CHAPTER 5

FLEXURAL VIBRATION OF PIEZOLAMINATED HOLLOW CYLINDER

5.1 INTRODUCTION

Piezoelectric composite becomes an important material due to its intrinsic energy electro-mechanical coupling behavior. The phenomenon and theory of piezoelectricity and pyroelectric materials can be found in the book by Cady (1964). Smart materials based on CFRP with embedded PZT sensors and actuators are expected to be a favorite composite for vibration damping and noise reduction. Gerhardmook (2003) has discussed about the imaging techniques using the active piezoceramic as transmitters of acoustic, electromagnetic and thermal fields. Damage detection and vibration control of a new smart board designed by mounting piezoelectric fibers with metal cores on the surface of a CFRP composite was studied by Takagi kiyoshi (2005). Toshio Tanimoto (2002) has discussed the passive damping of CFRP cantilever beam, surface-bonded by piezoelectric ceramics. The origin of Polymer piezoelectric materials behavior and their application to ultrasonic NDE was discussed by Yoseph Bar-Cohen (1996). A brief review of the representative works on vibration of piezoelectric crystals was studied by Dokmechi (1980). Paul (1966) has discussed the exact frequency equation for piezoelectric circular cylindrical shell of hexagonal (6 mm) class. Paul and Nelson (1994, 1995, 1996 and 1996a) have studied free vibration of piezocomposite plates and cylinders by embedding LEMV layer between piezoelectric layers.
A general frequency equation is derived for flexural vibration of an infinite laminated hollow cylinder. Both the inner and outer surfaces are traction free and connected with electrodes and that are shorted. Numerical calculations are carried out for PZT4/CFRP/PZT4. The attenuation effect is considered through the imaginary part of the dimensionless complex frequency (Sinha et al. 1992).

5.2 FUNDAMENTAL EQUATIONS AND METHOD OF ANALYSIS

The cylindrical polar coordinate system \((r, \theta, z)\) is used with the \(z\)-axis along the axis of the piezocomposite cylinder. In the flexural mode of vibration the displacement components \(u^\ell, v^\ell, w^\ell\) and the potentials \(\phi^\ell\) are functions of \(r, \theta\) and \(z\). The superscripts \(\ell = 1, 2\) are taken for the inner and the outer hollow piezoelectric cylinder respectively.

The governing equations for hexagonal (6mm) class are Paul and Nelson (1996a)

\[
\begin{align*}
&c_{11}\left(u_{,rr}^\ell + r^{-1}u_{,r}^\ell - r^{-2}u^\ell\right) + c_{66}^\ell r^{-2}u_{,\theta\theta}^\ell + c_{44}^\ell u_{,zz}^\ell + (c_{66}^\ell + c_{12}^\ell) r^{-1}v_{,\theta}^\ell \\
&\quad - (c_{66}^\ell + c_{11}^\ell) r^{-2}v_{,\theta}^\ell + (c_{44}^\ell + c_{13}^\ell) w_{,rz}^\ell + (c_{31}^\ell + c_{13}^\ell) \phi_{,rz}^\ell = \rho^\ell u_{,tt}^\ell
\end{align*}
\]

\[
\begin{align*}
&(c_{66}^\ell + c_{12}^\ell) r^{-1}u_{,\theta\theta}^\ell + (c_{11}^\ell + c_{66}^\ell) r^{-2}u_{,\theta\theta}^\ell + c_{66}^\ell (v_{,rr}^\ell + r^{-1} v_{,r}^\ell - r^{-2} v^\ell) \\
&c_{11}^\ell r^{-2}v_{,\theta\theta}^\ell + c_{44}^\ell v_{,zz}^\ell + (c_{44}^\ell + c_{13}^\ell) r^{-1} w_{,oz}^\ell + (c_{31}^\ell + c_{13}^\ell) r^{-1} \phi_{,oz}^\ell = \rho^\ell v_{,tt}^\ell
\end{align*}
\]

\[
\begin{align*}
&(c_{44}^\ell + c_{13}^\ell)(u_{,rz}^\ell + r^{-1}u_{,r}^\ell + r^{-1}v_{,oz}^\ell) + c_{44}^\ell (w_{,rr}^\ell + r^{-1}w_{,r}^\ell + r^{-2}w_{,00}^\ell) \\
&+ c_{33}^\ell w_{,zz}^\ell + c_{15}^\ell (\phi_{,rr}^\ell + r^{-1}\phi_{,r}^\ell + r^{-2}\phi_{,00}^\ell) + c_{33}^\ell \phi_{,zz}^\ell = \rho^\ell w_{,tt}^\ell
\end{align*}
\]

and
Here \( c_{ij}^\varepsilon \) - Elastic constants
\( e_{ij}^\varepsilon \) - Piezoelectric constants
\( \varepsilon_{ij}^\varepsilon \) - Dielectric constants
\( \rho^\varepsilon \) - Density of the materials

The comma followed by superscripts denotes the partial differentiation with respect to those variables and \( t \) is the time.

