CHAPTER 4

FLEXURAL VIBRATION OF PIEZOCOMPOSITE SOLID CYLINDER

4.1 INTRODUCTION


A general frequency equation is derived for flexural vibration of an infinite laminated inner solid and outer hollow cylinder. Both the inner and outer surfaces are traction free and connected with electrodes and that are shorted. Numerical calculations are carried out for PZT4/CFRP/PZT4. The attenuation effect is considered through the imaginary part of the dimensionless complex frequency (Sinha et al 1992).

4.2 FUNDAMENTAL EQUATIONS AND METHOD OF ANALYSIS

The cylindrical polar coordinate system \((r, \theta, z)\) is used with the z-axis along the axis of the piezocomposite cylinder. In the flexural mode of vibration, the displacement components \(u^\ell, v^\ell, w^\ell\) and the potentials \(\phi^\ell\) are functions of \(r, \theta\) and \(z\). The superscripts \(\ell = 1, 2\) are taken for the inner solid and the outer hollow piezoelectric cylinders respectively.

The governing equations for hexagonal (6mm) class are Paul (1966)

\[
\begin{align*}
    c_{11}^\ell (u_{rr}^\ell + r^{-1} u_{,r}^\ell - r^{-2} u^\ell) + c_{66}^\ell r^{-2} u_{,00}^\ell + c_{44}^\ell u_{,zz}^\ell + (c_{66}^\ell + c_{12}^\ell) r^{-1} v^\ell_{,00} &
    - (c_{66}^\ell + c_{11}^\ell) r^{-2} v^\ell_{,00} + (c_{44}^\ell + c_{13}^\ell) w^\ell_{,zz} + (e_{31}^\ell + e_{15}^\ell) \phi^\ell_{,zz} = \rho^\ell u_{,tt}^\ell \\
    (c_{66}^\ell + c_{12}^\ell) r^{-1} u_{,r0}^\ell + (c_{11}^\ell + c_{66}^\ell) r^{-2} u_{,0}^\ell + c_{66}^\ell (v^\ell_{,tt} + r^{-1} v^\ell_{,zz} - r^{-2} v^\ell) &
    - c_{11}^\ell r^{-2} v_{,00}^\ell + c_{44}^\ell v_{,zz}^\ell + (c_{44}^\ell + c_{13}^\ell) r^{-1} w_{,oz}^\ell + (e_{31}^\ell + e_{15}^\ell) r^{-1} \phi^\ell_{,oz} = \rho^\ell v^\ell_{,tt} \\
\end{align*}
\]
\[ (c_{44}^t + c_{11}^t)(u_{zz}^t + r^{-1}u_{x}^t + r^{-1}v_{0y}^t) + c_{44}^t (w_{rr}^t + r^{-1}w_{x}^t + r^{-2}w_{00}^t) \\
+ c_{33}^t w_{zz}^t + e_{15}^t (\phi_{rr}^t + r^{-1}\phi_{x}^t + r^{-2}\phi_{00}^t) + e_{33}^t \phi_{zz}^t = \rho^t w_{tt}^t \]

and

\[ (e_{31}^t + e_{15}^t)(u_{zz}^t + r^{-1}u_{x}^t + r^{-1}v_{0y}^t) + e_{15}^t (w_{rr}^t + r^{-1}w_{x}^t + r^{-2}w_{00}^t) \\
+ e_{33}^t w_{zz}^t - e_{11}^t (\phi_{rr}^t + r^{-1}\phi_{x}^t + r^{-2}\phi_{00}^t) - e_{33}^t \phi_{zz}^t = 0 \]

(4.1)

Here  
\( c_{ij}^t \) - Elastic constants  
\( e_{ij}^t \) - Piezoelectric constants  
\( \varepsilon_{ij}^t \) - Dielectric constants  
\( \rho^t \) - Density of the materials

The comma followed by superscripts denotes the partial differentiation with respect to those variables and t is the time.

