Chapter 2

Oscillation of Certain Third Order Nonlinear Differential Equation with Neutral Terms

CHAPTER 2

OSCILLATION OF CERTAIN THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION WITH NEUTRAL TERMS

2.1 Introduction

This chapter is concerned with oscillation behavior of a class of third order NDE with several neutral term

$$\left[r(t)\left[\left(x(t)+\sum_{i=1}^{n}p_i(t)x(\eta_i(t))\right)''\right]^{\gamma}\right]'+q(t)x^{\gamma}(\sigma(t))=0, \tag{2.1.1}$$

where $n>0$ is an integer, $q(t)$, $\sigma(t)$, $p_i(t)$ and $\eta_i(t)$ are continuous differentiable on $[t_0,+\infty)$. Throughout this chapter it always assume the following conditions hold:

(C_1) $\gamma>0$, $r,q>0$, $0\le p_i\le a_i<\infty$ for $i=1,2,...,n$;

(C_2) $\eta_i\circ\sigma=\sigma\circ\eta_i$, $\eta_i'(t)\ge\lambda_i>0$ for $i=1,2,...,n$; and $\lim\limits_{t\to+\infty}\sigma(t)=\infty$, $\sigma(t)<t$;

$$D(t)=\int_{t_0}^{t}\frac{1}{r^{1/\gamma}(s)}ds=\infty \quad \text{as} \quad t\to\infty. \tag{2.1.2}$$

We set

$$N(t)=\begin{cases} x(t)+\sum\limits_{i=1}^{n}p_i(t)x(\eta_i(t)), & n>1\ ;\\ x(t)+p(t)x(\eta(t)), & n=1\ .\end{cases} \tag{2.1.3}$$

By a solution to (2.1.1), we mean a function $x(t)$ in $C^2[T_x,\infty)$ for which $a(t)(N''(t))^{\gamma}$ is in $C^1[T_x,\infty)$ and (2.1.1) is satisfied on some interval $[T_x,\infty)$, where $T_x\ge t_0$. We consider only solutions $x(t)$ for which $\sup\{|x(t)|:t\ge T\}>0$ for all $T\ge T_x$. A proper solution of (2.1.1) is called oscillatory if it has no largest zero on $[T_x,\infty)$ and non-oscillatory otherwise. The equation itself is called oscillatory if all its solutions are oscillatory.

Recently, Zhang et al. [105] studied the oscillatory behavior of the following second order NDE

$$(r(t)|N'(t)|^{\alpha-1}N'(t))' + q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t)) = 0$$

where $N(t)$ is defined in (2.1.3), under the assumption of (2.1.2) and Li et al. [75] also studied the above equation in the noncanonical form of $r(t)$. In third order, Baculíková et al. [9], Thandapani et al. [90] and Dzurina et al. [23] were studied several oscillation results for equation

$$\left(r(t)(N''(t))^{\gamma}\right)' + q(t)x^{\gamma}(\sigma(t)) = 0,$$

by the condition of (2.1.2). Baculíková and Džurina [11] investigated oscillatory theorems of above neutral equation if $\gamma = 1$ via comparison principle.

The aim of this chapter establishing some new comparison theorems essentially simplify the examination of the studied equations and enable us also to eliminate some conditions imposed in the cited papers on the coefficients of Eq.(2.1.1). Also we improve and extend the main results of [11] and use the techniques in [105].

Remark 2.1.1. *All functional inequalities are considered in this thesis are assumed to support eventually, that is, they are satisfied for all t large enough.*

Remark 2.1.2. *Without loss of generality, we can deal only with the positive solutions of* (2.1.1).

2.2 Third-order Differential Equation with Several Neutral Terms

[1] For our further reference, let us denote

$$\begin{aligned} Q(t) &= \min\{q(t), q(\eta_1(t)), ..., q(\eta_n(t))\}, && (2.2.1)\\ D_1[t,t_1] &= D(t) - D(t_1), && (2.2.2)\\ D_2[t,t_1] &= \int_{t_1}^{t} D_1[s,t_1]\,ds && (2.2.3) \end{aligned}$$

To obtain sufficient conditions for the oscillation of solutions to (2.1.1), we need the the following lemma.