The solution of equation (5.1) is taken in the form.

\[
\begin{align*}
\mathbf{u}^\varepsilon &= \left( u_{r}^\varepsilon + r^{-1} v_{\theta}^\varepsilon \right) \exp i(kz + pt) \\
\mathbf{v}^\varepsilon &= \left( r^{-1} u_{\theta}^\varepsilon - v_{r}^\varepsilon \right) \exp i(kz + pt) \\
\mathbf{w}^\varepsilon &= \left( i / h \right) w^\varepsilon \exp i(kz + pt) \\
\phi^\varepsilon &= i\left( c_{44}^\varepsilon / h e_{33}^\varepsilon \right) \phi^\varepsilon \exp i(kz + pt)
\end{align*}
\]  

where \( p \) is angular frequency, \( k \) wave number \( h = h_3 - h_0 \) (\( h_0, h_3 \) are inner and outer radius of the cylinders) thickness of the composite hollow cylinder.

Substituting equation (5.2) along with the dimensionless variables \( x = r/h \) and \( \varepsilon = kh \) (\( k = 2\pi/\text{wavelength} \)) in equation (5.1) yield the following equation for the inner and outer cylinder.
\[
\begin{vmatrix}
\xi_{11} \nabla^2 + F_1' & -F_2' & -F_3' \xi_{44} \\
F_2' \nabla^2 & \xi_{44} \nabla^2 + F_4' & (\xi_{15} \nabla^2 - F_5') \xi_{44} \\
F_3' \nabla^2 & \xi_{15} \nabla^2 - F_5' & -(k_{13}^{-2})' \nabla^2 + F_6'
\end{vmatrix}
(u', w', \phi') = 0
\tag{4.3}
\]

\[\left[\xi_{66} \nabla^2 + F_1'\right]v' = 0
\tag{5.4}\]

where
\[
\xi_{ij}' = \frac{\xi_{ij}'}{\xi_{44}'} , \quad v_{ij}' = \frac{v_{ij}'}{v_{33}'}, 
\]

\[
\left(k_{13}'\right)' = \left(\frac{e_{33}'}{e_{44}' e_{11}'}\right) , \quad ch^2 = \frac{\rho_1 \rho_2}{c_{44}'} h^2
\]

And
\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x}
\]

\[
F_1' = \frac{\rho_1'}{\rho_1} (ch)^2 - \varepsilon^2 \xi_{44}' , \quad F_2' = \varepsilon (\xi_{44}' + \xi_{13}')
\]

\[
F_3' = \varepsilon (\xi_{31}' + \xi_{15}') , \quad F_4' = \frac{\rho_1'}{\rho_1} (ch)^2 - (\xi_{33}' e^2)
\]

\[
F_5' = \varepsilon^2 , \quad F_6' = (k_{33}^{-2})' \varepsilon^2
\]

Equation (5.3) can be expressed as

\[
(V^6 + P' V^4 + Q' V^2 + R')(u', w', \phi') = 0
\tag{5.5}\]

where
\[
P' = \xi_{11}' [(k_{13}^{-2})' F_4' - \xi_{44}' F_6' - 2 \xi_{44}' F_5' \xi_{15}'] + \xi_{44}' (k_{13}^{-2})' + (\xi_{13}')'
\]

\[
+ F_6' [(k_{13}^{-2})' F_2' + \xi_{44}' F_5' \xi_{44}'] + \xi_{44}' F_5' (\xi_{15}' F_2' - \xi_{44}' F_5') / \xi_{11}' \xi_{44}' [(k_{13}^{-2})' + (\xi_{13}')']
\]

\[
Q' = \xi_{11}' (\xi_{44}' F_6'^2 - F_4' F_6' F_5') + F_5' [(k_{13}^{-2})' F_2' - \xi_{44}' F_6' - 2 \xi_{44}' F_5' \xi_{15}']
\]

