Chapter 7

Oscillation Theorems for Third-Order Retarded Differential Equations with a Sublinear Neutral Term

CHAPTER 7

OSCILLATION THEOREMS FOR THIRD-ORDER RETARDED DIFFERENTIAL EQUATIONS WITH A SUBLINEAR NEUTRAL TERM

7.1 Introduction

In the present chapter, we concerned with the third order non-linear NDE's

$$\Big(c_1(t)\big(c_2(t)\left[x(t)+p(t)x^{\alpha}(\tau(t))\right]'\big)'\Big)'+q(t)f(x(\sigma(t)))=0, \tag{7.1.1}$$

and

$$\Big(c_1(t)\big(c_2(t)\left[x(t)+p(t)x^{\alpha}(t-\tau)\right]'\big)'\Big)'+q(t)x(t-\sigma)=0, \tag{7.1.2}$$

where $t \geq t_0 > 0$ and $0 < \alpha \leq 1$ is a ratio of odd natural numbers. Throughout this chapter, without further mention, let us assume

(A_1) $c_i > 0$, $q(t) > 0$, $c_i \in C([t_0, +\infty))$ for $i = 1, 2$ and p, $q \in C([t_0, +\infty))$;

(A_2) $\tau(t) \in C([t_0, +\infty))$,$\tau(t) \leq t$, $\sigma(t) \in C([t_0, +\infty))$, $\sigma(t) \leq t$;

(A_3) f is nondecreasing and $uf(u) \geq k > 0$ for $u \neq 0$ and $\lim\limits_{t\to+\infty} \tau(t) = \lim\limits_{t\to+\infty} \sigma(t) = \infty$.

By a solution of equation (7.1.1) (or (7.1.2)) we understand a nontrivial real valued function $x(t) \in C([T_x, \infty)), T_x \geq t_0$, which satisfies (7.1.1) (or (7.1.2)) on $[T_x, \infty)$. In the sequel, we assume that solutions of $x(t)$ of (7.1.1) (or (7.1.2)) exist for all $T \geq T_x$. A solution of (7.1.1) (or (7.1.2)) is called oscillatory if it does not have the largest zero and otherwise, it is said to be non-oscillatory. Equation (7.1.1) (or (7.1.2)) is called almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

A number of authors including Baculíková and Džurina [9], Candan and Dahiya [19, 20], Graef et al. [46], and Thandapani and Li [90] have studied the oscillatory behavior of solutions of third order NDE's in the form of equation (7.1.1) when $\alpha = 1$.

Recently, Lin and Tang [78] explored the oscillation of first-order NDE with a super-linear neutral term

$$[x(t) - p(t)x^{\alpha}(t-\tau)]' + q(t)\prod_{j=1}^{m}|x(t-\sigma_j)|^{\beta_j} sgn[x(t-\sigma_j)] = 0,$$

where $\alpha > 1$. Agarwal et al. [2] concerned with oscillation of a certain class of second-order NDEs with a sub-linear neutral term

$$\Big(a(t)\left[x(t) + p(t)x^{\alpha}(\tau(t))\right]'\Big)' + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where the positive constant $0 < \alpha \leq 1$ and Thandapani et al.[89] studied oscillatory results of solutions of a second-order nonlinear NDE

$$\Big(a(t)\left[x(t) + p(t)x^{\alpha}(\tau(t))\right]'\Big)' + \; q(t)x^{\beta}(\sigma(t)) = 0,$$

for $t \geq t_0 > 0$ where α and β are positive constants. The above observation shows that this chapter extend the results in third order.

This chapter, further investigation of the oscillations of (7.1.1) and (7.1.2). The following two cases:

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{c_1(s)}ds = \infty, \quad \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{c_2(s)}ds = \infty, \tag{7.1.3}$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{c_1(s)}ds < \infty, \quad \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{c_2(s)}ds = \infty, \tag{7.1.4}$$

are studied.

7.2 Differential Equation with a Sublinear Neutral Term I

[1] In this section, we state and prove our main results of equation (7.1.1). Define

$$N(t) := x(t) + p(t)x^{\alpha}(\tau(t)).$$

[1]On the oscillation of a third order nonlinear differential equations with neutral type (a part of this Section is published in the *Ural Mathematical Journal (UMJ)*, **3** (2) (2017), 122-129).