[1]Oscillation of certain third order nonlinear differential equation with neutral terms (a part of this Section is published in the *Bangmod International Journal of Mathematics and Computer Science*, **3** (1-2) (2017), 53-60).

Lemma 2.2.1. *Let* $\gamma \geq 1$. *Assume* $u_i \geq 0$ *for* $i = 1, 2, ..., n$. *Then*

$$\left(\sum_{i=1}^{n} u_i\right)^{\gamma} \leq (n+1)^{\gamma-1} \sum_{i=1}^{n} u_i^{\gamma}. \tag{2.2.4}$$

Proof. Consider a function $g(u) = u^{\gamma}$. Since $g'' > 0$ for $u > 0$, function $g(u)$ is convex, hence using Jensen's inequality, we obtain (2.2.4). □

We now give oscillation results when (2.1.2) holds.

Theorem 2.2.2. *Let* $\eta_i(t) \geq t$ *for* $i = 1, 2, ..., n$ *and* $\gamma \geq 1$. *Assume that*

$$\int_{t_1}^{\infty} \int_{v}^{\infty} \frac{1}{r^{1/\gamma}(u)} \Big(\int_{u}^{\infty} Q(s)\, ds \Big)^{1/\gamma} du\, dv = \infty, \tag{2.2.5}$$

and the first order DDE

$$w'(t) + \frac{Q(t)}{(n+1)^{\gamma-1}} \frac{\big(D_2[\sigma(t), t_1]\big)^{\gamma}}{\big(1 + \sum_{i=1}^{n} a_i^{\gamma} \lambda_i^{-1}\big)} w(\sigma(t)) = 0 \tag{2.2.6}$$

is oscillatory, then every non-oscillatory solution of (2.1.1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume, for sake of contradiction, $x(t)$ has an eventually positive solution of (2.1.1) on $[t_0, \infty)$. Then from (C_1) and (C_2) the corresponding function $N(t)$ satisfies

$$\begin{aligned} N^{\gamma}(\sigma(t)) &= \left[x(\sigma(t)) + \sum_{i=1}^{n} p_i(\sigma(t)) x(\eta_i(\sigma(t))) \right]^{\gamma} \\ &\leq \left[x(\sigma(t)) + \sum_{i=1}^{n} a_i x(\eta_i(\sigma(t))) \right]^{\gamma} \\ &\leq \frac{1}{(n+1)^{1-\gamma}} \left[x^{\gamma}(\sigma(t)) + \sum_{i=1}^{n} a_i^{\gamma} x^{\gamma}(\sigma(\eta_i(t))) \right]. \end{aligned} \tag{2.2.7}$$

On the other hand, it follows from (2.1.1) that

$$\Big(r \big(N''\big)^{\gamma} \Big)'(t) + q(t) x^{\gamma}(\sigma(t)) = 0, \tag{2.2.8}$$

which yields,

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\eta_i'(t)} \Big(r(\eta_i(t)) \big(N''(\eta_i(t))\big)^{\gamma} \Big)' + \sum_{i=1}^{n} a_i^{\gamma} q(\eta_i(t)) x^{\gamma}(\sigma(\eta_i(t))) \\ &\geq \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i} \Big(r(\eta_i(t)) \big(N''(\eta_i(t))\big)^{\gamma} \Big)' + \sum_{i=1}^{n} a_i^{\gamma} q(\eta_i(t)) x^{\gamma}(\sigma(\eta_i(t))). \end{aligned} \tag{2.2.9}$$

