Chapter 3

Asymptotic Behavior of Solutions of Third-Order Neutral Differential Equations with Discrete and Distributed Delay

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THIRD-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH DISCRETE AND DISTRIBUTED DELAY

3.1 Introduction

The purpose of this chapter, we are concerned with third-order NDE's with discrete and distributed delay

$$\left(a_2(t)\left[\left(a_1(t)N'(t)\right)'\right]^{\lambda}\right)' + q_1(t)y^{\lambda}(t-k_1) + q_2(t)y^{\lambda}(t+k_1) = 0, \tag{3.1.1}$$

and

$$\left(a_2(t)\left[\left(a_1(t)N'(t)\right)'\right]^{\lambda}\right)' + \int_c^d \tilde{q}_1(t,\xi)y^{\lambda}(t-\xi)\,d\xi + \int_c^d \tilde{q}_2(t,\xi)y^{\lambda}(t+\xi)\,d\xi = 0, \tag{3.1.2}$$

where $N(t) = y(t) + p_1(t)y(t-m_1) + p_2(t)y(t+m_2)$, $c < d$ and $\lambda \geq 1$. Now onwards, we assume that, $a_i(t), p_i(t) \in C([t_0, +\infty))$, $a_i(t) > 0$, $p_i(t) > 0$ for $i = 1, 2$ and $0 \leq p_i(t) \leq \mu_i$, $\mu_1 + \mu_2 < 1$ where μ_i are constants, $q_i \in C\big([t_0, +\infty), \mathbb{R}^+\big)$, $\tilde{q}_i(t, \xi)$ $\in C\big([t_0, +\infty) \times [c, d], \mathbb{R}^+\big)$ for $i = 1, 2$, and not identically zero on $[t_*, +\infty) \times [c, d]$, $t_* \geq t$, constants $m_i \geq 0$, for $i = 1, 2$, and the integral of (3.1.2) is take in the sense of Riemann-Stieltjes.

Let us recall that, a solution $y(t) \in C([T_y, \infty), \mathbb{R})$ of (3.1.1) (or (3.1.2)) is a non-trivial or $y(t) \neq 0$ with $T_y \geq t_0$, if the functions $N \in C^1([T_y, \infty), \mathbb{R})$, $a_1N' \in C^2([T_y, \infty), \mathbb{R})$ and $a_2[(a_1N')']^{\lambda} \in C^1([T_y, \infty), \mathbb{R})$ for certain $T_y \geq t_0$ which satisfies (3.1.1) (or (3.1.2)). Our attention is restricted to those solutions of (3.1.1) (or (3.1.2)) which exist on half-line $[T_y, \infty)$ and the condition $\sup\{|y(t)| : t > T_*\} > 0$ satisfies for any $T_* \geq t_y$.

We define the operators,

$$L^{[0]}N = N, \quad L^{[1]}N = N', \quad L^{[2]}N = (a_1 L^{[1]}N)',$$

$$L^{[3]}N = a_2\Big[L^{[2]}N\Big]^{\lambda}, \quad L^{[4]}N = (L^{[3]}N)'.$$

We shall consider the two cases,

$$\pi_1[t_0,t] = \int_{t_0}^{t} a_2^{-1/\lambda}(s)ds, \quad \pi_2[t_0,t] = \int_{t_0}^{t} a_1^{-1}(s)ds.$$

$$\pi_1[t_0,t] = \infty, \quad \pi_2[t_0,t] = \infty \text{ as } t \to \infty, \tag{3.1.3}$$

and

$$\pi_1[t_0,t] < \infty, \quad \pi_2[t_0,t] = \infty \text{ as } t \to \infty. \tag{3.1.4}$$

It is prudential to say that Mathematical modeling with DDE's have drawn obvious regards because of their potential applications in diverse fields, including biological sciences, physical sciences, gas and fluid mechanics, signal processing, robotics and traffic systems, engineering, population dynamics, medicine (see for examples [13, 15, 30]). It is now realized that the oscillation and asymptotic solutions of various classes of DE's is an important areas of investigation and its theory is a lot richer than the qualitative theory of DE's. The problem of oscillatory and non-oscillatory of solutions of various classes of second / third order DE's with delayed and mixed arguments has been widely investigated in the literature (see for examples [9, 16, 21, 49, 59, 60, 72, 73, 82, 83, 85, 103]), various types of techniques appeared for investigations of such equations.

Recently, Candan [16] investigated the oscillatory behavior of solutions of (3.1.1) and (3.1.2) by using the Riccati substitution techniques, he presented some new oscillation criteria for (3.1.1) and (3.1.2) by the assumption of condition (3.1.3). We notice that in [16], no criteria were found for (3.1.1) (or (3.1.2)) to be oscillatory for the assumption of condition (3.1.4). It would be interesting to improve and extend them with the condition (3.1.4).

In this chapter, we present several oscillatory criteria for (3.1.1) and (3.1.2), by applying three Riccati substitution techniques, integral averaging techniques and comparison principles. We present two examples in order to illustrate the main results at the end.