For convenience, we use the notations

$$p_*(t) = \left(1 - \frac{p(\sigma(t))}{M^{1-\alpha}}\right), \quad G(t) = \frac{\int_{t_2}^{\sigma(t)} \frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} ds}{\int_{t_1}^{t} \frac{1}{c_1(u)} du}. \tag{7.2.1}$$

Theorem 7.2.1. *Let* $0 \le p(t) \le p_1 \le 1$. *If* (7.1.3) *holds, there exists a positive function* $\phi \in C^1([t_0, \infty), \mathbb{R})$*, such that for all* $t_3 > t_2 > t_1 (\ge t_0)$*, we have*

$$\int_{t_3}^{\infty} \left(\phi(s) k q(s) p_*(s) G(s) - \frac{c_1(s)(\phi'(s))^2}{4\phi(s)}\right) ds = \infty, \tag{7.2.2}$$

and

$$\int_{t_0}^{\infty} \frac{1}{c_2(v)} \int_{v}^{\infty} \frac{1}{c_1(u)} \left[\int_{u}^{\infty} q(s) ds\right] du\, dv = \infty, \tag{7.2.3}$$

holds for all constants $M > 0$*, then* (7.1.1) *is almost oscillatory.*

Proof: Assume, for sake of contradiction, that equation (7.1.1) has an eventually positive solution $x(t)$. By condition (7.1.3), there exist two possible cases:

(1) $N(t) > 0$, $N'(t) > 0$, $(c_2(t)N'(t))' > 0$, $(c_1(t)(c_2(t)N'(t))')' < 0$,

(2) $N(t) > 0$, $N'(t) < 0$, $(c_2(t)N'(t))' > 0$, $(c_1(t)(c_2(t)N'(t))')' < 0$, for $t \ge t_1$, t_1 is large enough.

It follows from (7.1.1),

$$\Big(c_1(t)\big(c_2(t)N'(t)\big)'\Big)' = -q(t) f(x(\sigma(t))) \le -kq(t)x(\sigma(t)) < 0. \tag{7.2.4}$$

Assume case (1) holds. If there exists $t \ge t_1$ such that $N(t) > 0$, $N(\sigma(t)) > 0$, $N'(t) > 0$. Then, $N(t)$ is considered monotonically increasing, there exists a constant $M > 0$ such that $N(t) \ge M$ and the definition of N, we have

$$x(t) = N(t) - p(t)x^{\alpha}(\sigma(t)) \ge N(t) - p(t)N^{\alpha}(\sigma(t)) \ge \left(1 - \frac{p(\sigma(t))}{M^{1-\alpha}}\right) N(t) = p_*(t)N(t), \tag{7.2.5}$$

where $p_*(t)$ is defined in (7.2.1). Set

$$\omega(t) = \phi(t) \frac{c_1(t)(c_2(t)N'(t))'}{c_2(t)N'(t)} \tag{7.2.6}$$

and $\omega(t) > 0$ for $t \ge t_1$. Differentiating (7.2.6), we obtain

$$\begin{aligned} \omega'(t) = \phi'(t)\frac{c_1(t)(c_2(t)N'(t))'}{c_2(t)N'(t)} &+ \phi(t)\frac{(c_1(t)(c_2(t)N'(t))')'}{c_2(t)N'(t)} \\ &- \phi(t)\frac{c_1(t)(c_1(t)(c_2(t)N'(t))')(c_2(t)N'(t))'}{(c_2(t)N'(t))^2} \end{aligned} \tag{7.2.7}$$

Since $(c_1(t)(c_2(t)N'(t))')' < 0$ is decreasing, so

$$c_2(t)N'(t) \geq \int_{t_1}^{t} \frac{c_1(s)(c_2(s)N'(s))'}{c_1(s)} ds \geq c_1(t)(c_2(t)N'(t))' \int_{t_1}^{t} \frac{1}{c_1(s)} ds, \qquad (7.2.8)$$

which implies that

$$\left(\frac{c_2(t)N'(t)}{\int_{t_1}^{t} \frac{1}{c_1(s)} ds} \right)' \leq 0. \qquad (7.2.9)$$

Thus,

$$N(t) = N(t_2) + \int_{t_2}^{t} \frac{c_2(s)N'(s)}{\int_{t_1}^{s} \frac{1}{c_1(u)} du} \frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} ds \geq \frac{c_2(t)N'(t)}{\int_{t_1}^{t} \frac{1}{c_1(u)} du} \int_{t_2}^{t} \frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} ds, \qquad (7.2.10)$$

for $t \geq t_2 \geq t_1$.