Combining (2.2.8) and (2.2.9), we are led to,

$$\Big(r(t)\big(N''(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{n}\frac{a_i^{\gamma}}{\lambda_i}\Big(r(\eta_i(t))\big(N''(\eta_i(t))\big)^{\gamma}\Big)' + q(t)x^{\gamma}(\sigma(t)) + \sum_{i=1}^{n} a_i^{\gamma} q(\eta_i(t))x^{\gamma}(\sigma(\eta_i(t))) \le 0,$$

$$\Big(r(t)\big(N''(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{n}\frac{a_i^{\gamma}}{\lambda_i}\Big(r(\eta_i(t))\big(N''(\eta_i(t))\big)^{\gamma}\Big)' + \min\{q(t), q(\eta_1(t)), ..., q(\eta_n(t))\}\left[x^{\gamma}(\sigma(t)) + \sum_{i=1}^{n} a_i^{\gamma}\, x^{\gamma}(\sigma(\eta_i(t)))\right] \le 0,$$

that is,

$$\Big(r(t)\big(N''(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{n}\frac{a_i^{\gamma}}{\lambda_i}\Big(r(\eta_i(t))\big(N''(\eta_i(t))\big)^{\gamma}\Big)' + \frac{Q(t)}{(n+1)^{\gamma-1}}N^{\gamma}(\sigma(t)) \le 0, \quad (2.2.10)$$

where $Q(t)$ defined in (2.2.1). Assumption of (2.1.2), there exists following cases

$$N > 0, \quad N' > 0, \quad N'' > 0, \quad \text{and} \quad (r(N'')^{\gamma})' \le 0, \quad (2.2.11)$$

or

$$N > 0, \quad N' < 0, \quad N'' > 0, \quad \text{and} \quad (r(N'')^{\gamma})' \le 0, \quad (2.2.12)$$

for $t \ge t_1$, t_1 is large enough. Assume (2.2.11) holds. Since $N(t) > 0$ and $y = r(N'')^{\gamma} > 0$ is decreasing, we obtain

$$N'(t) \ge \int_{t_1}^{t} \frac{1}{r^{1/\gamma}(s)}[r(s)(N''(s))^{\gamma}]^{1/\gamma}\, ds \ge y^{1/\gamma}(t)\int_{t_1}^{t}\frac{ds}{r^{1/\gamma}(s)} = y^{1/\gamma}(t)D_1[t, t_1].$$

Integrate above from t_1 to t, yields

$$N(t) \ge \int_{t_1}^{t} y^{1/\gamma}(s)D_1[s, t_1]\, ds \ge y^{1/\gamma}(t)\int_{t_1}^{t} D_1[s, t_1]\, ds.$$

That is,

$$N^{\gamma}(\sigma(t)) \ge y(\sigma(t))\left(\int_{t_1}^{\sigma(t)} D_1[s, t_1]\, ds\right)^{\gamma} = y(\sigma(t))\big(D_2[\sigma(t), t_1]\big)^{\gamma}. \quad (2.2.13)$$

Combining (2.2.13) together with (2.2.10), we get that $y(t)$ is a positive solution of

$$\left(y(t) + \sum_{i=1}^{n}\frac{a_i^{\gamma}}{\lambda_i}y(\eta_i(t))\right)' + \frac{Q(t)}{(n+1)^{\gamma-1}}\big(D_2[\sigma(t), t_1]\big)^{\gamma}y(\sigma(t)) \le 0. \quad (2.2.14)$$

Now, we define

$$w(t) = y(t) + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} y(\eta_i(t)). \tag{2.2.15}$$

Since the function $y(t)$ is decreasing and $\eta_i(t) \geq t$ that

$$w(t) \leq \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right) y(t).$$

Using latter inequality into (2.2.14), which yields

$$w'(t) + \frac{Q(t)}{(n+1)^{\gamma-1}} \frac{\big(D_2[\sigma(t), t_1]\big)^\gamma}{\left(1 + \sum_{i=1}^{n} a_i^\gamma \lambda_i^{-1}\right)} w(\sigma(t)) \leq 0,$$

it follows from Theorem 1 in [84], we get $w(t)$ is a positive solution which contradicts to (2.2.6).