The solution of equation (4.1) is taken in the form.

\[ u^t = (u_{x}^t + r^{-1}v_{0}^t) \exp i(kz + pt) \]
\[ v^t = (r^{-1}u_{0}^t - v_{x}^t) \exp i(kz + pt) \]
\[ w^t = (i/a)w^t \exp i(kz + pt) \]
\[ \phi^t = i(c_{44}^t / ae_{33}^t)\phi^t \exp i(kz + pt) \]

(4.2)

where p is angular frequency, k wave number and ‘a’ is the radius of the cylinder.
Substituting equation (4.2) along with the dimensionless variables

\[ x = r/a \] and \[ \varepsilon = ka(k=2\pi/wavelength) \] in equation (4.1) yield the following equation for the inner and outer cylinder.

\[
\begin{vmatrix}
\bar{c}^{ij} V^2 + F_1' & -F_2' & -F_3' \bar{c}_{44}' \\
F_2' V^2 & \bar{c}_{44}' V^2 + F_4' & (\bar{c}_{13}' V^2 - F_5') \bar{c}_{44}' \\
F_3' V^2 & \bar{c}_{13}' V^2 - F_5' & -(k_{13}'^{-2})' V^2 + F_6'
\end{vmatrix}
\]

\[ (u', w', \phi') = 0 \quad (4.3) \]

\[ [\bar{c}_{66}' V^2 + F_1'] v' = 0 \quad (4.4) \]

where

\[ \bar{c}_{ij}' = \frac{c_{ij}'}{c_{44}'}, \quad \bar{c}_{ij} = \frac{e_{ij}'}{e_{33}'} \]

\[ (k_{13}')' = \frac{(e_{33}')'}{(c_{44}'e_{11}')}, \quad c a^2 = \frac{\rho^1 \rho^2}{c_{44}'} a^2 \]

And

\[ V^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} \]

\[ F_1' = \frac{\rho^1'}{\rho^1} (ca)^2 - \varepsilon^2 \bar{c}_{44}', \quad F_2' = \varepsilon (\bar{c}_{44}' + \bar{c}_{13}') \]

\[ F_3' = \varepsilon (\bar{c}_{31}' + \bar{c}_{13}'), \quad F_4' = \frac{\rho^1'}{\rho^1} (ca)^2 - (\bar{c}_{33}' \varepsilon^2) \]

\[ F_5' = \varepsilon^2, \quad F_6' (k_{33}'^{-2})' \varepsilon^2 \]

Equation (4.3) can be expressed as

\[ (V^6 + P'V^4 + Q'V^2 + R')(u', w', \phi') = 0 \quad (4.5) \]
where \( P' = c_1 (k_{13}^{-2})' F_4 - c_4 F_6 - 2\varepsilon' F_5 c_1 (k_{13}^{-2})' + c_4 (k_{13}^{-2})' + (\varepsilon_1')^2 \)
+ \( F_2 [(k_{13}^{-2})' F_2 + c_4 (\varepsilon_1') F_4'] / c_4 (\varepsilon_1') (k_{13}^{-2})' + (\varepsilon_1')^2 \)

\[ Q' = (c_1 (\varepsilon_1')^2 - F_4') + F_1 [(k_{13}^{-2})' F_4 - c_4 (k_{13}^{-2})' F_6 - 2\varepsilon' F_5 c_1 (k_{13}^{-2})' + (\varepsilon_1')^2 \]

\[ -F_2 (F_2 F_6 c_4 (\varepsilon_1') - c_4 F_5 (F_2 F_5 + F_3 F_4') / c_1 (\varepsilon_1') (k_{13}^{-2})' + (\varepsilon_1')^2 \]

\[ R' = F_1 [c_4 (\varepsilon_1')^2 - F_4 F_6'] / c_1 (\varepsilon_1') (k_{13}^{-2})' + (\varepsilon_1')^2 \]  
(4.6)