It follows from (7.1.1), (7.2.5), and (7.2.6) that

$$\omega'(t) \leq \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)c_1(t)} - \phi(t)kq(t)p_*(t)\frac{N(\sigma(t))}{c_2(t)N'(t)}, \qquad (7.2.11)$$

that is,

$$\omega'(t) \leq \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)c_1(t)} - \phi(t)kq(t)p_*(t)\frac{N(\sigma(t))}{c_2(\sigma(t))N'(\sigma(t))} \frac{c_2(\sigma(t))N'(\sigma(t))}{c_2(t)N'(t)} \qquad (7.2.12)$$

which follows from (7.2.9) and (7.2.10) that

$$\begin{aligned}
\omega'(t) &\leq \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)c_1(t)} - \phi(t)kq(t)p_*(t) \frac{\int_{t_2}^{\sigma(t)} \left(\frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} \right) ds}{\int_{t_1}^{\sigma(t)} \frac{1}{c_1(u)} du} \frac{\int_{t_1}^{\sigma(t)} \frac{1}{c_1(u)} du}{\int_{t_1}^{t} \frac{1}{c_1(u)} du} \\
&= \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)c_1(t)} - \phi(t)kq(t)p_*(t) \frac{\int_{t_2}^{\sigma(t)} \left(\frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} \right) ds}{\int_{t_1}^{t} \frac{1}{c_1(u)} du} \\
&\leq -\left[\frac{\omega(t)}{\sqrt{\phi(t)c_1(t)}} - \frac{1}{2}\sqrt{\frac{c_1(t)}{\phi(t)}}\phi'(t) \right]^2 \\
&\qquad -\phi(t)q(t)kp_*(t)G(t) + \frac{c_1(t)(\phi'(t))^2}{4\phi(t)}, \qquad (7.2.13)
\end{aligned}$$

which implies,

$$\omega'(t) \leq -\phi(t)q(t)kp_*(t)G(t) + \frac{c_1(t)(\phi'(t))^2}{4\phi(t)}. \qquad (7.2.14)$$

Integrating the last inequality from t_3 $(> t_2)$ to t gives

$$\int_{t_3}^{t} \left(\phi(s)q(s)kp_*(s)G(s) - \frac{c_1(s)(\phi'(s))^2}{4\phi(s)} \right) ds \leq \omega(t_3), \qquad (7.2.15)$$

letting $t \to \infty$, which contradicts to (7.2.2).

Next, if case (2) holds. Using the similar proof of [[9], Lemma 2], we can get $\lim_{t\to\infty} x(t) = 0$ due to condition (7.2.3).

Theorem 7.2.2. *Let $0 \le p(t) \le p_1 \le 1$. If* (7.1.4) *holds and there exists a positive function $\varphi \in C^1([t_0, \infty), \mathbb{R})$, such that for all $t_3 > t_2 > t_1 (\ge t_0)$, one has* (7.2.2) *and* (7.2.3). *If*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left(\delta(s)q(s)kp_*(s) \left(\int_{t_1}^{\sigma(s)} \frac{dv}{c_2(v)} \right) - \frac{1}{4\delta(s)c_1(s)} \right) ds = \infty, \quad (7.2.16)$$

where

$$\delta(t) := \int_{t}^{\infty} \frac{1}{c_1(s)} ds, \quad (7.2.17)$$

holds for all constants $M > 0$, then (7.1.1) *is almost oscillatory.*

Proof: Assume, for sake of contradiction, that equation (7.1.1) has an eventually positive solution $x(t)$. By condition (7.1.4), there exist case (1), case (2) and

(3) $N(t) > 0$, $N'(t) > 0$, $(c_2(t)N'(t))' < 0$, $(c_1(t)(c_2(t)N'(t))')' < 0$, for $t \ge t_1$, t_1 is large enough.