Assume (2.2.12) hold. Since $N(t) > 0$ and $N'(t) < 0$, then there exists $\lim_{t\to\infty} N(t) = l \geq 0$. We claim that $l = 0$. If $l > 0$, then an integration of (2.2.10) from t to ∞ leads to

$$\begin{aligned} r(t)(N''(t))^\gamma + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} r(\eta_i(t))(N''(\eta_i(t)))^\gamma &\geq \int_t^\infty \frac{Q(s)}{(n+1)^{\gamma-1}} N^\gamma(\sigma(s))\, ds \\ &\geq \frac{l^\gamma}{(n+1)^{\gamma-1}} \int_t^\infty Q(s)\, ds. \quad (2.2.16) \end{aligned}$$

Since the function $y = r(N'')^\gamma$ decreasing and $\eta_i(t) \geq t$, then

$$r(t)(N''(t))^\gamma \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right) \geq r(t)(N''(t))^\gamma + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} r(\eta_i(t))(N''(\eta_i(t)))^\gamma,$$

which in view of (2.2.16) provides,

$$\begin{aligned} r(t)(N''(t))^\gamma &\geq \frac{l^\gamma}{(n+1)^{\gamma-1}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1} \int_t^\infty Q(s)\, ds \\ N''(t) &\geq \frac{l}{(n+1)^{\frac{\gamma-1}{\gamma}}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1/\gamma} \frac{1}{r^{1/\gamma}(t)} \left[\int_t^\infty Q(s)\, ds\right]^{1/\gamma}. \end{aligned}$$

Next, again integrating the above from t to ∞, we have

$$-N'(t) \geq \frac{l}{(n+1)^{\frac{\gamma-1}{\gamma}}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1/\gamma} \int_t^\infty \frac{1}{r^{1/\gamma}(u)} \left[\int_u^\infty Q(s)\, ds\right]^{1/\gamma} du.$$

Finally, integrating the above from t_1 to t, we obtain

$$N(t_1) \geq \frac{l}{(n+1)^{\frac{\gamma-1}{\gamma}}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1/\gamma} \int_{t_1}^t \int_v^\infty \frac{1}{r^{1/\gamma}(v)} \left[\int_u^\infty Q(s)\, ds\right]^{1/\gamma} du\, dv.$$

This contradicts (2.2.5) and hence $l = 0$. □

Corollary 2.2.3. *Let $\gamma \geq 1$, $\eta_i(t) \geq t$ for $i = 1, 2, ..., n$ and* (2.2.5) *holds. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)(D_2[\sigma(s), t_1])^\gamma \, ds > \frac{(n+1)^{\gamma-1}}{e}\Big(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\Big), \tag{2.2.17}$$

then every non-oscillatory solution $x(t)$ of (2.1.1) *satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. It follows from Theorem 2.1.1 in [67] that the related first order DDE (2.2.6) also has a positive solution, which contradicts the oscillatory nature of (2.2.6). □

Theorem 2.2.4. *Let $\gamma \geq 1$, $\sigma(t) \leq \eta_i(t) \leq t$ for $i = 1, 2, ..., n$ and* (2.2.5) *holds. Assume that the first order DDE*

$$w'(t) + \frac{Q(t)}{(n+1)^{\gamma-1}} \frac{\big(D_2[\sigma(t), t_1]\big)^\gamma}{\Big(1 + \sum_{i=1}^{n} a_i^\gamma \lambda_i^{-1}\Big)} w(\eta^{-1}(\sigma(t))) = 0, \tag{2.2.18}$$

where $\eta(t) = \min\{\eta_i(t), i = 1, 2, ..., n\}$ and $\eta^{-1}(t)$ is inverse function of $\eta(t)$, is oscillatory, then every non-oscillatory solution of (2.1.1) *satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. Assume, for sake of contradiction, $x(t)$ has an eventually positive solution of (2.1.1) on $[t_0, \infty)$. By the proof of Theorem 2.2.2, we assume that (2.2.11) holds, $y(t) = r(t)(N''(t))^\gamma > 0$ satisfies (2.2.14). Let us denote $w(t) = y(t) + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} y(\eta_i(t))$. Since $y(t)$ is decreasing and $\eta_i(t) \leq t$ that

$$w(t) = y(\eta(t))\Big(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\Big). \tag{2.2.19}$$

Substituting (2.2.19) into (2.2.14), it follows from Theorem 1 in [84], we get $w(t)$ is a positive solution which contradicts to (2.2.18). Next, Assume (2.2.12) holds, we get $\lim_{t\to\infty} x(t) = 0$. □

