Chapter 5.

*M|PH|1* retrial queue with service interruption and orbital search

Queueing system with service interruption have been introduced by White and Christie [73] who considered the problem as a pre-emptive priority system. Different types of interruptions have been extensively studied by many researchers. Service interruptions can be viewed as a special type of breakdown of the server in which the server is restarted instantaneously. See Aissani [1, 2], Aissani and Artalejo [3], Artalejo [4] and references therein for queueing system with breakdown. Queues with service interruption also fall into the category of queues with feedback (See Choi and Kulkarni [24]) and queues with disaster to the unit undergoing service (See A. Krishnamoorthy and P. V. Ushakumari [50]). Artalejo and Gomez-Correl [10] considered a retrial queueing system with two types of service interruptions. In their model, depending on the type of the interruption that the unit has encountered, it may rejoin the orbit for another attempt or leave the system for ever.

In this chapter, we consider a single server retrial queue in which the server is subject to service interruptions with auto repeat facilities and equipped with the ‘mechanism of search of customers from the orbit’ as we have introduced
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in chapter 2. The customer whose service is interrupted rejoins the orbit with some known probability \( q \) or leaves the system with the complimentary probability \((1 - q)\). At the departure epoch, (ie, the epoch at which the server becomes free either by a successful completion of a service or by a service interruption), the server goes for search of customers in the orbit with some known probability \( p \) or remains idle with the complimentary probability \( 1 - p \).

There are lots of real life situations which fit to our model. For example, consider the transmission of messages in fascimile 'networks' having the autorepeat facilities. If the transmission is not successfully completed for some reasons such as a power failure or a transmission error, then the message leaves the server and joins the buffer with some known probability and leaves the system with the complimentary probability. Immediately after a successful transmission or an interruption, instead of staying idle, the server goes for search of customers in the buffer with a known probability. By the introduction of the 'search mechanism', the idle time of the server is considerably reduced and there by attaining the optimum utilization of the server facility.

5.1 The mathematical model

We consider a single server retrial queueing system at which primary customers arrive according to a Poisson process with rate \( \lambda \). The retrial is assumed to be exponential with rate \( j \mu \), when there as \( j \) customers in the orbit (ie, the classical retrial policy). The service is interrupted at an exponential rate \( \sigma \). The interrupted customer goes back to the orbit with a known probability \( q \) or leaves the system with the complimentary probability \((1 - q)\). At the epoch when the server becomes free, (either by a service completion or by a service interruption) it goes for search of customers in the orbit with some known probability \( p \) or remains idle with the probability \( 1 - p \). The search time is assumed to be negligible. The service time is assume to follow a phase-type distribution \((PH\)-distribution) with representation
\( (\beta, S) \) of order \( m \).

\( PH \)-distributions and ‘\( PH \)-renewal processes’ were introduce by Neuts [59]. They have gained widespread attention in the area of stochastic modelling, particularly in queueing theory. A phase-type distribution on \([0, \infty)\) of order \( m \) is defined as the absorption-time distribution in a finite state Markov process with \( m \) transient states and one absorbing state as follows: Consider a Markov process on the states \( \{0, 1, \ldots, m\} \) with infinitesimal generator

\[
Q = \begin{bmatrix}
S & S^0 \\
0 & 0
\end{bmatrix}
\]

where the \( m \times m \) matrix \( S \) satisfies \( S_{ii} < 0 \), for \( 1 \leq i \leq m \), and \( S_{ij} \geq 0 \) for \( i \neq j \). Also \( Se + S^0 = 0 \) and the initial probability vector of \( Q \) is given by \( (\beta, \beta_0) \), with \( \beta e + \beta_0 = 1 \), where \( e \) is a column vector of 1's in all its \( m \) positions. It can be shown that the states 1, \ldots, \( m \) are transient if and only if the matrix \( S \) is non-singular. Also, the probability distribution \( H(\cdot) \) of the time until absorption in the state 0, corresponding to the initial probability vector \( (\beta, \beta_0) \), is given by

\[
H(x) = 1 - \beta \exp(Sx)e, \quad \text{for } x \geq 0
\]