Assume that case (1) and case (2) holds, respectively. We can obtain the conclusion of Theorem 7.2.2 by applying the proof of Theorem 7.2.1. Assume that case (3) holds, $(c_1(t)(c_2(t)N'(t))')' < 0$ and $c_1(t)(c_2(t)N'(t))'$ is nonincreasing. Thus, we get

$$c_1(s)(c_2(s)N'(s))' \le c_1(t)(c_2(t)N'(t))', \quad s \ge t \ge t_1. \quad (7.2.18)$$

Dividing the above inequality by $c_1(s)$ and integrating from t to l, we obtain

$$c_2(l)N'(l) \le c_2(t)N'(t) + c_1(t)(c_2(t)N'(t))' \int_{t}^{l} \frac{1}{c_1(s)} ds. \quad (7.2.19)$$

Letting $l \to \infty$, we have

$$0 \le c_2(t)N'(t) + c_1(t)(c_2(t)N'(t))' \int_{t}^{\infty} \frac{1}{c_1(s)} ds, \quad (7.2.20)$$

that is,

$$-\frac{c_1(t)(c_2(t)N'(t))'}{c_2(t)N'(t)} \int_{t}^{\infty} \frac{ds}{c_1(s)} \le 1. \quad (7.2.21)$$

Now define φ by,

$$\varphi(t) := \frac{c_1(t)(c_2(t)N'(t))'}{c_2(t)N'(t)}, \quad t \ge t_1. \quad (7.2.22)$$

Then, $\varphi(t) < 0$ for $t \ge t_1$. Therefore, by (7.2.21) and (7.2.22), we obtain

$$-\delta(t)\varphi(t) \le 1. \quad (7.2.23)$$

Differentiating (7.2.22) gives,

$$\varphi'(t) = \frac{(c_1(t)(c_2(t)N'(t))')'}{c_2(t)N'(t)} - \frac{c_1(t)(c_2(t)N'(t))'(c_2(t)N'(t))'}{(c_2(t)N'(t))^2}. \tag{7.2.24}$$

Now $N'(t) > 0$, so from (7.1.1) and (7.2.5),

$$\varphi'(t) \leq -q(t)kp_*(t)\frac{N(\sigma(t))}{c_2(t)N'(t)} - \frac{c_1(t)(c_2(t)N')'(c_2(t)N'(t))'}{(c_2(t)N'(t))^2}. \tag{7.2.25}$$

In view of case (3), we see that

$$N(t) \geq c_2(t)\int_{t_1}^{t} \frac{ds}{c_2(s)}N'(t). \tag{7.2.26}$$

Thus, $\left(\frac{N(t)}{\int_{t_1}^{t}\frac{ds}{c_2(s)}}\right)' \leq 0$ implies

$$\frac{N(\sigma(t))}{N(t)} \geq \frac{\int_{t_1}^{\sigma(t)}\frac{1}{c_2(s)}ds}{\int_{t_1}^{t}\frac{1}{c_2(s)}ds}. \tag{7.2.27}$$

From (7.2.22) and (7.2.25)-(7.2.27),

$$\varphi'(t) \leq -q(t)kp_*(t)\int_{t_1}^{\sigma(t)}\frac{ds}{c_2(s)} - \frac{\varphi^2(t)}{c_1(t)}.$$

Multiplying the above inequality by $\delta(t)$ and integrating it from $t_2(> t_1)$ to t,

$$\varphi(t)\delta(t) - \varphi(t_2)\delta(t_2) + \int_{t_2}^{t}\delta(s)kq(s)p_*(s)\Big(\int_{t_1}^{\sigma(s)}\frac{dv}{c_2(v)}\Big)ds + \int_{t_2}^{t}\frac{\varphi^2(s)\delta(s)}{c_1(s)}ds + \int_{t_2}^{t}\frac{\varphi(s)}{c_1(s)}ds \leq 0,$$