Corollary 2.2.5. *Let $\gamma \geq 1$, $\sigma(t) \leq \eta_i(t) \leq t$ for $i = 1, 2, ..., n$ and (2.2.5) holds. If*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} Q(s)(D_2[\sigma(s), t_1])^\gamma \, ds > \frac{(n+1)^{\gamma-1}}{e}\Big(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\Big), \tag{2.2.20}$$

then every non-oscillatory solution $x(t)$ of (2.1.1) *satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. It follows from Theorem 2.1.1 in [67] that the related first order DDE (2.2.6) also has a positive solution, which contradicts the oscillatory nature of (2.2.18). □

Remark 2.2.6. *If $\gamma = 1$ and $n = 1$, then all the above results are reduce to [11].*

2.3 Third-order Differential Equation with Neutral Term

[2] If $n = 1$, equation (2.1.1) reduces to a third order nonlinear NDE

$$\left[r(t)\left[(x(t)+p(t)x(\eta(t)))''\right]^{\gamma}\right]' + q(t)x^{\gamma}(\sigma(t)) = 0, \tag{2.3.1}$$

where $0 \le p(t) \le p_1 < \infty$. In this section we establish several new comparison theorems for oscillation of solutions to Equ. (2.3.1). For our further reference, let us denote

$$\begin{aligned} P(t) &= \min\{q(t), q(\eta(t))\}, \\ P_1(t) &= P(t)\left(\int_{t_1}^{\sigma(t)} (D(s)-D(t_1))\, ds\right)^{\gamma}. \end{aligned}$$

To obtain sufficient conditions for the oscillation of solutions to Eq. (2.3.1) when $n = 1$.

Lemma 2.3.1. *Let $0 < \gamma \le 1$. Assume $u_i \ge 0$ for $i = 1, 2$. Then*

$$(u_1 + u_2)^{\gamma} \le u_1^{\gamma} + u_2^{\gamma}$$

Lemma 2.3.2. *Let $\gamma \ge 1$. Assume $u_i \ge 0$ for $i = 1, 2$. Then*

$$(u_1 + u_2)^{\gamma} \le 2^{\gamma-1}(u_1^{\gamma} + u_2^{\gamma})$$

Theorem 2.3.3. *Let $\eta(t) \ge t$ and $0 < \gamma \le 1$. Assume that*

$$\int_{t_1}^{\infty}\int_{v}^{\infty} \frac{1}{r^{1/\gamma}(u)}\Big(\int_{u}^{\infty} P(s)\, ds\Big)^{1/\gamma} du\, dv = \infty, \tag{2.3.2}$$

and the first order DDE

$$w'(t) + \frac{\eta_0}{\eta_0 + p_1^{\gamma}} P_1(t) w(\sigma(t)) = 0 \tag{2.3.3}$$

is oscillatory, then every nonoscillatory solution of Eq. (2.3.1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Corollary 2.3.4. *Let $\eta(t) \ge t$. Assume that $0 < \gamma \le 1$ and* (2.3.2) *holds. Further, assume that*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} P_1(s)\, ds > \frac{\eta_0 + p_1^{\gamma}}{\eta_0\, e}, \tag{2.3.4}$$

then every nonoscillatory solution $x(t)$ of Eq. (2.3.1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

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[2] On the oscillatory behavior of solutions of third order nonlinear neutral differential equations (a part of this Section is published in the *Malaya Journal of Mathematik*, 2019, (In Press)).