\[
- F_6' (F_2' F_6' + \xi_{44}' F_3' F_5') - \xi_{44}' F_5' (F_2' F_5' + F_3' F_4') / \xi_{11}' \xi_{44}' [(k_{13}^{-2})' + (\xi_{13}')']
\]
The solutions of the Eqn (5.5) are taken in terms of Bessel functions of first kind ($J_n$) and second kind ($Y_n$) to suit the wave propagation in cylindrical polar coordinate system as

\[ u^\ell = \sum_{j=1}^{3} [A_j^\ell J_n (\alpha_j^\ell x) + B_j^\ell y_n (\alpha_j^\ell x)] \cos n\theta \]

\[ w^\ell = \sum_{j=1}^{3} d_j^\ell [A_j^\ell J_n (\alpha_j^\ell x) + B_j^\ell y_n (\alpha_j^\ell x)] \cos n\theta \]

\[ \phi^\ell = \sum_{j=1}^{3} e_j^\ell [A_j^\ell J_n (\alpha_j^\ell x) + B_j^\ell y_n (\alpha_j^\ell x)] \cos n\theta \]

Here $(\alpha_j^\ell)^2$ are the non-zero roots of

\[(\alpha_j^\ell)^2 - P^\ell (\alpha_j^\ell)^4 + Q^\ell (\alpha_j^\ell)^2 - R^\ell = 0 \]

The arbitrary constants $d_j^\ell$ and $e_j^\ell$ are given by

\[ F_2^\ell d_j^\ell + F_3^\ell c_{44}^\ell e_j^\ell = F_4^\ell - c_{11}^{-1}(\alpha_j^\ell)^2 \]

\[-[e_{15}^\ell (\alpha_j^\ell)^2 + F_6^\ell] d_j^\ell + [(k_{13}^2)^\ell (\alpha_j^\ell)^2 + F_6^\ell] e_j^\ell = F_5^\ell (\alpha_j^\ell)^2 \]

Also for equation (5.4)

\[ v^\ell = [A_4^\ell J_n (\alpha_4^\ell x) + B_4^\ell y_n (\alpha_4^\ell x)] \sin n\theta \]

where \((\alpha_4)^2 = \frac{F_1^\ell}{\bar{c}_{66}^\ell}\)
For isotropic materials, the governing equations are

\[ \mu \nabla^2 \bar{u} + (\lambda + \mu) \nabla \nabla \cdot \bar{u} = \rho \bar{u}_{,tt} \]  

(5.11)

where \( \bar{u} \) is the displacement vector

\[ \lambda = c_{12}, \quad \mu = (c_{11} - c_{12}) / 2 \]  

are Lame’s constants

\( \rho \) is the mass density and \( t \) is the time.

The solution of equation (5.11) is taken as

\[ u = (u_x + r^{-1}v_\theta) \exp i(kz + pt) \]
\[ v = (r^{-1}u_\theta - v_x) \exp i(kz + pt) \]
\[ w = (i/h^2)w \exp i(kz + pt) \]  

(5.12)

Using the solution in equation (5.12) and the dimensionless variables \( x \) and \( \varepsilon \), equation (5.11) can be simplified as

\[ \left[ (\lambda + 2\mu)\nabla^2 + F_1 \right] \frac{-F_2}{\mu \nabla^2 + F_3} (u, w) = 0 \]  

(5.13)

\[ (\mu \nabla^2 + F_3)v = 0 \]  

(5.14)

where \( \lambda = \lambda / c_{44}^4, \mu = \mu / c_{44}^4 \)

\[ F_1 = (\rho / \rho^1)(ch)^2 - \mu \varepsilon^2 \]
\[ F_2 = (\lambda + \mu)\varepsilon, \quad F_3 = (\rho / \rho^1)(ch)^2 - (\lambda + 2\mu)\varepsilon^2 \]

The equation (5.11) can be written as

\[ (\nabla^4 + PV^2 + Q)(u, w) = 0 \]  

(5.15)
where \( P = \frac{[(\mu + 2\mu)F_2 + \mu F_1 + (F_2)²]}{(\mu + 2\mu)\mu} \)

\[
Q = \frac{[F_1 F_2]}{(\mu + 2\mu)\mu} 
\]  

(5.16)