$$\varphi(t)\delta(t) - \varphi(t_2)\delta(t_2) + \int_{t_2}^{t}\delta(s)kq(s)p_*(s)\Big(\int_{t_1}^{\sigma(s)}\frac{dv}{c_2(v)}\Big)ds - \int_{t_2}^{t}\frac{1}{4\delta(s)c_1(s)} + \int_{t_2}^{t}\left[\sqrt{\frac{\delta(s)}{c_1(s)}}\varphi(s) + \frac{1}{2}\frac{1}{\sqrt{c_1(s)\delta(s)}}\right]^2 ds \leq 0,$$

from which follows that

$$\int_{t_2}^{t}\left(\delta(s)q(s)kp_*(s)\left(\int_{t_1}^{\sigma(s)}\frac{dv}{c_2(v)}\right) - \frac{1}{4\delta(s)c_1(s)}\right)ds \leq 1 + \varphi(t_2)\delta(t_2) \tag{7.2.28}$$

due to (7.2.23). Letting $t \to \infty$, which contradicts to (7.2.16). Then the result follows.

7.3 Differential Equation with a Sublinear Neutral Term II

[2] In this section, we present sufficient conditions for the oscillation of all solutions of equation (7.1.2). Define

$$\hat{N}(t) := x(t) + p(t)x^{\alpha}(t-\tau).$$

Theorem 7.3.1. *Let* $0 \le p(t) \le p_1 \le 1$. *If* (7.1.4) *holds and there exists a positive function* $\rho \in C^1([t_0,\infty),\mathbb{R})$, *such that for all* $t_3 > t_2 > t_1(\ge t_0)$, *we have*

$$\int_{t_3}^{\infty}\left(\rho(s)q(s)\psi(s-\sigma)\frac{\int_{t_2}^{s-\sigma}\left(\frac{\int_{t_1}^{v}\frac{1}{c_1(u)}du}{c_2(u)}\right)dv}{\int_{t_1}^{s}\frac{1}{c_1(u)}du} - \frac{c_1(s)(\rho'(s))^2}{4\rho(s)}\right)ds = \infty, \tag{7.3.1}$$

$$\int_{t_0}^{\infty}\frac{1}{c_2(v)}\int_{v}^{\infty}\frac{1}{c_1(u)}\int_{u}^{\infty}q(s)ds\,du\,dv = \infty, \tag{7.3.2}$$

and

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\delta(s)q(s)\psi(s-\sigma)\int_{t_1}^{s-\sigma}\frac{dv}{c_2(v)} - \frac{1}{4\delta(s)c_1(s)}\right)ds = \infty, \tag{7.3.3}$$

where

$$\delta(t) := \int_{t}^{\infty}\frac{1}{c_1(s)}ds, \quad \psi(t) = 1 - \frac{p(t)}{M^{1-\alpha}} \tag{7.3.4}$$

holds for all constants $M > 0$, *then* (7.1.2) *is almost oscillatory.*

Proof. Assume, for sake of contradiction, that equation (7.1.2) has an eventually positive solution $x(t)$. By condition (7.1.4) and from Theorem 7.2.2, there exist three cases. Assume case (i) holds, there exists $t \ge t_1$ such that $\hat{N}(t) > 0$, $\hat{N}(t-\tau) > 0$, $\hat{N}(t-\sigma) > 0$, $\hat{N}'(t) > 0$. Then, $\hat{N}(t)$ is considered monotonically increasing, there exists a constant $M > 0$ such that $\hat{N}(t) \ge M$ and the definition of $\hat{N}$, we have

$$x(t) = \hat{N}(t) - p(t)x^{\alpha}(t-\sigma) \ge \hat{N}(t) - p(t)\hat{N}^{\alpha}(t-\sigma) \ge \left(1 - \frac{p(t)}{M^{1-\alpha}}\right)\hat{N}(t) \tag{7.3.5}$$

The function ω is defined as follows,

$$\omega(t) = \rho(t)\frac{c_1(t)(c_2(t)\hat{N}'(t))'}{c_2(t)\hat{N}'(t)}, \quad t \ge t_1. \tag{7.3.6}$$

[2] Oscillation theorems for third-order retarded differential equations with a sublinear neutral term (a part of this Section is published in the *International Journal of Pure and Applied Mathematics*, **114** (5) (2017), 63-71).