3.2 Preliminary

In this section, we present some basic lemmas for helping to prove the main results. We use throughout this section the following notations for convenience and for shortening the equations:

$$L_k^{[0]}N(t) = N(t+k), \quad L_k^{[1]}N(t) = N'(t+k), \quad L_k^{[2]}N(t) = (a_1(t+k)N'(t+k))',$$
$$L_k^{[3]}N(t) = a_2(t+k)\left[L_k^{[2]}N(t)\right]^{\lambda}, \quad L_k^{[4]}N(t) = \left(L_k^{[3]}N(t)\right)', \quad A(t) = \int_{t_0}^{t} \frac{\pi_1[t_0,s]}{a_1(s)}ds.$$

Lemma 3.2.1. *Let $\lambda \geq 1$, assume $u \geq 0$. Then*

$$(u_1+u_2+u_3)^{\lambda} \leq 3^{\lambda-1}\left(u_1^{\lambda}+u_2^{\lambda}+u_3^{\lambda}\right). \tag{3.2.1}$$

Lemma 3.2.2. *Let $\lambda \leq 1$, assume $u \geq 0$. Then*

$$(u_1+u_2+u_3)^{\lambda} \leq \left(u_1^{\lambda}+u_2^{\lambda}+u_3^{\lambda}\right). \tag{3.2.2}$$

Lemma 3.2.3. *If $\lambda > 0$ and $X, Y > 0$, then*

$$Yv - Xv^{\frac{\lambda+1}{\lambda}} \leq \frac{\lambda^{\lambda}}{(1+\lambda)^{1+\lambda}}\frac{Y^{1+\lambda}}{X^{\lambda}}. \tag{3.2.3}$$

Lemma 3.2.4. *Assume that* (3.1.3) *holds. Furthermore, assume that y is an eventually positive solution of* (3.1.1) *(or* (3.1.2)*). Then N for $t_1 \in [t_0, \infty)$ satisfies, eventually of the following cases:*

$$(C_1) : L^{[0]}N(t) > 0, \quad L^{[1]}N(t) > 0, \quad \textit{and} \quad L^{[2]}N(t) > 0;$$
$$(C_2) : L^{[0]}N(t) > 0, \quad L^{[1]}N(t) < 0, \quad \textit{and} \quad L^{[2]}N(t) > 0;$$

and if (3.1.4) *holds, then also*

$$(C_3) : L^{[0]}N(t) > 0, \quad L^{[1]}N(t) > 0, \quad \textit{and} \quad L^{[2]}N(t) < 0.$$

Lemma 3.2.5. *Assume that N satisfies (C_1) for $t \geq t_0$. Then*

$$N'(t) \quad \geq \quad \frac{(L^{[3]}N(t))^{1/\lambda}}{a_1(t)}\pi_1[t_0,t] \tag{3.2.4}$$

and

$$N(t) \quad \geq \quad (L^{[3]}N(t))^{1/\lambda}A(t). \tag{3.2.5}$$

Proof. Since $L^{[4]}N(t) \leq 0$, $L^{[3]}N(t)$ is nondecreasing. Then we have

$$a_1(t)N'(t) \geq a_1(t)N'(t) - a_1(t_0)N'(t_0) = \int_{t_0}^{t} \frac{a_2^{1/\lambda}(s)L^{[2]}N(s)}{a_2^{1/\lambda}(s)}ds \geq a_2^{1/\lambda}(t)L^{[2]}N(t)\,\pi_1[t_0,t].$$

Again integrate, we get

$$N(t) \geq (L^{[3]}N(t))^{1/\lambda}\int_{t_0}^{t}\frac{\pi_1[t_0,s]}{a_1(s)}ds = (L^{[3]}N(t))^{1/\lambda}A(t).$$

□

Lemma 3.2.6 (See [16]). *Assume that* N *is a solution of* (3.1.1) *which satisfies* (C_2) *in Lemma 3.2.4. Furthermore,*

$$\int_{t_4}^{\infty} a_1^{-1}(v) \int_{v}^{\infty} a_2^{-1/\lambda}(u) \left(\int_{u}^{\infty} (q_1(s) + q_2(s))\, ds \right)^{1/\lambda} du\, dv = \infty. \tag{3.2.6}$$

Then, there is $\lim_{t\to\infty} N(t) = 0$.

Lemma 3.2.7 (See [16]). *Assume that* N *is a solution of* (3.1.2) *which satisfies* (C_2) *in Lemma 3.2.4. Furthermore,*

$$\int_{t_4}^{\infty} a_1^{-1}(v) \int_{v}^{\infty} a_2^{-1/\lambda}(u) \left(\int_{u}^{\infty} \int_{a}^{b} (\tilde{q}_1(s,\xi) + \tilde{q}_2(s,\xi))\, d\xi\, ds \right)^{1/\lambda} du\, dv = \infty. \tag{3.2.7}$$

Then, there is $\lim_{t\to\infty} N(t) = 0$.

3.3 Mixed Neutral Equations with Discrete Delay

In this section, we will establish several oscillation criteria for (3.1.1). The following notations for convenience and for shortening the equations:

$$\begin{aligned} &P_1(t) = \min\{q_1(t),\, q_1(t-m_1),\, q_1(t+m_2)\}, \\ &P_2(t) = \min\{q_2(t),\, q_2(t-m_1),\, q_2(t+m_2)\}, \\ &P(t) = P_1(t) + P_2(t), \quad B(t) = \int_{t_0}^{t} \frac{\int_s^\infty \frac{du}{a_2^{1/\lambda}(u)}}{a_1(s)} ds. \end{aligned}$$

Theorem 3.3.1. *Let* (3.1.3) *hold and* $k_1 \geq m_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.6) *and*

$$\int_{t_3}^{\infty} \left[m(s)\frac{P(s)}{3^{\lambda-1}} - \left(1 + \mu_1^\lambda + \mu_2^\lambda\right) \left(\frac{(m'(s))_+}{(\lambda+1)} \right)^{\lambda+1} \left(\frac{a_1(s-k_1)}{m(s)\pi_1[t_0, s-k_1]} \right)^{\lambda} \right] ds = \infty, \tag{3.3.1}$$

where $(m'(t))_+ = \max\{0, m'(t)\}$*, then every solution* $y(t)$ *of* (3.1.1) *is either oscillatory or tends to* 0.