Similar to the assumption of solution of Eqn. (5.5), the solutions of the Eqn. (5.13) are taken as

\[
u = \sum_{j=1}^{3} [A_j J_n(\alpha_j x) + B_j y_n(\alpha_j x)] \sin n\theta
\]

\[
w = \sum_{j=1}^{3} d_j [A_j J_n(\alpha_j x) + B_j y_n(\alpha_j x)] \cos n\theta
\]  

(5.17)

where \((\alpha_j)^2\) is the non zero roots of

\[
(\alpha)^4 + P(\alpha)^2 - Q = 0
\]  

(5.18)

And the arbitrary constants \(d_j\) are obtained from

\[
d_j = \frac{-[(\mu + 2\mu)(\alpha_j)^2 + F_1]}{F_2}
\]  

(5.19)

The solution of equation (5.14) is

\[
v = [A_4 J_n(\alpha_4 x) + B_4 y_n(\alpha_4 x)] \sin n\theta
\]  

(5.20)

where \((\alpha_4)^2 = \frac{F_1}{\mu}\)  

(5.21)

### 5.3 BOUNDARY – INTERFACE CONDITIONS AND FREQUENCY EQUATIONS

The frequency equations can be obtained using the following boundary and interface conditions.

i) On the traction free outer surfaces
\[ T_{rr}^2 = 0 \]
\[ T_{r\theta}^2 = 0 \]
\[ T_{rz}^2 = 0 \]
\[ \phi^2 = 0 \]

ii) At the interface between (outer and middle) cylinders

\[ T_{rr}^2 = T_{rr}^1 \]
\[ T_{r\theta}^2 = T_{r\theta}^1 \]
\[ T_{rz}^2 = T_{rz}^1 \]
\[ U^2 = U^1 \]
\[ V^2 = V^1 \]
\[ W^2 = W^1 \]
\[ \phi^2 = 0 \]

iii) At the interface between (middle and inner) cylinders

\[ T_{rr} = T_{rr}^1 \]
\[ T_{r\theta} = T_{r\theta}^1 \]
\[ T_{rz} = T_{rz}^1 \]
\[ U = U^1 \]
\[ V = V^1 \]
\[ W = W^1 \]
\[ \phi^1 = 0 \]

iv) On the traction free inner surfaces
\[ T_{rr}^1 = 0 \]
\[ T_{r\theta}^1 = 0 \]
\[ T_{r\zeta}^1 = 0 \]
\[ \phi^1 = 0 \]

The frequency equation is obtained as a \(22 \times 22\) determinantal equation by substituting the solutions (5.7), (5.10), (5.17) and (5.20) in the boundary – interface conditions. It is written as

\[
|D(i,j)| = 0, \quad (i, j = 1, 2...22) \quad (5.22)
\]

And the non-zero elements by varying \(j\) from 1 to 3 and \(i\) varies from 1 to 2 are