The solutions of the Eqn (4.5) are taken in terms of Bessel functions of first kind \( (J_n) \) and second kind \( (Y_n) \) to suit the wave propagation in cylindrical polar coordinate system as

\[ u' = \sum_{j=1}^{3} [A_j J_n (\alpha_j ax) + B_j y_n (\alpha_j ax)] \cos n\theta \]

\[ w' = \sum_{j=1}^{3} d_j [A_j J_n (\alpha_j ax) + B_j y_n (\alpha_j ax)] \cos n\theta \]

\[ \phi' = \sum_{j=1}^{3} e_j [A_j J_n (\alpha_j ax) + B_j y_n (\alpha_j ax)] \cos n\theta \]  
(4.7)

Here \( (\alpha_j a)^2 \) are the non-zero roots of

\[ (\alpha_j a)^6 - P' (\alpha_j a)^4 + Q' (\alpha_j a)^2 - R' = 0 \]  
(4.8)

The arbitrary constants \( d_j' \) and \( e_j' \) are given by

\[ F_2 d_j' + F_3 c_4 e_j' = F_1' - c_1^{-1} (\alpha_j a)^2 \]

\[ -[e_1 (\alpha_j a)^2 + F_5'] d_j' + [k_{13}^{-2} (\alpha_j a)^2 + F_6'] e_j' = F_3' (\alpha_j a)^2 \]  
(4.9)

Also for equation (4.4)
\[ v^e = [A'_4 J_n (\alpha_4' ax) + B'_4 y_n (\alpha_4' ax)] \sin \theta \]

where \( (\alpha_4 ax)^2 = \frac{F'_{\theta}}{c_{66}} \) \hspace{1cm} (4.10)

For isotropic materials, the governing equations are

\[ \mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla \nabla \vec{u} = \rho \vec{u}, \quad \mu \nabla \nabla \vec{u} = \rho \vec{u}, \quad \mu \nabla \nabla \vec{u} = \rho \vec{u} \] \hspace{1cm} (4.11)

where \( \vec{u} \) is the displacement vector

\[ \lambda = c_{12}, \quad \mu = (c_{11} - c_{12}) / 2 \] are Lame’s constants

\( \rho \) is the mass density and \( t \) is the time.

The solution of equation (4.11) is taken as

\[ u = (u_r + r^{-1} v_\theta) \exp i(kz + pt) \]
\[ v = (r^{-1} u_\theta - v_r) \exp i(kz + pt) \]
\[ w = (i/a^2) w \exp i(kz + pt) \] \hspace{1cm} (4.12)

Using the solution in equation (4.12) and the dimensionless variables \( x \) and \( \varepsilon \), equation (4.11) can be simplified as

\[ \begin{vmatrix} (\lambda + 2\mu) \nabla^2 + F_1 & -F_2 \\ F_2 \nabla^2 & \mu \nabla^2 + F_3 \end{vmatrix} (u, w) = 0 \] \hspace{1cm} (4.13)

\[ (\mu \nabla^2 + F_3) v = 0 \] \hspace{1cm} (4.14)

where \( \lambda = \lambda / c_{44}^t, \mu = \mu / c_{44}^t \)

\[ F_1 = (\rho / \rho') (ca)^2 - \mu \varepsilon^2 \]
\[ F_2 = (\lambda + \mu)\epsilon, \quad F_4 = (\rho/\rho^1)(\kappa_a)^2 - (\lambda + 2\mu)\epsilon^2 \]

The equation (4.11) can be written as

\[
(\nabla^4 + PV^2 + Q)(u, w) = 0 \tag{4.15}
\]

where

\[
P = [(\lambda + 2\mu)F_4 + \mu F_1 + (F_2)^2]/(\lambda + 2\mu)\mu
\]

\[
Q = [F_1 F_4]/(\lambda + 2\mu)\mu \tag{4.16}
\]

Similar to the assumption of solution of Eqn. (4.5), the solutions of the Eqn. (4.13) are taken as

\[
u = \sum_{j=1}^{3} d_j [A_{j} J_n (\alpha_{j} ax) + B_{j} y_n (\alpha_{j} ax)] \sin n\theta \tag{4.20}
\]

where \((\alpha_{j} a)^2\) is the non zero roots of

\[
(\alpha a)^4 + P(\alpha a)^2 - Q = 0 \tag{4.18}
\]