and note that $\omega(t) > 0$ for $t \geq t_1$. Since

$$c_2(t)\hat{N}'(t) \geq \int_{t_1}^{t} \frac{c_1(s)(c_2(s)\hat{N}'(s))'}{c_1(s)} ds \geq c_1(t)(c_2(t)\hat{N}'(t))' \int_{t_1}^{t} \frac{1}{c_1(s)} ds, \quad (7.3.7)$$

we have that

$$\left(\frac{c_2(t)\hat{N}'(t)}{\int_{t_1}^{t} \frac{1}{c_1(s)} ds} \right)' \leq 0. \quad (7.3.8)$$

Thus, we get

$$\hat{N}(t) = \hat{N}(t_2) + \int_{t_2}^{t} \frac{c_2(s)\hat{N}'(s)}{\int_{t_1}^{s} \frac{1}{c_1(u)} du} \frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} ds \geq \frac{c_2(t)\hat{N}'(t)}{\int_{t_1}^{t} \frac{1}{c_1(u)} du} \int_{t_2}^{t} \frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} ds, \quad (7.3.9)$$

for $t \geq t_2 \geq t_1$. Differentiating (7.3.6), we obtain

$$\begin{aligned} \omega(t) = \rho'(t) \frac{c_1(t)(c_2(t)\hat{N}'(t))'}{c_2(t)\hat{N}'(t)} &+ \rho(t) \frac{c_1(t)(c_2(t)\hat{N}'(t))'}{c_2(t)\hat{N}'(t)} \\ &- \rho(t) \frac{c_1(t)(c_1(t)(c_2(t)\hat{N}')'(c_2(t)\hat{N}'(t))'}{(c_2(t)\hat{N}'(t))^2}. \end{aligned} \quad (7.3.10)$$

It follows from (7.1.2), (7.3.5), and (7.3.6) that

$$\begin{aligned} \omega'(t) \leq\ & \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\omega^2(t)}{\rho(t)c_1(t)} \\ & -\rho(t)q(t)\psi(s-\sigma) \frac{\hat{N}(t-\sigma)}{c_2(t-\sigma)\hat{N}'(t-\sigma)} \frac{c_2(t-\sigma)\hat{N}'(t-\sigma)}{c_2(t)\hat{N}'(t)}, \end{aligned} \quad (7.3.11)$$

which follows from (7.3.8) and (7.3.9) that

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t)q(t)\psi(s-\sigma) \frac{\int_{t_2}^{t-\sigma} \frac{\int_{t_1}^{s} \frac{1}{c_1(u)} du}{c_2(s)} ds}{\int_{t_1}^{t} \frac{1}{c_1(u)} du} + \frac{c_1(t)(\rho'(t))^2}{4\rho(t)}.$$

Integrating the last inequality from t_3 $(> t_2)$ to t gives

$$\int_{t_3}^{t} \left(\rho(s)q(s)\psi(s-\sigma) \frac{\int_{t_2}^{t-\sigma} \frac{\int_{t_1}^{v} \frac{1}{c_1(u)} du}{c_2(v)} dv}{\int_{t_1}^{t} \frac{1}{c_1(u)} du} - \frac{c_1(t)(\rho'(t))^2}{4\rho(t)} \right) ds \leq \omega(t_3),$$

which contradicts (7.3.1).

Next, assume that case (2) holds. Using the similar proof of [[9], Lemma 2], we can get $\lim_{t\to\infty} x(t) = 0$ due to condition (7.3.2).

Finally, Assume that case (3) holds, $(c_1(t)(c_2(t)\hat{N}'(t))')' < 0$, $c_1(t)(c_2(t)\hat{N}'(t))'$ is nonincreasing. Thus, we get

$$c_1(s)(c_2(s)\hat{N}'(s))' \leq c_1(t)(c_2(t)\hat{N}'(t))', \quad s \geq t \geq t_1. \quad (7.3.12)$$

Dividing the above inequality by $a(s)$ and integrating from t to l, we obtain

$$c_2(l)\hat{N}'(l) \leq c_2(t)\hat{N}'(t) + c_1(t)(c_2(t)\hat{N}'(t))' \int_t^l \frac{1}{c_1(s)} ds. \tag{7.3.13}$$