Proof. Suppose that (3.1.1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-m_1) > 0$, $y(t+m_2) > 0$, $y(t-k_1) > 0$ and $y(t+k_1) > 0$ for $t \geq t_1 \geq t_0$. Since $y(t) > 0$ for all $t \geq t_1$, in view of (3.1.1), we have

$$L^{[4]}N(t) = -q_1(t)y^\lambda(t-k_1) - q_2(t)y^\lambda(t+k_1) \leq 0. \tag{3.3.2}$$

Assumption of (3.1.3), by Lemma 3.2.4 there exists two cases (C_1) and (C_2). If (C_2) holds, then by Lemma 3.2.6, $\lim_{t\to\infty} N(t) = 0$. If (C_1) holds.

$$\begin{aligned} L^{[4]}N(t) + q_1(t)y^\lambda(t-k_1) + q_2(t)y^\lambda(t+k_1) + \mu_1^\lambda L^{[4]}_{-m_1}N(t) \\ +\mu_1^\lambda q_1(t-m_1)y^\lambda(t-m_1-k_1) + \mu_1^\lambda q_2(t-m_1)y^\lambda(t-m_1+k_1) + \mu_2^\lambda L^{[4]}_{m_2}N(t) \\ +\mu_2^\lambda q_1(t+m_2)y^\lambda(t+m_2-k_1) + \mu_2^\lambda q_2(t+m_2)y^\lambda(t+m_2+k_1) = 0. \end{aligned} \quad (3.3.3)$$

Furthermore, from Lemma 3.2.1, we get

$$\begin{aligned} q_1(t)y^\lambda(t-k_1) + \mu_1^\lambda q_1(t-m_1)y^\lambda(t-m_1-k_1) \\ +\mu_1^\lambda q_1(t+m_2)y^\lambda(t+m_2-k_1) \ge \frac{P_1(t)}{3^{\lambda-1}} N^\lambda(t-k_1). \end{aligned} \quad (3.3.4)$$

Similarly, we get

$$\begin{aligned} q_2(t)y^\lambda(t+k_1) + \mu_2^\lambda q_2(t-m_1)y^\lambda(t-m_1+k_1) \\ +\mu_2^\lambda q_2(t+m_2)y^\lambda(t+m_2+k_1) \ge \frac{P_2(t)}{3^{\lambda-1}} N^\lambda(t+k_1). \end{aligned} \quad (3.3.5)$$

Substituting (3.3.4), (3.3.5) into (3.3.3), we have

$$L^{[4]}N(t)+\mu_1^\lambda L^{[4]}_{-m_1}N(t)+\mu_2^\lambda L^{[4]}_{m_2}N(t)+\frac{P_1(t)}{3^{\lambda-1}} N^\lambda(t-k_1)+\frac{P_2(t)}{3^{\lambda-1}} N^\lambda(t+k_1) \le 0. \quad (3.3.6)$$

Using the fact of $L^{[1]}N(t) > 0$, we obtain

$$L^{[4]}N(t) + \mu_1^\lambda L^{[4]}_{-m_1}N(t) + \mu_2^\lambda L^{[4]}_{m_2}N(t) + \frac{P(t)}{3^{\lambda-1}} N^\lambda(t-k_1) \le 0. \quad (3.3.7)$$

Define

$$w_1(t) = m(t)\frac{L^{[3]}N(t)}{N^\lambda(t-k_1)}. \quad (3.3.8)$$

We obtain $w_1(t) > 0$, then

$$w_1'(t) = m'(t)\frac{L^{[3]}N(t)}{N^\lambda(t-k_1)} + m(t)\frac{L^{[4]}N(t)}{N^\lambda(t-k_1)} - \lambda m(t)\frac{L^{[3]}N(t)N'(t-k_1)}{N^{\lambda+1}(t-k_1)}. \quad (3.3.9)$$

By Lemma 3.2.5, one gets $N'(t-k_1) \ge \frac{a_2^{1/\lambda}(t)}{a_1(t-k_1)}\pi_1[t_0, t-k_1]\, L^{[2]}N(t)$. Therefore

$$w_1'(t) \le m'(t)\frac{L^{[3]}N(t)}{N^\lambda(t-k_1)}+m(t)\frac{L^{[4]}N(t)}{N^\lambda(t-k_1)}-\lambda m(t)\frac{a_2^{\frac{\lambda+1}{\lambda}}(t)\pi_1[t_0,t-k_1]L^{[2]}N(t)N'(t-k_1)}{N^{\lambda+1}(t-k_1)a_1(t-k_1)}. \quad (3.3.10)$$

Using (3.3.8) in (3.3.10), we obtain

$$w_1'(t) \le \frac{(m'(t))_+}{m(t)}w_1(t) + m(t)\frac{L^{[4]}N(t)}{N^\lambda(t-k_1)} - \lambda\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}. \quad (3.3.11)$$

Next, define

$$w_2(t) = m(t)\frac{L^{[3]}_{-m_1}N(t)}{N^{\lambda}(t-k_1)}. \tag{3.3.12}$$

We obtain $w_2(t) > 0$, then

$$w_2'(t) = m'(t)\frac{L^{[3]}_{-m_1}N(t)}{N^{\lambda}(t-k_1)} + m(t)\frac{L^{[4]}_{-m_1}N(t)}{N^{\lambda}(t-k_1)} - \lambda m(t)\frac{L^{[3]}_{-m_1}N(t)N'(t-k_1)}{N^{\lambda+1}(t-k_1)}. \tag{3.3.13}$$

By Lemma 3.2.5, one gets $N'(t-k_1) \geq \frac{a_2^{1/\lambda}(t-m_1)}{a_1(t-k_1)}\pi_1[t_0, t-k_1]\, L^{[2]}_{-m_1}N(t)$ and using (3.3.12) in (3.3.13), we have

$$w_2'(t) \leq \frac{(m'(t))_+}{m(t)}w_2(t) + m(t)\frac{L^{[4]}_{-m_1}N(t)}{N^{\lambda}(t-k_1)} - \lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}. \tag{3.3.14}$$