And the arbitrary constants \(d_j\) are obtained from

\[
d_j = [-(\lambda + 2\mu)(\alpha_{j} a)^2 + F_1]/F_2 \tag{4.19}
\]

The solution of equation (4.14) is

\[
v = [A_{4} J_n (\alpha_{4} ax) + B_{4} J_n (\alpha_{4} ax)] \sin n\theta
\]

where \((\alpha_{4} a)^2 = F_1/\mu \tag{4.19} \]

\[
(\alpha_{4} a)^2 = F_1/\mu \tag{4.21} \]
4.3 BOUNDARY – INTERFACE CONDITIONS AND FREQUENCY EQUATIONS

The frequency equations can be obtained using the following boundary and interface conditions.

i) The outer surface of the cylinder is traction free

\[ T_{rr}^2 = 0 \]
\[ T_{r\theta}^2 = 0 \]
\[ T_{rz}^2 = 0 \]
\[ \phi^2 = 0 \]

ii) At the interface between (outer and middle) cylinders

\[ T_{rr}^2 = T_{rr} \]
\[ T_{r\theta}^2 = T_{r\theta} \]
\[ T_{rz}^2 = T_{rz} \]
\[ U^2 = U \]
\[ V^2 = V \]
\[ W^2 = W \]
\[ \phi^2 = 0 \]

iii) At the interface between (middle and inner) cylinders

\[ T_{rr} = T_{rr}^1 \]
\[ T_{r\theta} = T_{r\theta}^1 \]
\[ T_{rz} = T_{rz}^1 \]
\[ U = U^1 \]
\[ V = V^1 \]
\[ W = W^1 \]
\[ \phi^1 = 0 \]
The frequency equation is obtained as $18 \times 18$ determinant equation by substituting the solutions (4.3), (4.4), (4.7) and (4.10) in the boundary interface conditions. It is written as,

$$|D(i,j)| = 0, \quad (i, j = 1, 2\ldots18) \quad (4.22)$$

and the non-zero elements by varying $j$ from 1 to 3 and $i$ varies from 1 to 2 are

$$D(1, j) = 2\overline{c}_{i6}^j (\alpha_j^1 a / x_1) J_{n+1}(\alpha_j^1 a x_1) + [2\overline{c}_{66}^j n(n-1)/(x_1)^2 - \overline{c}_{i1}^j (\alpha_j^1 a)^2$$

$$- \overline{c}_{i1}^j \varepsilon d_j - \overline{c}_{i1}^j \varepsilon e_j] J_n(\alpha_j^1 a x_1)$$

$$D(1, 4) = -2n\overline{c}_{66}^4 (\alpha_4^1 a / x_1) J_{n+1}(\alpha_4^1 a x_1) + [2\overline{c}_{66}^4 n(n-1)/(x_1)^2] J_n(\alpha_4^1 a x_1)^2$$

$$D(1,i+4) = -[2\overline{\mu}(\alpha_j^1 a / x_1) J_{n+1}(\alpha_j^1 a x_1) + [-\overline{\lambda} + 2\overline{\mu})(\alpha_j^1 a)^2 + e_j$$

$$- \overline{\lambda}(\alpha_j^1 a / x_1) + 2n(n-1) (\overline{\mu}/(x_1)^2] J_n(\alpha_j^1 a x_1)$$

$$D(1, 7) = 2n\overline{\mu}[(\alpha_4^1 a / x_1) J_{n+1}(\alpha_4^1 a x_1) + [(n-1)/(x_1)^2] J_n(\alpha_4^1 a x_1)]$$

$$D(2, j) = 2n\overline{c}_{i6}^j [(\alpha_j^1 a / x_1) J_{n+1}(\alpha_j^1 a x_1) + [(n-1)/(x_1)^2] J_n(\alpha_j^1 a x_1)]$$

$$D(2, 4) = \overline{c}_{i6}^4 [-2(\alpha_4^1 a / x_1) J_{n+1}(\alpha_4^1 a x_1) + [(\alpha_4^1 a)^2 - 2n(n-1)/(x_1)^2] J_n(\alpha_4^1 a x_1)]$$