\[
D(1, j) = 2c_{66}^1 (\alpha_j^1 / x_0) J_{n+1}(\alpha_j^1 x_0) + [2c_{66}^1 n(n-1)/(x_0)^2 - \bar{c}_{11}^1(\alpha_j^1)^2
\]
\[-\bar{c}_{11}^1 \varepsilon d_j - \bar{c}_{31}^1 \varepsilon e_j] J_n(\alpha_j^1 x_0)
\]
\[
D(1, 4) = -2n c_{66}^1 (\alpha_4^1 / x_0) J_{n+1}(\alpha_4^1 x_0) + [2c_{66}^1 n(n-1)/(x_0)^2] J_n(\alpha_4^1 x_0)
\]
\[
D(2, j) = -(d_j^1 + \varepsilon \bar{c}_{15}^1 e_j^1) [(\alpha_j^1) J_{n+1}(\alpha_j^1 x_0) - [(n/(x_0))] J_n(\alpha_j^1 x_0)]
\]
\[
D(2, 4) = \varepsilon(n/x_0) J_n(\alpha_4^1 x_0)
\]
\[
D(3, j) = 2n c_{66}^1 [(\alpha_j^1 / x_0) J_{n+1}(\alpha_j^1 x_0) + [(n-1)/(x_0)^2] J_n(\alpha_j^1 x_0)]
\]
\[
D(3, 4) = \bar{c}_{66}^1 [(-2(\alpha_4^1 / x_0) J_{n+1}(\alpha_4^1 x_0) + [(\alpha_4^1)^2 - 2n(n-1)/(x_0)^2] J_n(\alpha_4^1 x_0)]
\]
\[
D(4, j) = e_j^1 J_n(\alpha_j^1 x_0)
\]
\[
D(5, j) = 2c_{66}^1 (\alpha_j^1 / x_1) J_{n+1}(\alpha_j^1 x_1) + [2c_{66}^1 n(n-1)/(x_1)^2 - \bar{c}_{11}^1(\alpha_j^1)^2
\]
\[-\bar{c}_{11}^1 \varepsilon d_j - \bar{c}_{31}^1 \varepsilon e_j] J_n(\alpha_j^1 x_1)
\]
\[
D(5, 4) = -2n c_{66}^1 (\alpha_4^1 / x_1) J_{n+1}(\alpha_4^1 x_1) + [2c_{66}^1 n(n-1)/(x_1)^2] J_n(\alpha_4^1 x_1)
\]
\[
D(5,i+8) = -[2\bar{\mu} (\alpha_j^1 / x_1) J_{n+1}(\alpha_j^1 x_1) [-\bar{\lambda} + 2\bar{\mu}) (\alpha_j^1)^2 - \bar{\lambda} \varepsilon d_j
\]
\[+ 2n(n-1) (\bar{\mu}(x_1)^2] J_n(\alpha_j^1 x_1)]
\]
\[ D(5, 11) = 2n\bar{\mu} \left[ \left( \alpha_4/x_1 \right) J_{n+1}(\alpha_4 x_1) - \frac{1}{2} (n-1)/(x_1)^2 \right] J_n(\alpha_4 x_1) \]

\[ D(6, j) = (d_j^1 + \varepsilon + c_j^1 d_j^1) \left[ -\left( \alpha_j^1 \right) J_{n+1}(\alpha_j^1 x_1) + \frac{1}{2} (n-1)/(x_1)^2 \right] J_n(\alpha_j^1 x_1) \]

\[ D(6, 4) = \varepsilon \left( \frac{n}{x_1} \right) J_n(\alpha_4 x_1) \]

\[ D(6, i+8) = -\bar{\mu} (\varepsilon + d_j^1) \left[ -\left( \alpha_j \right) J_{n+1}(\alpha_j x_1) + \frac{1}{2} (n-1)/(x_1)^2 \right] J_n(\alpha_j x_1) \]

\[ D(6, 11) = -\bar{\mu} \left( \frac{n}{x_1} \right) \varepsilon J_n(\alpha_4 x_1) \]

\[ D(7, j) = 2n\bar{c}_{6j} \left[ \left( \alpha_j^1/x_1 \right) J_{n+1}(\alpha_j^1 x_1) + \frac{1}{2} (n-1)/(x_1)^2 \right] J_n(\alpha_j^1 x_1) \]

\[ D(7, 4) = c_{6j} \left[ -2\left( \alpha_4/x_1 \right) J_{n+1}(\alpha_4 x_1) + \frac{1}{2} (n-1)/(x_1)^2 \right] J_n(\alpha_4 x_1) \]

\[ D(7, i+8) = -2n\bar{\mu} \left[ \left( \alpha_j \right) J_{n+1}(\alpha_j x_1) - \frac{1}{2} (n-1)/(x_1)^2 \right] J_n(\alpha_j x_1) \]

\[ D(7, 11) = 2\bar{\mu} \left[ \left( \alpha_4/x_1 \right) J_{n+1}(\alpha_4 x_1) + \frac{1}{2} (n-1)/(x_1)^2 \right] - \frac{1}{2} \left( \alpha_4 \right)^2 J_n(\alpha_4 x_1) \]

\[ D(8, j) = -\left( \alpha_j^1 \right) J_{n+1}(\alpha_j^1 x_1) + \frac{1}{2} (n-1)/(x_1)^2 J_n(\alpha_j^1 x_1) \]

\[ D(8, 4) = \left( \frac{n}{x_1} \right) J_n(\alpha_4 x_1) \]

\[ D(8, i+8) = \alpha_j J_{n+1}(\alpha_j x_1) - \frac{1}{2} (n-1)/(x_1)^2 J_n(\alpha_j x_1) \]

\[ D(8, 11) = -\left( \frac{n}{x_1} \right) J_n(\alpha_4 x_1) \]

\[ D(9, j) = -\left( \frac{n}{x_1} \right) J_n(\alpha_j^1 x_1) \]