Finally, define

$$w_3(t) = m(t)\frac{L^{[3]}_{m_2}N(t)}{N^{\lambda}(t-k_1)}. \tag{3.3.15}$$

We obtain $w_3(t) > 0$, then

$$w_3'(t) = m'(t)\frac{L^{[3]}_{m_2}N(t)}{N^{\lambda}(t-k_1)} + m(t)\frac{L^{[4]}_{m_2}N(t)}{N^{\lambda}(t-k_1)} - \lambda m(t)\frac{L^{[3]}_{m_2}N(t)N'(t-k_1)}{N^{\lambda+1}(t-k_1)}. \tag{3.3.16}$$

By Lemma 3.2.5, one gets $N'(t-k_1) \geq \frac{a_2^{1/\lambda}(t+m_2)}{a_1(t-k_1)}\pi_1[t_0, t-k_1]\, L^{[2]}_{m_2}N(t)$ and using (3.3.15) in (3.3.16), we get

$$w_3'(t) \leq \frac{(m'(t))_+}{m(t)}w_3(t) + m(t)\frac{L^{[4]}_{m_2}N(t)}{N^{\lambda}(t-k_1)} - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}. \tag{3.3.17}$$

From (3.3.8), (3.3.10) and (3.3.15), we have

$$\begin{aligned}
w_1'(t) + \mu_1^{\lambda}w_2'(t) + \mu_2^{\lambda}w_3'(t) \leq{}& m(t)\left[\frac{L^{[4]}N(t) + \mu_1^{\lambda}L^{[4]}_{-m_1}N(t) + \mu_2^{\lambda}L^{[4]}_{m_2}N(t)}{N^{\lambda}(t-k_1)}\right] \\
&+\left[\frac{(m'(t))_+}{m(t)}w_1(t) - \lambda\,\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}\right] \\
&+\mu_1^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_2(t) - \lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}\right] \\
&+\mu_2^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_3(t) - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}\right].
\end{aligned} \tag{3.3.18}$$

Using (3.3.7) in (3.3.18), we have

$$\begin{aligned} w_1'(t)+\mu_1^\lambda w_2'(t)+\mu_2^\lambda w_3'(t) &\le -m(t)\frac{P(t)}{3^{\lambda-1}}+\left[\frac{(m'(t))_+}{m(t)}w_1(t)-\lambda\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}\right] \\ &\quad +\mu_1^\lambda\left[\frac{(m'(t))_+}{m(t)}w_2(t)-\lambda\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}\right] \\ &\quad +\mu_2^\lambda\left[\frac{(m'(t))_+}{m(t)}w_3(t)-\lambda\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}\right]. \quad (3.3.19)\end{aligned}$$

Applying Lemma 3.2.3, we get

$$\frac{(m'(t))_+}{m(t)}w_i(t)-\frac{\lambda\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}(w_i(t))^{\frac{\lambda+1}{\lambda}}=\left(\frac{(m'(t))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(t-k_1)}{m(t)\pi_1[t_0,t-k_1]}\right)^{\lambda},$$

for $i=1,2,3$ and we conclude from (3.3.19), we obtain

$$\begin{aligned} w_1'(t)+\mu_1^\lambda w_2'(t)+\mu_2^\lambda w_3'(t)\le -m(t)\frac{P(t)}{3^{\lambda-1}}+\left(1+\mu_1^\lambda+\mu_2^\lambda\right)\left(\frac{(m'(t))_+}{(\lambda+1)}\right)^{\lambda+1} \\ \times\left(\frac{a_1(t-k_1)}{m(t)\pi_1[t_0,t-k_1]}\right)^{\lambda}. \quad (3.3.20)\end{aligned}$$

Integrating (3.3.20) from t_3 to t, we have

$$\begin{aligned} \int_{t_3}^{t}\left[m(s)\frac{P(s)}{3^{\lambda-1}}-\left(1+\mu_1^\lambda+\mu_2^\lambda\right)\left(\frac{(m'(s))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(s-k_1)}{m(s)\pi_1[t_0,s-k_1]}\right)^{\lambda}\right]ds \\ \le w_1'(t_3)+\mu_1^\lambda w_2'(t_3)+\mu_2^\lambda w_3'(t_3).\end{aligned}$$

we get a contradiction to (3.3.1). □

Theorem 3.3.2. *Let* (3.1.3) *hold and* $m_1\ge k_1$. *If there exists an* $m(t)\in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.6) *and*

$$\int_{t_3}^{\infty}\left[m(s)\frac{P(s)}{3^{\lambda-1}}-\left(1+\mu_1^\lambda+\mu_2^\lambda\right)\left(\frac{(m'(s))_+}{\lambda+1}\right)^{\lambda+1}\left(\frac{a_1(s-m_1)}{m(s)\pi_1[t_0,s-m_1]}\right)^{\lambda}\right]ds=\infty, \quad (3.3.21)$$

then every solution $y(t)$ *of* (3.1.1) *is either oscillatory or tends to* 0.