$$D(2, i+4) = -2n\overline{\mu}[(\alpha_j^1 a / x_1) J_{n+1}(\alpha_j^1 a x_1) - [(n-1)/(x_1)^2] J_n(\alpha_j^1 a / x_1)]$$

$$D(2, 7) = 2\overline{\mu}[(\alpha_4^1 a / x_1) J_{n+1}(\alpha_4^1 a x_1) + [n(n-1)/(x_1)^2] - ((\alpha_4^1 a)^2 / 2) J_n(\alpha_4^1 a x_1)]$$

$$D(3, j) = (d_j^1 + \varepsilon + \overline{c}_{i3}^j e_j) [-(\alpha_j^1 a) J_{n+1}(\alpha_j^1 a x_1) + [(n/x_1)] J_n(\alpha_j^1 a x_1)]$$

$$D(3, 4) = \varepsilon (n / x_1) J_n(\alpha_4^1 a x_1)$$

$$D(3, i+4) = -\overline{\mu} (\varepsilon + d_{i+4}) [-(\alpha_j^1 a) J_{n+1}(\alpha_j^1 a x_1) + (n / x_1) J_n(\alpha_j^1 a x_1)]$$

$$D(3, 7) = -\overline{\mu} (n / x_1) \varepsilon J_n(\alpha_4^1 a x_1)$$

$$D(4, j) = -(\alpha_j^1 a) J_{n+1}(\alpha_j^1 a x_1) + (n / x_1) J_n(\alpha_j^1 a x_1)$$

$$D(4, 4) = (n / x_1) J_n(\alpha_4^1 a x_1)$$
\[ D(4, i+4) = (\alpha_j a) J_{n+1}(\alpha_j ax_i) - (n/x_i) J_n(\alpha_j ax_i) \]

\[ D(4, 7) = -(n/x_i) J_n(\alpha_j ax_i) \]

\[ D(5, j) = -(n/x_i) J_n(\alpha_j^1 ax_i) \]

\[ D(5, 4) = (\alpha_j^1 a) J_{n+1}(\alpha_j^1 ax_i) - (n/x_i) J_n(\alpha_j ax_i) \]

\[ D(5, i+4) = (n/x_i) J_n(\alpha_j ax_i) \]

\[ D(5, 7) = -((\alpha_j a) J_{n+1}(\alpha_j ax_i) - (n/x_i) J_n(\alpha_j ax_i)) \]

\[ D(6, j) = d_j^1 J_n(\alpha_j^1 ax_i) \]

\[ D(6, i+4) = -d_j J_n(\alpha_j ax_i) \]

\[ D(7, j) = e_j^1 J_n(\alpha_j^1 ax_i) \]

And other elements at the interface, \( x = x_1 \) can be obtained on replacing \( J_n \) and \( J_{n+1} \) by \( Y_n \) and \( Y_{n+1} \) in the above elements and are given by \( D(k, i + 7), D(k, 10) \) \((k = 1, 2, 3, 4, 5, 6; i = 1, 2)\).

At the interface \( x = x_2 \), non zero elements along the following rows \( D(k, j), (k = 8, 9, 10, 11, \ldots \ldots \ldots \ldots \ldots 14) \) \((j = 5, 6, 7, \ldots \ldots \ldots \ldots 18)\) are obtained on replacing \( x_1 \) by \( x_2 \) and superscript 1 by 2 in order. Similarly at the outer surface \( x = x_3 \), the non zero elements are, \( D(k, j), (i = 15, 16, 17, 18) \) and \((j = 11, \ldots \ldots \ldots \ldots 18) \) are obtained on replacing \( x_1 \) by \( x_3 \) and super script 1 by 2 in order. The frequency equations derived above are valid for different inner and outer materials of 6 mm class and arbitrary thickness of layers.