Letting $l \to \infty$, we have

$$0 \leq c_2(t)\hat{N}'(t) + c_1(t)(c_2(t)\hat{N}'(t))' \int_t^\infty \frac{1}{c_1(s)} ds, \tag{7.3.14}$$

that is,

$$-\frac{c_1(t)(c_2(t)\hat{N}'(t))'}{c_2(t)\hat{N}'(t)} \int_t^\infty \frac{ds}{c_1(s)} \leq 1. \tag{7.3.15}$$

Now define ϕ by,

$$\phi(t) := \frac{c_1(t)(c_2(t)\hat{N}'(t))'}{c_2(t)\hat{N}'(t)}, \quad t \geq t_1. \tag{7.3.16}$$

Then, $\phi(t) < 0$ for $t \geq t_1$. Hence, by (7.3.15) and (7.3.16), we obtain

$$-\delta(t)\phi(t) \leq 1. \tag{7.3.17}$$

Differentiating (7.3.16) gives,

$$\phi' = \frac{c_1(t)(c_2(t)\hat{N}'(t))'}{c_2(t)\hat{N}'(t)} - \frac{c_1(t)(c_1(t)(c_2(t)\hat{N}')'(c_2(t)\hat{N}'(t))'}{(c_2(t)\hat{N}'(t))^2}. \tag{7.3.18}$$

Now $\hat{N}'(t) > 0$, so from (7.1.2) and (7.3.5), we have

$$\phi'(t) \leq -q(t)\psi(s-\sigma)\frac{\hat{N}(t-\sigma)}{c_2(t)\hat{N}'(t)} - \frac{c_1(t)(c_1(t)(c_2(t)\hat{N}')'(c_2(t)\hat{N}'(t))'}{(c_2(t)\hat{N}'(t))^2}. \tag{7.3.19}$$

In view of case (3), we see that

$$\hat{N}(t) \geq c_2(t) \int_{t_1}^t \frac{ds}{c_2(s)} \hat{N}'(t). \tag{7.3.20}$$

Hence, $\left(\frac{\hat{N}(t)}{\int_{t_1}^t \frac{ds}{c_2(s)}} \right)' \leq 0$, which implies that

$$\frac{\hat{N}(t-\sigma)}{\hat{N}(t)} \geq \frac{\int_{t_1}^{t-\sigma} \frac{1}{c_2(s)} ds}{\int_{t_1}^t \frac{1}{c_2(s)} ds}. \tag{7.3.21}$$

From (7.3.16) and (7.3.19)-(7.3.21), we obtain

$$\phi'(t) \leq -q(t)\psi(s-\sigma) \int_{t_1}^{t-\sigma} \frac{ds}{c_2(s)} - \frac{\phi^2(t)}{c_1(t)}.$$

Multiplying the above inequality by $\delta(t)$ and integrating it from $t_2(> t_1)$ to t, we have

$$\phi(t)\delta(t) - \phi(t_2)\delta(t_2) + \int_{t_2}^{t} \delta(s)q(s)\psi(s-\sigma)\int_{t_1}^{s-\sigma}\frac{dv}{c_2(v)}ds + \int_{t_2}^{t}\frac{\phi^2(s)\delta(s)}{c_1(s)}ds + \int_{t_2}^{t}\frac{\phi(s)}{c_1(s)}ds \le 0,$$

from which follows that

$$\int_{t_2}^{t}\left(\delta(s)q(s)\psi(s-\sigma)\int_{t_1}^{s-\sigma}\frac{dv}{c_2(v)} - \frac{1}{4\delta(s)c_1(s)}\right)ds \le 1 + \phi(t_2)\delta(t_2)$$

due to (7.3.17). This contradicts (7.3.3) and completes the proof of the theorem. □