Proof. Suppose that (3.1.1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t)>0$, $y(t-m_1)>0$, $y(t+m_2)>0$, $y(t-k_1)>0$ and $y(t+k_1)>0$ for $t\ge t_1\ge t_0$. Assumption of (3.1.3), by Lemma 3.2.4 there exists two cases (C_1) and (C_2). If (C_2) holds, then by Lemma 3.2.6, $\lim_{t\to\infty}N(t)=0$. We only consider (C_1), by using the fact that $N'(t) > 0$ and $m_1 \geq k_1$, we obtain that Using the fact of $L^{[1]}N(t) > 0$, we obtain

$$L^{[4]}N(t) + \mu_1^{\lambda}\, L^{[4]}_{-m_1}N(t) + \mu_2^{\lambda}\, L^{[4]}_{m_2}N(t) + \frac{P(t)}{3^{\lambda-1}}\, N^{\lambda}(t-m_1) \leq 0. \qquad (3.3.22)$$

Next, we categorize the functions as $w_1(t) = m(t)\frac{L^{[3]}N(t)}{N^{\lambda}(t-m_1)}$, $w_2(t) = m(t)\frac{L^{[3]}_{-m_1}N(t)}{N^{\lambda}(t-m_1)}$ and $w_3(t) = m(t)\frac{L^{[3]}_{m_2}N(t)}{N^{\lambda}(t-m_1)}$ respectively. The rest of the proof is similar to that of Theorem 3.3.1, therefore, it is omitted. □

Next, we analyse the oscillation results of solutions of (3.1.1) by Philos-type. Let $\mathbb{S}_0 = \{(t,s) : a \leq s < t < +\infty\}$, $\mathbb{S} = \{(t,s) : a \leq s \leq t < +\infty\}$ the continuous function $H(t,s)$, $H : \mathbb{S} \to \mathbb{R}$ belongs to the class function $\Re$,

(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $(t,s) \in \mathbb{S}_0$,

(ii) $\frac{\partial H(t,s)}{\partial s} \leq 0$, $(t,s) \in \mathbb{S}_0$ and some locally integrable function $h(t,s)$ such that

$$-\frac{\partial}{\partial s}H(t,s) - H(t,s)\frac{m'(t)}{m(t)} = \frac{h(t,s)(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(t)} \quad \text{for all } (t,s) \in \mathbb{S}_0.$$

Theorem 3.3.3. *Let* (3.1.3) *hold and* $k_1 \geq m_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.6) *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_3)} \int_{t_3}^{t} \left[H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}} - \frac{1+\mu_1^{\lambda}+\mu_2^{\lambda}}{(\lambda+1)^{\lambda+1}} \left(\frac{|h(t,s)|a_1(s-k_1)}{m(s)\pi_1[t_0,s-k_1]} \right)^{\lambda} \right] ds = \infty, \qquad (3.3.23)$$

then every solution $y(t)$ *of* (3.1.1) *is either oscillatory or tends to* 0.

Proof. Proceeding by the same argument as in proof of Theorem 3.3.1, we obtain the inequality (3.3.19),

$$\begin{aligned} m(t)\frac{P(t)}{3^{\lambda-1}} &\leq -w_1'(t) - \mu_1^{\lambda}w_2'(t) - \mu_2^{\lambda}w_3'(t) + \frac{(m'(t))_+}{m(t)}w_1(t) \\ &\quad -\lambda\,\frac{\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}(w_1(t))^{\frac{\lambda+1}{\lambda}} \\ &\quad +\mu_1^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_2(t) - \lambda\,\frac{\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}(w_2(t))^{\frac{\lambda+1}{\lambda}}\right] \\ &\quad +\mu_2^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_3(t) - \lambda\,\frac{\pi_1[t_0,t-k_1]}{(m(t))^{1/\lambda}a_1(t-k_1)}(w_3(t))^{\frac{\lambda+1}{\lambda}}\right]. \end{aligned} \qquad (3.3.24)$$

Multiply $H(t,s)$ and integrate (3.3.24) from t_3 to t, one can get that

$$\begin{aligned}\int_{t_3}^{t} H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}}ds \leq& -\int_{t_3}^{t} H(t,s)w_1'(s)ds - \mu_1^\lambda \int_{t_3}^{t} H(t,s)w_2'(s)ds\\ &-\mu_2^\lambda \int_{t_3}^{t} H(t,s)w_3'(s)ds + \int_{t_3}^{t} H(t,s)\frac{(m'(s))_+}{m(s)}w_1(s)\,ds\\ &-\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\ +\mu_1^\lambda \int_{t_3}^{t} H(t,s)\frac{(m'(s))_+}{m(s)}w_2(s)ds &- \mu_1^\lambda \int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\ +\mu_2^\lambda \int_{t_3}^{t} H(t,s)\frac{(m'(s))_+}{m(s)}w_3(s)ds &- \mu_2^\lambda \int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.\end{aligned} \tag{3.3.25}$$

Thus, we obtain

$$\begin{aligned}\int_{t_3}^{t} H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}}ds \leq& H(t,t_3)w_1(t_3) + \mu_1^\lambda H(t,t_3)w_2(t_3) + \mu_2^\lambda H(t,t_3)w_3(t_3)\\ &-\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s) - H(t,s)\frac{m'(s)}{m(s)}\right]w_1(s)\,ds\\ &-\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\ &-\mu_1^\lambda\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s) - H(t,s)\frac{m'(s)}{m(s)}\right]w_2(s)ds\\ &-\mu_1^\lambda\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\ &-\mu_2^\lambda\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s) - H(t,s)\frac{m'(s)}{m(s)}\right]w_3(s)ds\\ &-\mu_2^\lambda\int_{t_3}^{t} H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.\end{aligned} \tag{3.3.26}$$

Then

$$\begin{aligned}\int_{t_3}^{t} H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}}ds \leq& H(t,t_3)w_1(t_3) + \mu_1^\lambda H(t,t_3)w_2(t_3) + \mu_2^\lambda H(t,t_3)w_3(t_3)\\ +&\int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_1(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_1(s))^{\frac{\lambda+1}{\lambda}}\right]ds\\ +\mu_1^\lambda&\int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_2(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_2(s))^{\frac{\lambda+1}{\lambda}}\right]ds\\ +\mu_2^\lambda&\int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_3(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}(w_3(s))^{\frac{\lambda+1}{\lambda}}\right]ds.\end{aligned} \tag{3.3.27}$$