The probability distribution \( H(\cdot) \) on \([0, \alpha)\) is a \( PH \)-distribution if and only if it is the distribution of the time until absorption in a finite Markov chain of the type defined in (5.1). Note that the distribution \( H(\cdot) \) has a jump of height \( \beta_0 \) at \( x = 0 \) and its density \( H'(x) \) on \((0, \infty)\) is given

\[
H'(x) = \beta \exp(Sx)S^0
\]

The Laplace-Stieltjes transform \( f(s) \) of \( H(\cdot) \) is given by

\[
f(s) = \beta_0 + \beta(sI - S)^{-1}S^0 \quad \text{for Re } s \geq 0
\]
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The non-central moments $\mu'_i$ of $H(\cdot)$ are all finite and is given by

$$\mu'_i = (-1)^i i! (\beta S^{-i} - e), \text{ for } i \geq 0 \quad (5.5)$$

Discrete $PH$-distributions are defined by considering an $(m+1)$-state Markov chain $P$ of the form

$$P = \begin{bmatrix} S & S^0 \\ 0 & 1 \end{bmatrix}$$

where $S$ is a substochastic matrix, such that $I - S$ is non-singular. The initial probability vector is $([\beta, \beta_0])$. The probability density $\{p_k\}$ of phase type is given by

$$p_0 = \beta_0, \quad p_k = \beta S^{k-1} S^0; \quad k \geq 1$$

Its probability generating function $P(z) = \beta_0 + z\beta(I - zS)^{-1}S^0$ and the factorial moments are given by

$$P^{(k)}(1) = k! \beta S^{k-1} (I - S)^{-k}e$$

5.2 Algorithmic solution

Our model may be studied as a level dependent quasi birth-and-death process (LDQBD) with the state space given by

$$E = \{j, \bar{j}; \ j \geq 0\}, \text{ where } j = \{(j, 0); \ j \geq 0\} \text{ and } \bar{j} = \{(j, 1, k); j \geq 0, \ 1 \leq k \leq m\}.$$
The states \( j = (j, 0), j \geq 0 \) correspond to the idle server; with \( j \) customers in the orbit; the states \( j = (j, 1, k), j \geq 0, 1 \leq k \leq m \) correspond to the busy server with the service process in the phase \( k \), and \( j \) customers in the orbit.

The generator \( Q \) is given by

\[
Q = \begin{bmatrix}
B_0 & A_0 & 0 & 0 & 0 & 0 & \ldots \\
A_{21} & A_{11} & A_0 & 0 & 0 & 0 \\
0 & A_{22} & A_{12} & A_0 & 0 & 0 \\
0 & 0 & A_{23} & A_{13} & A_0 & 0 \\
\ldots
\end{bmatrix}
\]

Where

\[
B_0 = \begin{bmatrix}
-\lambda & \lambda \beta \\
\sigma (1 - q)e + S^0 & S - (\lambda + \sigma)I + \sigma pq e \beta
\end{bmatrix}
\]

\[
A_0 = \begin{bmatrix}
0 & 0 \\
\sigma q (1 - p)e & \lambda I
\end{bmatrix}
\]

\[
A_{2i} = \begin{bmatrix}
0 & i \mu \beta \\
0 & S^0 p \beta + \sigma p (1 - q) e \beta
\end{bmatrix} \quad ; \quad i \geq 1
\]

and

\[
A_{1i} = \begin{bmatrix}
-(\lambda + i \mu) & \lambda \beta \\
S^0 (1 - p) + \sigma (1 - q) (1 - p) e & S - (\lambda + \sigma)I + \sigma pq e \beta
\end{bmatrix} \quad ; \quad i \geq 1
\]

Let \( x = (x(0), x(1), \ldots) \) be the steady state probability vector associated with \( Q \). That is, when the queue is stable, \( x \) is the unique solution to

\[
xQ = 0 \text{ and } xe = 1 \quad (5.6)
\]