7.4 Examples

In this section, we will present some examples to illustrate the main results

Example 7.4.1. *Consider a third-order NDE*

$$\left(t^{-1/2}\left(t^{1/2}\left[x(t) + \frac{1}{4}x^{3/5}(t-1)\right]'\right)'\right)' + \frac{\lambda}{t^{1/2}}x(t-2) = 0, \quad t \ge 1, \tag{7.4.1}$$

where $\lambda > 0$ *is a constant. Let* $\alpha = 3/5$, $c_1(t) = t^{-1/2}$, $c_2(t) = t^{1/2}$, $p(t) = 1/4$, $q(t) = \frac{\lambda}{t^{1/2}}$, $\tau(t) = t-1$, *and* $\sigma(t) = t-2$. *We obtain* $p_*(t) = 1 - \frac{1/4}{M^{2/5}}$,

$$\int_{t_0}^{\infty}\frac{1}{c_2(v)}\int_{v}^{\infty}\frac{1}{c_1(u)}\left[\int_{u}^{\infty}\frac{\lambda}{s^{1/2}}ds\right]du\,dv = \infty.$$

and

$$G(t) = \frac{\int_{t_2}^{\sigma(t)}\left(\frac{\int_{t_1}^{s}\frac{1}{c_1(u)}du}{c_2(s)}\right)ds}{\int_{t_1}^{t}\frac{1}{c_1(u)}du} = \frac{(t-2)^3 + 6(t-2)^{1/2}t_1^{1/2} - 3c}{t^{3/2} - t_1^{3/2}},$$

where $c = \frac{t_2^3}{3} + 2t_1^{3/2}t_2^{1/2}$. *Pick* $\phi(t) = 1$, *then*

$$\int_{t_3}^{\infty} q(s)p_*(s)G(s)ds = \frac{\lambda}{3}\Big(1 - \frac{1/4}{M^{2/5}}\Big)\int_{t_3}^{\infty}\frac{(s-2)^3 + 6(s-2)^{1/2}t_1^{1/2} - 3c}{s^2 - s^{1/2}t_1^{3/2}} = \infty,$$

if $\frac{1}{4} < M^{2/5}$. *Hence, by Theorem 7.2.1, every solution of equation* (7.4.1) *is either oscillatory or converges to zero as* $t \to \infty$ *when* $\frac{1}{4} < M^{2/5}$.

Example 7.4.2. *Consider a third-order NDE*

$$\left(t^2\left[x(t) + \frac{1}{2}x^{1/3}(t/8)\right]''\right)' + \frac{1}{t}\Big(1 + \frac{2}{27}t^{2/3}\Big)x(t/2) = 0 \quad, t \ge 1. \tag{7.4.2}$$

Let $\alpha = 1/3$, $c_1(t) = t^2$, $c_2(t) = 1$, $p(t) = \frac{1}{2}$, $q(t) = \frac{1}{t}\Big(1 + \frac{2}{27}t^{2/3}\Big)$, $\tau(t) = t/8$ *and* $\sigma(t) = t/2$. *We obtain* $p_*(t) = 1 - \frac{1/2}{M^{2/3}}$, $\delta(t) = \int_t^\infty \frac{1}{s^2}ds = \frac{1}{t}$ *and* $\int_{t_1}^{t/2} \frac{ds}{c_2(s)} = \frac{1}{2}(t - 2t_1)$, *then*

$$\int_{t_2}^{\infty} \left(\delta(s)q(s)p_*(s) \left(\int_{t_1}^{\sigma(s)} \frac{dv}{c_2(v)} \right) - \frac{1}{4\delta(s)c_1(s)} \right) ds$$
$$= \int_{t_2}^{\infty} \left(\left(1 - 0.5M^{-2/3}\right) \left[\frac{1}{2s} + \frac{1}{27}s^{-1/3} - \frac{t_1}{s^2} - \frac{2t_1}{27}s^{-4/3} \right] - \frac{1}{4s} \right) ds = \infty,$$

if $0.5 < M^{2/3}$. *Hence, by Theorem 7.2.2, every solution of equation* (7.4.2) *is either oscillatory or converges to zero as* $t \to \infty$ *when* $0.5 < M^{2/3}$ *and* $x(t) = t^{-1}$ *is a such solution of* (7.4.2).

7.5 Conclusion

In this chapter, we have used Riccati substitution techniques, integral averaging technique and some new oscillation and asymptotic theorems for (7.1.1) and (7.1.2) under the conditions (7.1.3) and (7.1.4) have been established. Our result improves and complement results in the cited papers. It is interesting to find a method for the case when

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{c_1(s)} ds < \infty \quad \text{and} \quad \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{c_2(s)} ds < \infty,$$

to study some sufficient conditions which guarantee that every solution of (7.1.1) (or (7.1.2)) is oscillatory.