Setting $Y = \frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}$, $X = \frac{H(t,s)\,\lambda\pi_1[t_0,s-k_1]}{(m(s))^{1/\lambda}a_1(s-k_1)}$ and $u = w_i(t)$ for $i = 1,2,3$. By using the Lemma 3.2.3, we conclude that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t}\left[H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}} - \frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}}\left(\frac{|h(t,s)|a_1(s-k_1)}{m(s)\pi_1[t_0,s-k_1]}\right)^{\lambda}\right]ds$$
$$\leq w_1(t_3)+\mu_1^\lambda w_2(t_3)+\mu_2^\lambda w_3(t_3), \qquad (3.3.28)$$

which contradicts condition (3.3.23). □

Theorem 3.3.4. *Let* (3.1.4) *hold and* $k_1 \geq m_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (3.2.6), (3.3.1) *and*

$$\int_{t_3}^{\infty}\left[\pi_*^\lambda(s+m_2)\frac{P(s)}{3^{\lambda-1}}\Big(\int_{t_2}^{s+k_1}\frac{du}{a_1(u)}\Big)^{\lambda} - \left(\frac{\lambda}{1+\lambda}\right)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+m_2+k_1)}{a_2^{1+\frac{1}{\lambda}}(s)\pi_*^\lambda(s+m_2)}\right]ds = \infty, \quad (3.3.29)$$

where $\pi_*(t) = \int_{t+k_1}^{\infty} a_2^{-1/\lambda}(s)ds$, *then every solution* $y(t)$ *of* (3.1.1) *is either oscillatory or tends to* 0.

Proof. Suppose that (3.1.1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-m_1) > 0$, $y(t+m_2) > 0$, $y(t-k_1) > 0$ and $y(t+k_1) > 0$ for $t \geq t_1 \geq t_0$. Since $y(t) > 0$ for all $t \geq t_1$. Assumption of (3.1.4), by Lemma 3.2.4 there exists three cases (C_1), (C_2) and (C_3). If case (C_1) and (C_2) holds, the proof is follows from Theorem 3.3.1.

If case (C_3) holds, $N'(t-k_1) < 0$ for $t \geq t_1$. The facts that $N'(t) < 0$, $c+d \geq 0$ and (3.3.6), we obtain

$$L^{[4]}N(t) + \mu_1^\lambda L_{-m_1}^{[4]}N(t) + \mu_2^\lambda L_{m_2}^{[4]}N(t) + \frac{P(t)}{3^{\lambda-1}}N^\lambda(t+k_1) \leq 0. \qquad (3.3.30)$$

Define

$$w_*(t) = \frac{L^{[3]}N(t)}{(a_1(t+k_1)N'(t+k_1))^\lambda}. \qquad (3.3.31)$$

We obtain $w_*(t) < 0$ for $t \geq t_2$. Noting that $L^{[3]}N(t)$ is decreasing, we obtain

$$a_2(s)\Big[L^{[2]}N(s)\Big]^\lambda \leq a_2(t)\Big[L^{[2]}N(t)\Big]^\lambda \qquad (3.3.32)$$

for $s \geq t \geq t_2$. Dividing (3.3.32) by $a_2(s)$ and integrating from $t+k_1$ to l ($l \geq t$), we get

$$a_1(l)N'(l) \leq a_1(t+k_1)N'(t+k_1) + a_2^{1/\lambda}(t)\Big[L^{[2]}N(t)\Big]\int_{t+k_1}^{l} a_2^{-1/\lambda}(s)ds.$$

Letting $l \to \infty$, we get

$$-1 \le \frac{a_2^{1/\lambda}(t)\Big[L^{[2]}N(t)\Big]}{a_1(t+k_1)N'(t+k_1)}\pi_*(t), \tag{3.3.33}$$

for $t \ge t_2$. From (3.3.31), we have

$$-1 \le w_*(t)\pi_*^{\lambda}(t) \le 0. \tag{3.3.34}$$

By (3.3.2) we have $a_1(t+k_1)N'(t+k_1) \le a_1(t)N'(t)$. Differentiating (3.3.31) gives,

$$w_*'(t) \le \frac{(L^{[3]}N(t))'}{(a_1(t+k_1)N'(t+k_1))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}N(t)}{a_1(t+k_1)N'(t+k_1)}\Big]^{\lambda+1}. \tag{3.3.35}$$

Using (3.3.31) in (3.3.35), we have

$$w_*'(t) \le \frac{L^{[4]}N(t)}{(a_1(t+k_1)N'(t+k_1))^{\lambda}} - \lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{3.3.36}$$

Again, we define

$$w_{**}(t) = \frac{L^{[3]}_{-m_1}N(t)}{(a_1(t+k_1)N'(t+k_1))^{\lambda}}. \tag{3.3.37}$$

We obtain $w_{**}(t) < 0$ and $w_{**}(t) \ge w_*(t)$ for $t \ge t_2$. By (3.3.34), we obtain

$$-1 \le w_{**}(t)\pi_*^{\lambda}(t) \le 0. \tag{3.3.38}$$

By (3.3.2) we have $a_1(t+k_1)N'(t+k_1) \le a_1(t-m_1)N'(t-m_1)$. Differentiating (3.3.37) gives,

$$w_{**}'(t) \le \frac{(L^{[3]}_{-m_1}N(t))'}{(a_1(t+k_1)N'(t+k_1))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}_{-m_1}N(t)}{a_1(t+k_1)N'(t+k_1)}\Big]^{\lambda+1}. \tag{3.3.39}$$

Using (3.3.37) in (3.3.39), we have

$$w_{**}'(t) \le \frac{L^{[4]}_{-m_1}N(t)}{(a_1(t+k_1)N'(t+k_1))^{\lambda}} - \lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{3.3.40}$$