\[ D(9, 4) = \alpha_j^1 J_{n+1}(\alpha_j^1 x_1) - \frac{1}{2} (n-1)/(x_1)^2 J_n(\alpha_j^1 x_1) \]

\[ D(9, i+8) = \left( \frac{n}{x_1} \right) J_n(\alpha_j x_1) \]

\[ D(9, 11) = -\left[ \alpha_4 J_{n+1}(\alpha_4 x_1) - \frac{1}{2} (n-1)/(x_1)^2 \right] J_n(\alpha_4 x_1) \]

\[ D(10, j) = d_j^1 J_n(\alpha_j^1 x_1) \]

\[ D(10, i+8) = -d_j^1 J_n(\alpha_j x_1) \]

\[ D(11, j) = e_j^1 J_n(\alpha_j^1 x_1) \]

And other elements at the interface \( x = x_1 \), can be obtained on replacing \( J_n \) and \( J_{n+1} \) by \( Y_n \) and \( Y_{n+1} \) in the above elements and are given by

\[ D(k, j+4) \quad (k = 1, 2, 3 \ldots 11) \quad (j = 1, 2, 3) \]

\[ D(i, 8) \quad (i = 1, 2, 3, 5, 6, 7, 8, 9) \]

\[ D(i, i+11) \quad (j = 5, 6, 7, 8, 9) \]

\[ D(11, j) \quad (j = 5, 6, 7, 8, 9) \]
At the interface \( x = x_2 \), non zero elements along the following rows 
\[ D(k, j), \quad (k = 12, 13, \ldots, 18) \quad (j = 9, 10, \ldots, 22) \] 
are obtained on replacing \( x_1 \) by \( x_2 \) and super script 1 by 2 in order. Similarly at the outer surface \( x = x_3 \), the non zero elements are, 
\[ D(k, j), \quad (i = 19, 20, 21, 22) \quad (j = 15, 16, \ldots, 22) \] 
are obtained on replacing \( x_0 \) by \( x_3 \) from the first four rows and super script 1 by 2 in order. The frequency equations derived above are valid for different inner and outer materials of 6mm class and arbitrary thickness of layers.

5.4 NUMERICAL RESULTS

Zeros of the frequency equations are evaluated using Muller’s method (Jain et al 1987) as it evaluates the complex roots rapidly even in Pentium III processor over other root finding methods. Material Constants of CFRP bonding layer are taken from Ashby and Jones (1986). The elastic piezoelectric and dielectric constants of PZT4 are taken from Brelincourt et al (1964). The complex frequencies are calculated by fixing real wave numbers for the thin core of thickness 0.002 m, shell of thickness 0.03 m for hollow cylinder of piezoelectric material. The complex frequencies for the flexural waves in the first and second modes are given in Tables 5.1 and 5.2. The attenuation in the case of piezocomposite with LEMV as the middle core is more with the void volume factor \( N = 0.33 \) when compared to CFRP as core material in the present model. The dispersion curves for the real part of frequency against the dimensionless wave number are plotted for the first and second flexural mode in Figure 5.1 and Figure 5.2. The bold, thin, dotted and discontinuous lines indicate the dispersion curves in the flexural vibrations of the piezolaminated-CFRP, piezoelectric and piezolaminated-LEMV with \( N=0.33 \) and \( N=0.0 \) cylinders.
Table 5.1  Different value of complex frequencies for real wave numbers in the first flexural mode of the piezocomposite hollow cylinder