### 4.4 NUMERICAL RESULTS

Zeros of the frequency equations are evaluated using Muller’s method (Jain et al 1987) as it evaluates the complex roots rapidly even in
Pentium III processor over other root finding methods. Material Constants of CFRP bonding layer are taken from Ashby and Jones (1986). The elastic piezoelectric and dielectric constants of PZT4 are taken from Brelincourt et al (1964). The calculations are carried out by fixing the real wave number and varying frequencies. The complex frequencies for the flexural waves in the first and second modes are given in Tables 4.1 and 4.2. The attenuation in the case of piezocomposite with LEMV as the middle core is more when compared to CFRP as core material in the present model. The dispersion curves for the real part of frequency against the dimensionless wave number are plotted for the first and second flexural mode in Figure 4.1 and Figure 4.2. The bold, dotted and discontinuous lines indicate the dispersion curves in the flexural vibrations of the piezolaminated-CFRP, piezolaminated- LEMV (with $N=0.66$ and $N=0.33$) cylinders. Due to the presence of voids in CFRP as well as in LEMV both composite behaves like a thermo piezoelectric material. For all numerical calculations, the inner/outer and middle/outer radius of the cylinders are taken as $a/c = 0.6666$ and $b/c = 0.7333$ respectively.
Table 4.1 Different value of complex frequencies for real wave numbers in the first flexural mode of piezocomposite solid cylinder

| Wave no (ε) | Frequencies |  |
|-------------|-------------|  |
|             | With middle core LEMV (N = 0.33) | With middle core LEMV (N=0.66) | With middle core CFRP |
| 0.1         | 0.2993+i 0.0000 | 0.3015+i 0.0000 | 0.1667+i 0.0511 |
| 0.4         | 0.7055+i 0.2263 | 0.8868+i 0.0000 | 0.4000+i 0.0000 |
| 0.8         | 0.9531+i 0.0402 | 1.3000+i 0.0004 | 0.6952+i 0.0000 |
| 1.2         | 1.3725+i 0.3240 | 1.4000+i 0.0000 | 1.0126+i 0.0012 |
| 1.6         | 1.4138+i 0.0000 | 1.9000+i 0.0579 | 1.3103+i 0.0006 |
| 2.0         | 1.9023+i 0.0030 | 2.0995+i 0.0024 | 1.5623+i 0.0127 |
| 2.4         | 2.4965+i 0.0000 | 2.5000+i 0.0003 | 2.2007+i 0.0005 |
| 2.8         | 2.6951+i 0.0050 | 2.7004+i 0.0000 | 2.5985+i 0.0181 |
| 3.0         | 2.8891+i 0.0010 | 3.0000+i 0.0002 | 2.6657+i 0.0199 |
Table 4.2  Different value of complex frequencies for real wave numbers in the second flexural mode of piezocomposite solid cylinder

<table>
<thead>
<tr>
<th>Wave no</th>
<th>Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>With middle core LEMV (N = 0.33)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3094+i 0.0012</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9999+i 0.0000</td>
</tr>
<tr>
<td>0.8</td>
<td>1.3634+i 0.0032</td>
</tr>
<tr>
<td>1.2</td>
<td>1.4294+i 0.1824</td>
</tr>
<tr>
<td>1.6</td>
<td>1.6993+i 1.0731</td>
</tr>
<tr>
<td>2.0</td>
<td>2.0334+i 0.4423</td>
</tr>
<tr>
<td>2.4</td>
<td>2.5293+i 0.0037</td>
</tr>
<tr>
<td>2.8</td>
<td>2.7201+i 0.0076</td>
</tr>
<tr>
<td>3.0</td>
<td>3.1528+i 0.0015</td>
</tr>
</tbody>
</table>
Figure 4.1  Comparison of dispersion curves of piezocomposite solid cylinders: PZT4/CFRP/PZT4 (Bold line), PZT4/LEMV (N=0.66)/PZT4 (Dotted line) and PZT4/LEMV (N=0.33)/PZT4 (Discontinues line) in the first flexural mode
Figure 4.2 Comparison of dispersion curves of piezocomposite solid cylinders PZT4/CFRP/PZT4 (Bold line), PZT4/LEMV (N=0.66)/PZT4 (Dotted line) and PZT4/LEMV (N=0.33)/PZT4 (Discontinues line) in the second flexural mode