Finally, we define a function

$$w_{***}(t) = \frac{L^{[3]}_{m_2}N(t)}{(a_1(t+m_2+k_1)N'(t+m_2+k_1))^{\lambda}}. \tag{3.3.41}$$

We obtain $w_{***}(t) < 0$ and $w_{***}(t) = w_*(t+m_2)$ for $t \ge t_2$. By (3.3.34), we obtain

$$-1 \le w_{***}(t)\pi_*^{\lambda}(t+m_2) \le 0. \tag{3.3.42}$$

By (3.3.2) we have $a_1(t+m_2+k_1)N'(t+m_2+k_1) \leq a_1(t+m_2)N'(t+m_2)$. Differentiating (3.3.41) gives,

$$w'_{***}(t) \leq \frac{(L^{[3]}_{m_2}N(t))'}{(a_1(t+k_1)N'(t+k_1))^\lambda} - \lambda a_2(t)\Big[\frac{L^{[2]}_{m_2}N(t)}{a_1(t+m_2+k_1)N'(t+m_2+k_1)}\Big]^{\lambda+1} \quad (3.3.43)$$

Using (3.3.41) in (3.3.43), we have

$$w'_{***}(t) \leq \frac{L^{[4]}_{m_2}N(t)}{(a_1(t+k_1)N'(t+k_1))^\lambda} - \lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.3.44)$$

From (3.3.36), (3.3.40), (3.3.44) and (3.3.30) which implies

$$\begin{aligned} w'_*(t) + \mu_1^\lambda w'_{**}(t) + \mu_2^\lambda w'_{***}(t) &\leq -\frac{P(t)}{3^{\lambda-1}}\frac{N^\lambda(t+k_1)}{(a_1(t+k_1)N'(t+k_1))^\lambda} \\ &-\lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_1^\lambda\lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_2^\lambda\lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.3.45) \end{aligned}$$

In case (C_3), $(a_1(t)N'(t))' < 0$ we seen that

$$N(t) \geq a_1(t)N'(t)\int_{t_2}^t \frac{ds}{a_1(s)}. \quad (3.3.46)$$

Using (3.3.46) in (3.3.45), we get

$$\begin{aligned} w'_*(t) + \mu_1^\lambda w'_{**}(t) + \mu_2^\lambda w'_{***}(t) &\leq -\frac{P(t)}{3^{\lambda-1}}\Big(\int_{t_2}^{t+k_1}\frac{ds}{a_1(s)}\Big)^\lambda \\ &-\lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_1^\lambda\lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_2^\lambda\lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.3.47) \end{aligned}$$

Multiplying $\pi_*^\lambda(t+m_2)$ and integrating from t_3 $(t_3 > t_2)$ to t, yields

$$\begin{aligned} &\pi_*^\lambda(t+m_2)w_*(t) - \pi_*^\lambda(t_3+m_2)w_*(t_3) + \pi_*^\lambda(t+m_2)\,\mu_1^\lambda\,w_{**}(t) \\ &-\pi_*^\lambda(t_3+m_2)\,\mu_1^\lambda\,w_{**}(t_3) + \pi_*^\lambda(t+m_2)\,\mu_2^\lambda\,w_{***}(t) - \pi_*^\lambda(t_3+m_2)\,\mu_2^\lambda\,w_{***}(t_3) \\ &-\lambda\int_{t_3}^t\left[\frac{\pi_*^{\lambda-1}(s+m_2)(-w_*(s))}{a_2^{1/\lambda}(s+m_2)} - \frac{\pi_*^\lambda(s+m_2)(-w_*(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\right]ds \\ &-\lambda\,\mu_1^\lambda\int_{t_3}^t\left[\frac{\pi_*^{\lambda-1}(s+m_2)(-w_{**}(s))}{a_2^{1/\lambda}(s+m_2)} - \frac{\pi_*^\lambda(s+m_2)(-w_{**}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\right]ds \\ &-\lambda\,\mu_2^\lambda\int_{t_3}^t\left[\frac{\pi_*^{\lambda-1}(s+m_2)(-w_{***}(s))}{a_2^{1/\lambda}(s+m_2)} - \frac{\pi_*^\lambda(s+m_2)(-w_{***}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\right]ds \\ &+\int_{t_3}^t\pi_*^\lambda(s+m_2)\frac{P(s)}{3^{\lambda-1}}\Big(\int_{t_2}^{s+k_1}\frac{du}{a_1(u)}\Big)^\lambda ds \leq 0. \quad (3.3.48) \end{aligned}$$

Applying Lemma 3.2.3, we conclude that

$$\int_{t_3}^{t}\Bigg[\pi_*^\lambda(s+m_2)\frac{P(s)}{3^{\lambda-1}}\Big(\int_{t_2}^{s+k_1}\frac{du}{a_1(u)}\Big)^\lambda-\Big(\frac{\lambda}{1+\lambda}\Big)^{1+\lambda}$$

$$\times\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+m_2+k_1)}{a_2^{1+\frac{1}{\lambda}}(s)\pi_*^\lambda(s+m_2)}\Bigg]ds$$

$$\leq-\Big[\pi_*^\lambda(t+m_2)w_*(t)+\mu_1^\lambda\pi_*^\lambda(t+m_2)w_{**}(t)+\mu_2^\lambda\pi_*^\lambda(t+m_2)w_{***}(t)\Big]. \quad (3.3.49)$$

Using the fact of $\pi_*^\lambda(t+m_2)\leq\pi_*^\lambda(t)$ in (3.3.34), (3.3.38), (3.3.42) and (3.3.49) imply that

$$\int_{t_3}^{t}\Bigg[\pi_*^\lambda(s+m_2)\frac{P(s)}{3^{\lambda-1}}\Big(\int_{t_2}^{s+k_1}\frac{du}{a_1(u)}\Big)^\lambda-\Big(\frac{\lambda}{1+\lambda}\Big)^{1+\lambda}$$

$$\times\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+m_2+k_1)}{a_2^{1+\frac{1}{\lambda}}(s)\pi_*^\lambda(s+m_2)}\Bigg]ds\leq 1+\mu_1^\lambda+\mu_2^\lambda. \quad (3.3.50)$$

a contradiction to (3.3.29). □

Finally, we establish new comparison theorems for (3.1.1) under the case when (3.1.4) holds.