Even though \( Q \) is highly structured, \( x \) cannot be expressed in a tractable analytical form. So we propose an algorithmic solution based on the Neuts-Rao truncation
approach as described in section 2 of chapter 4. Recall that, if the number of customers in the orbit is small, the likelihood of an idle server and therefore that of a retrial request being successful is not small. When the number of customers in the orbit is sufficiently large, a majority of the retrial requests fail to find a free server and do not result in a change of state. Therefore, if the number of customers in the orbit who can generate retrial requests is restricted to an appropriately chosen number $N$, then the effect on the system dynamics and the equilibrium probability vector is minimal. That approximation modifies the infinitesimal generator $Q$ to that given below:

$$
\bar{Q} = \begin{bmatrix}
B_0 & A_0 & 0 & \cdots & 0 \\
A_{21} & A_{11} & A_0 & 0 & \cdots \\
A_{22} & A_{12} & A_0 & 0 & \cdots \\
& \ddots & \ddots & \ddots & \ddots \\
& & A_{2,N-1} & A_{1,N-1} & A_0 & \cdots & 0 \\
& & & A_2 & A_1 & A_0 & \cdots \\
& & & & A_2 & A_1 & A_0 & \cdots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots & \ddots \\
\end{bmatrix}
$$

where $A_1 = A_{1N}$, and $A_2 = A_{2N}$.

### 5.2.1 Stability condition

The steady state probability vector exists if and only if

$$
\pi A_0 e < \pi A_2 e. 
$$

(5.7)

where $\pi$ is the invariant probability vector of the matrix $A = A_0 + A_1 + A_2$ given by

G[856]
\[ A = \begin{bmatrix} -\lambda - N\mu & (\lambda + N\mu)\beta \\ (1 - p)S^0 + \sigma(1 - p) & S^0(p\beta + S + \sigma p e\beta - \sigma I) \end{bmatrix} \quad (5.8) \]

Let \( \bar{x} = (\bar{x}_0, \bar{x}_1) \) where \( \bar{x}_0 \) is a scalar and \( \bar{x}_1 \) is a vector of order \( m \).

\[ \bar{x}A = 0, \text{ subject to } \bar{x}_0 + \bar{x}_1e = 1 \implies \]

\[ \bar{x}_1 = (\lambda + N\mu)\bar{x}_0\beta[\sigma I - pS^0\beta - S - p\sigma e\beta]^{-1} \]

Then from (5.7), after some algebra we get the stability condition as

\[ (\lambda + N\mu)\beta[\sigma I - pS^0\beta - S - p\sigma e\beta]^{-1}[(\lambda + \sigma(q - p))e - ps^0] < N\mu \]

### 5.2.2 Computation of the vector \( x \)

Because of the special structure of \( \bar{Q}, x \) can be expressed as

\[ x(i + N - 1) = x(N - 1)R^i, \quad i \geq 0 \quad (5.9) \]

where the matrix \( R \) is the unique non-negative solution with spectral radius less than 1 of the equation

\[ R^2A_2 + RA_1 + A_0 = 0 \quad (5.10) \]

The vectors \( x(0), \ldots, x(N - 1) \) can be obtained by solving the following equations.

\[ x(i - 1)A_0 + x(i)A_{1i} + x(i + 1)A_{2i+1} = 0, \quad 1 \leq i \leq N - 1 \quad (5.11) \]

\[ x(N - 2)A_0 + x(N - 1)[A_{1,N-1} + RA_2] = 0 \]
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Subject to the normalizing condition

$$\sum_{i=0}^{N-2} x(i) + x(N-1)(I - R)^{-1}e = 1$$  \hspace{1cm} (5.12)

Due to the special structure of $A_0$, the matrix $R$ can be computed as follows:

Rewrite the matrix $R$ as

$$R = \begin{bmatrix} 0 & 0 \\ R_0 & R_1 \end{bmatrix}$$  \hspace{1cm} (5.13)

where $R_0$ is a column vector of order $m$ in and $R_1$ is a square matrix of order $m$.