<table>
<thead>
<tr>
<th>Wave no (ε)</th>
<th>Frequencies</th>
<th>Piezoelectric cylinder</th>
<th>With middle core LEMV (N= 0)</th>
<th>With middle core LEMV (N = 0.33)</th>
<th>With middle core CFRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td></td>
<td>0.1000+i 0.0000</td>
<td>0.1000+i 0.0000</td>
<td>0.2552+i 0.0329</td>
<td>0.1249+i 0.0316</td>
</tr>
<tr>
<td>0.4</td>
<td></td>
<td>0.3405+i 0.0000</td>
<td>0.4291+i 0.0067</td>
<td>0.3780+i 0.0389</td>
<td>0.4000+i 0.0000</td>
</tr>
<tr>
<td>0.8</td>
<td></td>
<td>0.6811+i 0.0000</td>
<td>0.6814+i 0.0002</td>
<td>0.6833+i 0.0004</td>
<td>0.6102+i 0.0015</td>
</tr>
<tr>
<td>1.2</td>
<td></td>
<td>1.0217+i 0.0000</td>
<td>1.0626+i 0.0336</td>
<td>1.0779+i 0.0103</td>
<td>1.2000+i 0.0000</td>
</tr>
<tr>
<td>1.6</td>
<td></td>
<td>1.3623+i 0.0000</td>
<td>1.3640+i 0.0117</td>
<td>1.3964+i 0.0150</td>
<td>1.4205+i 0.0032</td>
</tr>
<tr>
<td>2.0</td>
<td></td>
<td>1.7029+i 0.0000</td>
<td>1.6813+i 0.0477</td>
<td>1.7506+i 0.0260</td>
<td>1.7332+i 0.0072</td>
</tr>
<tr>
<td>2.4</td>
<td></td>
<td>2.0435+i 0.0000</td>
<td>2.2238+i 0.0000</td>
<td>2.1240+i 0.0452</td>
<td>2.0029+i 0.0164</td>
</tr>
<tr>
<td>2.8</td>
<td></td>
<td>2.3841+i 0.0000</td>
<td>2.7929+i 0.0035</td>
<td>2.5103+i 0.0901</td>
<td>2.7979+i 0.0019</td>
</tr>
<tr>
<td>3.0</td>
<td></td>
<td>2.5544+i 0.0000</td>
<td>2.9982+i 0.0029</td>
<td>2.7102+i 0.1143</td>
<td>2.9823+i 0.0081</td>
</tr>
</tbody>
</table>
Table 5.2 Different value of complex frequencies for real wave numbers in the second flexural mode of the piezocomposite hollow cylinder

<table>
<thead>
<tr>
<th>Wave no</th>
<th>Frequencies</th>
<th>Piezoelectric cylinder</th>
<th>With middle core LEMV (N= 0)</th>
<th>With middle core LEMV (N = 0.33)</th>
<th>With middle core CFRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ε)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.1582+i 0.0000</td>
<td>0.2106+i 0.0010</td>
<td>0.3198+i 0.0518</td>
<td>0.2423+i 0.0017</td>
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</tr>
<tr>
<td>0.4</td>
<td>0.3999+i 0.0000</td>
<td>0.4462+i 0.0038</td>
<td>0.5642+i 0.1207</td>
<td>0.4039+i 0.0000</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.8000+i 0.0000</td>
<td>0.8000+i 0.0000</td>
<td>0.8340+i 0.0116</td>
<td>0.7512+i 0.0310</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>1.1999+i 0.0000</td>
<td>1.2000+i 0.0000</td>
<td>1.1750+i 0.0604</td>
<td>1.3217+i 0.0034</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>1.4757+i 0.0000</td>
<td>1.5999+i 0.0000</td>
<td>1.4353+i 0.0269</td>
<td>1.5010+i 0.0001</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>1.8990+i 0.0000</td>
<td>1.9670+i 0.0786</td>
<td>1.5300+i 0.0571</td>
<td>1.8240+i 0.0021</td>
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</tr>
<tr>
<td>2.4</td>
<td>2.4000+i 0.0000</td>
<td>2.3707+i 0.0208</td>
<td>2.2677+i 0.0504</td>
<td>2.4000+i 0.0000</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>2.8000+i 0.0000</td>
<td>2.9271+i 0.0000</td>
<td>2.7275+i 0.0333</td>
<td>2.8986+i 0.0004</td>
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<tr>
<td>3.0</td>
<td>2.9865+i 0.0000</td>
<td>3.1378+i 0.0000</td>
<td>2.9498+i 0.0266</td>
<td>3.0068+i 0.0088</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.1  Comparison of dispersion curves of piezocomposite hollow cylinders  PZT4/CFRP/PZT4  (Bold line), PZT4/LEMV (N=0)/PZT4  (Discontinuous line), PZT4/LEMV (N=0.33)/PZT4  (Dotted lines) and thin line for piezoelectric cylinder in the first flexural mode
Figure 5.2 Comparison of dispersion curves of piezocomposite hollow cylinders PZT4/CFRP/PZT4 (Bold line), PZT4/LEMV (N=0)/PZT4 (Discontinuous line), PZT4/LEMV (N=0.33)/PZT4 (Dotted lines) and thin line for piezoelectric cylinder in the second flexural mode