfied. Therefore, all the solutions of* (3.4.52) *is either oscillates or tends to* 0 *and* $y(t) = \cos t$ *is a such solution of* (3.4.52)*.*

3.5 Conclusion

In this chapter, we have used Riccati substitution techniques, integral averaging technique and some new oscillation and asymptotic theorems for (3.1.1) and (3.1.2) under the conditions (3.1.3) and (3.1.4) have been established. Additionally, we established new comparison theorem that permit to study properties of (3.1.1) regardless under the conditions (3.1.4). The results obtained indicated that it improved theorems reported by Candan [16]. Similar results can be presented under the assumption that $\lambda \leq 1$. In this case, using Lemma 3.2.2, one has to simply replace $3^{\lambda-1}$ by 1 and proceed as above. This results easily extended to the corresponding dynamic equations on time scales.

Chapter 4

Some New Oscillatory Behavior of Certain Third-Order Nonlinear Neutral Differential Equations of Mixed Type

CHAPTER 4

SOME NEW OSCILLATORY BEHAVIOR OF CERTAIN THIRD-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

4.1 Introduction

[1] In the present chapter, we have focused on the oscillation and asymptotic nature of third-order nonlinear DE's with mixed neutral type

$$\Big(b(t)\Big(\big[x(t)+p_1(t)x(t-m_1)+p_2(t)x(t+m_2)\big]''\Big)^{\beta}\Big)' \\ +\int_c^d q(t,\mu)x^{\beta}(t-\mu)\,d\mu+\int_c^d r(t,\mu)x^{\beta}(t+\mu)\,d\mu=0, \quad t\geq t_0>0, \tag{4.1.1}$$

and $c<d$. We put $N(t):=x(t)+p_1(t)x(t-m_1)+p_2(t)x(t+m_2)$. Now onwards the following hypotheses are tacitly supposed to hold:

(H_1) $\beta\geq 1$ is a ratio of odd natural numbers, $b(t),p_i(t)\in C\big([t_0,+\infty)\big)$ and $b(t)$, $p_i(t)>0$, $0\leq p_i(t)\leq \xi_i$, where ξ_i are constants for $i=1,2.$;

(H_2) $r(t,\mu),\ q(t,\mu)\in C\big([t_0,+\infty)\times[c,d],\ [0,+\infty)\big)$ and not identically zero on $[t_*,+\infty)\times[c,d]$, $t_*\geq t$;

(H_3) constants $m_i\geq 0$, for $i=1,2$, and the integral of (4.1.1) is take in the sense of Riemann-Stieltjes.

Assume that the function x is said to be a solution of (4.1.1) if the function $N(t)$, $N'(t)$ and $b(t)(N''(t))^{\beta}$ are continuous differential function and x satisfies equation (4.1.1). We shall considering two cases,

$$B[t_0,t]=\int_{t_0}^{t} b^{-1/\beta}(s)ds, \quad B[t_0,t]=\infty \text{ as } t\to\infty \tag{4.1.2}$$

[1]Some new oscillatory behavior of certain third-order nonlinear neutral differential equations of mixed type (a part of this chapter is published in the *International Journal of Applied and Computational Mathematics*, **4** (78), 2018, 1-14).

and

$$B[t_0, t] < \infty \text{ as } t \to \infty. \tag{4.1.3}$$

Of late, much attention is being paid in the research activities related to oscillation and asymptotic behavior of different kinds of DE's. We refer the readers to the books [1, 61, 66], the papers [8, 9, 16, 22, 29, 49, 59, 60, 72, 74, 76, 83, 85, 90, 92, 95, 103] and the references cited therein. The applications of NDE's are manifold. For example the equations are used for the study of distributed networks, automatic control, technology and natural, physical sciences. It is certain that the survey of third order DE's have not obtained remarkably so much response and documentation in the literature as the first/second order DE's have got.

Li et al. [72, 73] have established comparison theorems / Riccati substitution techniques for a second-order DE of mixed neutral type

$$(r(t)(x(t) + p_1(t)x(t-\sigma_1) + p_2(t)x(t-\sigma_2))')'_1(t)x(t-\sigma_3) + q_2(t)x(t-\sigma_4) = 0,$$

keeping the conditions of $\int_{t_0}^{\infty} dt/r(t) = \infty$ and $\int_{t_0}^{\infty} dt/r(t) < \infty$.

Qi and Yu [85] are obtained second-order DE of mixed neutral type

$$(r(t)((x(t) + p_1(t)x(t-\sigma_1) + p_2(t)x(t-\sigma_2))')^{\gamma})' + \int_a^b q_1(t,\xi)x^{\gamma}(t-\xi)d\xi + \int_a^b q_2(t,\xi)x^{\gamma}(t-\xi)d\xi = 0,$$

under $\gamma \geq 1$, $\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt = \infty$ and $\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt < \infty$. Zhang et al. [103] improved Qi and Yu [85] results.

Thandapani and Rama [92] concentrated on nonlinear third-order DE of mixed neutral type

$$(r(t)(x(t) + b(t)x(t-\tau_1) + c(t)x(t-\tau_2))'')' + q(t)x^{\alpha}(t-\sigma_1) + p(t)x^{\beta}(t-\sigma_2) = 0,$$

keeping the conditions of $\alpha = \beta \geq 1$ and $\int_{t_0}^{\infty} dt/r(t) = \infty$. If $\alpha = \beta = 1$, Han et al. [49] have presented some new oscillation and asymptotic nature for the above equation.

Till necessarily, there are no oscillation criteria to cover equation (4.1.1) under canonical and non-canonical form of operator $b(t)$. Therefore, we present some new results for all solutions of (4.1.1) to be is either oscillates or tends to 0 asymptotically by employing triple of Riccati substitutions technique under the cases of (4.1.2) and (4.1.3) and also present related examples.