Theorem 2.3.5. *Let $\sigma(t) \le \eta(t) \le t$. Assume that $0 < \gamma \le 1$*

$$\int_{t_1}^{t} \eta'(v) \int_{v}^{\infty} \left[\frac{\eta'(u)}{r(\eta(u))} \int_{u}^{\infty} P(s)\, ds \right]^{1/\gamma} du\, dv = \infty \tag{2.3.5}$$

and the first order DDE

$$w'(t) + \frac{\eta_0}{\eta_0 + p_1^\gamma} P_1(t) w(\eta^{-1}(\sigma(t))) = 0, \tag{2.3.6}$$

where $\eta^{-1}(t)$ is inverse function of $\eta(t)$, is oscillatory, then every nonoscillatory solution of Eq. (2.3.1) *satisfies $\lim\limits_{t\to\infty} x(t) = 0$.*

Corollary 2.3.6. *Let $\eta(t) \le t$. Assume that $0 < \gamma \le 1$ and (2.3.5) holds. Further, assume that*

$$\liminf_{t\to\infty} \int_{\eta^{-1}(\sigma(t))}^{t} P_1(s)\, ds > \frac{\eta_0 + p_1^\gamma}{\eta_0\, e}, \tag{2.3.7}$$

then every nonoscillatory solution $x(t)$ of Eq. (2.3.1) *satisfies $\lim\limits_{t\to\infty} x(t) = 0$.*

Remark 2.3.7. *The results of this section, we establish some new oscillation theorems for Eq.* (2.3.1) *in the case* (2.1.2) *where $0 \le p(t) \le p_1 < \infty$. The criteria obtained and extend the results in [11]. If $\gamma \ge 1$, using Lemma 2.3.2, the proof is similar one has to replace $P(t)$ with $P(t)/2^{\gamma-1}$ and proceed as above.*

2.4 Examples

In this section, we give three examples to illustrate our results.

Example 2.4.1. *Consider,*

$$\left(x(t) + \frac{1}{2} x(t - \frac{3\pi}{2}) + \frac{1}{3} x(t + \frac{\pi}{2}) \right)''' + \frac{1}{6} x(t - \pi) = 0, \quad t \ge 1, \tag{2.4.1}$$

where $r(t) = 1$, $\gamma = 1$, $q(t) = 1/6$, $\sigma(t) = t - \pi$, $p_1(t) = 1/2$, $p_2(t) = 1/3$, $\eta_1 = t - 3\pi/2$ and $\eta_2 = t + \pi/2$. It is not hard to verify all conditions of Corollary 2.2.3 are satisfied. Hence, every non-oscillatory solution of (2.4.1) *satisfies $\lim_{t\to\infty} x(t) = 0$.*

Example 2.4.2. *Consider,*

$$\left[e^{-t} \left[\left(x(t) + \frac{1}{3} x(t/2) + \frac{1}{5} x(2t/3) \right)'' \right]^3 \right]' + \frac{e^{2t}}{t^5} x^3(t/2) = 0, \quad t \ge 1, \tag{2.4.2}$$

where $r(t) = e^{-t}$, $\gamma = 3$, $q(t) = e^{2t}/t^5$, $\sigma(t) = t/2$, $p_1(t) = 1/3$, $p_2(t) = 1/5$, $\eta_1 = t/2$ and $\eta_2 = 2t/3$. It is not hard to verify all conditions of Corollary 2.2.5 are satisfied. Hence, every non-oscillatory solution of (2.4.2) *satisfies $\lim_{t\to\infty} x(t) = 0$.*

Example 2.4.3. *Consider the third order nonlinear DE*

$$\left(t^{1/2}\big[\big(x(t)+\frac{1}{2}x(\eta_0 t)\big)''\big]^{1/2}\right)' + \frac{r}{t^{5/2}}x^{1/2}(\beta t) = 0, \tag{2.4.3}$$

$r > 0$, $t \geq 1$, *where* $\gamma = 1/2$, $0 < \beta < 1$, $p(t) = 1/2$, $q(t) = \frac{r}{t^{5/2}}$, $\eta(t) = \eta_0 t$, $\sigma(t) = \beta t$ *and* $r(t) = t^{1/2}$. *If* $\eta_0 \geq 1$, *then* $P(t) = \frac{r}{\eta_0 t^{5/2}}$ *it is easy to see that all conditions of Corollary 2.3.4 are satisfied. If* $\gamma < \eta_0 \leq 1$, *then* $P(t) = \frac{r}{t^{5/2}}$ *it is easy to see that all conditions of Corollary 2.3.6 are satisfied. Hence every solution of equation* (2.4.3) *is either oscillatory or converges to zero as* $t \to \infty$.