Theorem 3.3.5. *Let* (3.1.4), (3.2.6) *hold and* $k_1>m_1$, $k_1>m_2$. *If the first-order differential inequality*

$$\psi'(t)+\frac{P_1(t)}{3^{\lambda-1}}\frac{A^\lambda(t-k_1)}{1+\mu_1^\lambda+\mu_2^\lambda}\psi(t-k_1+m_1)\leq 0, \quad (3.3.51)$$

for $t\geq t_0$, *has no positive nonincreasing solution and the first-order differential inequality*

$$\psi'(t)-\frac{P_2(t)}{3^{\lambda-1}}\frac{B^\lambda(t+k_1)}{1+\mu_1^\lambda+\mu_2^\lambda}\psi(t-m_2+k_1)\geq 0, \quad (3.3.52)$$

for $t\geq t_0$, *has no positive nondecreasing solution. Then Eq.* (3.1.1) *oscillatory.*

Proof. Suppose that (3.1.1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t)>0$, $y(t-m_1)>0$, $y(t+m_2)>0$, $y(t-k_1)>0$ and $y(t+k_1)>0$ for $t\geq t_1\geq t_0$. Since $y(t)>0$ for all $t\geq t_1$. Assumption of (3.1.4), by Lemma 3.2.4, there exists three cases (C_1), (C_2) and (C_3). If case (C_2) hold, the proof is follows from Theorem 3.3.1.

If case (C_1) holds, we have $L^{[2]}N(t)>0$, from (3.3.6), we obtain

$$L^{[4]}N(t)+\mu_1^\lambda L^{[4]}_{-m_1}N(t)+\mu_2^\lambda L^{[4]}_{m_2}N(t)+\frac{P_1(t)}{3^{\lambda-1}}N^\lambda(t-k_1)\leq 0. \quad (3.3.53)$$

By Lemma 3.2.5, one gets $N(t-k_1) \geq (L^{[3]}_{-k_1}N(t))^{1/\lambda}A(t-k_1)$ and using in (3.3.53), we have

$$\Big(L^{[3]}N(t) + \mu_1^\lambda\, L^{[3]}_{-m_1}N(t) + \mu_2^\lambda\, L^{[3]}_{m_2}N(t)\Big)' + \frac{P_1(t)}{3^{\lambda-1}}\, L^{[3]}_{-k_1}N(t)A^\lambda(t-k_1) \leq 0. \quad (3.3.54)$$

Now, set

$$\psi(t) = L^{[3]}N(t) + \mu_1^\lambda\, L^{[3]}_{-m_1}N(t) + \mu_2^\lambda\, L^{[3]}_{m_2}N(t).$$

Then $\psi(t) > 0$ and the fact that $L^{[3]}N(t)$ is nonincreasing, we have

$$\psi(t) \leq L^{[3]}_{-m_1}N(t)\Big(1 + \mu_1^\lambda + \mu_2^\lambda\Big). \quad (3.3.55)$$

Using (3.3.55) and (3.3.54), we see that $\psi(t)$ is a nonincreasing positive solution of the first order differential inequality

$$\psi'(t) + \frac{P_1(t)}{3^{\lambda-1}}\frac{A^\lambda(t-k_1)}{1+\mu_1^\lambda+\mu_2^\lambda}\psi(t-k_1+m_1) \leq 0, \quad (3.3.56)$$

which is contradiction to (3.3.51).
If case (C_3) holds, we have $L^{[2]}N(t) < 0$, from (3.3.6), we obtain

$$L^{[4]}N(t) + \mu_1^\lambda\, L^{[4]}_{-m_1}N(t) + \mu_2^\lambda\, L^{[4]}_{m_2}N(t) + \frac{P_2(t)}{3^{\lambda-1}}\, N^\lambda(t+k_1) \leq 0. \quad (3.3.57)$$

Since $L^{[3]}N(t)$ is nondecreasing . Then we get

$$L^{[3]}N(s) \leq L^{[3]}N(t) \quad \text{for all} \quad s \geq t \geq t_1 \geq t_0.$$

Integrating above inequality from t to l, we get

$$\begin{aligned} a_1(l)N'(l) &\leq a_1(t)N'(t) + \int_t^l \frac{a_2^{1/\lambda}(t)L^{[2]}N(t)}{a_2^{1/\lambda}(s)}ds \\ &\leq a_1(t)N'(t) + \Big(L^{[3]}N(s)\Big)^{1/\lambda}\int_t^l \frac{ds}{a_2^{1/\lambda}(s)}. \end{aligned}$$

Letting $l \to \infty$, we get

$$-a_1(t)N'(t) \leq \Big(L^{[3]}N(s)\Big)^{1/\lambda}\int_t^\infty \frac{ds}{a_2^{1/\lambda}(s)}.$$

Again integrating, we get

$$N(t) \geq -(L^{[3]}N(t))^{1/\lambda}\int_{t_0}^t \frac{\int_t^\infty \frac{du}{a_2^{1/\lambda}(u)}}{a_1(s)}ds = -(L^{[3]}N(t))^{1/\lambda}B(t). \quad (3.3.58)$$