Then (5.10) yields

$$-(\lambda + N\mu)R_0 + (1 - p)R_1S^0 + \sigma(1 - p)(1 - q)R_1e + \sigma q(1 - p)e = 0$$

and

$$N\mu R_1 R_0 \beta + p R_1^2 S^0 \beta + \sigma(1 - q)p R_1^2 e_\beta + \lambda R_0 \beta + R_1[S - (\lambda + \sigma)I + \sigma q e] + \lambda I = 0$$

Thus we obtain $R_0$ and $R_1$ as

$$R_0 = \frac{(1 - p)}{(\lambda + N\mu)} \{ R_1[S^0 + \sigma(1 - q)e] + \sigma q e \}$$

and

$$R_1 = \{ N\mu R_1 R_0 \beta + p R_1^2 S^0 \beta + \sigma(1 - q)p R_1^2 e_\beta + \lambda R_0 \beta + \lambda I \}$$

$$\{ (\lambda + \sigma)I - S - \sigma q e \beta \}^{-1}$$  \hspace{1cm} (5.14)

Partition the components $x(i)$ of the vector $x$ as $x(i) = (x_i(0), x_i(1))$, $i \geq 0$. Where $x_i(0)$ is a scalar and $x_i(1)$ is vector of order $m$. 
From (5.9), we get

\[
(x_{i+N-1}(0), x_{i+N-1}(1)) = (x_{N-1}(0), x_{N-1}(1)) \begin{bmatrix} 0 & 0 \\ R_{i}^{-1} & R_{i} \end{bmatrix}
\]

which yields

\[
x_{i+N-1}(j) = x_{N-1}(1)R_{i}^{-1}R_{j}, \quad i \geq 1, j \in \{0, 1\}
\] (5.15)

Now from (5.11),

\[
x(0) = x(1)A_{21}(-B_{0})^{-1} = x(1)A_{21}(-A_{0}')^{-1}
\]

and

\[
x(1) = -x(2)A_{22}[A_{11} + A_{21}(-A_{0}')^{-1}A_{0}]^{-1}
\]

\[= x(2)A_{22}(-A_{0}')^{-1}
\]

In general,

\[
x(i) = x(i + 1)A_{2,i+1}(-A_{0}')^{-1}; \quad 0 \leq i \leq N - 1
\] (5.16)

where

\[
A_{i}' = \begin{cases} B_{0} & i = 0 \\ A_{11} + A_{21}(-A_{0}')^{-1}A_{0} & 1 \leq i \leq N \end{cases}
\] (5.17)

Now, by applying block Gaussian elimination, the partitioned subvector

\[(x(N), x(N + 1), \ldots)\]

corresponding to non-boundary states, satisfies the relation.

\[
\begin{bmatrix} A_{N}' & A_{0} \\ A_{2} & A_{1} & A_{0} \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} = 0
\] (5.18)
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Let

\[ \delta = \sum_{i=N}^{\infty} x(i)e \]  
and \[ y(i) = \delta^{-1} x(N + i), \quad i \geq 0 \]

From (5.18), we get

\[ x(N)A_N' + x(N + 1)A_2 = 0 \]
\[ x(N + i) = x(N + i - 1)R, \quad i \geq 1 \]

which implies

\[ y(0)A_N' + y(1)A_2 = 0 \]
and

\[ y(i) = y(i - 1)R, \quad i \geq 1 \]

Since \( \sum_{i=0}^{\infty} y(i)e = 1 \), we get

\[ y(0)(I - R)^{-1}e = 1 \]

Thus, \( x(i) = \delta y(0)R^{i-N}, \quad i \geq N \).

Again by (5.16), we get

\[ x(i) = \delta y(0) \prod_{j=i}^{N} A_{2j}(-A_{j-1}')^{-1}, \quad 0 \leq i \leq N - 1 \]

Therefore,

\[ x(i) = \begin{cases} 
\delta y(0) \prod_{j=i}^{N} A_{2j}(-A_{j-1}')^{-1}, & 0 \leq i \leq N - 1 \\
\delta y(0)R^{i-N} & i \geq N 
\end{cases} \]  
(5.21)
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where \( y(0) \) is the unique solution of the system

\[
y(0)(A_N' + RA_2) = 0
\]
\[
y(0)(I - R)^{-1} e = 1
\]

(5.22)

Now \( xe = 1 \) implies

\[
\delta y(0) \sum_{i=0}^{N-1} \prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1} e + \delta y(0) \sum_{i=N}^{\infty} R^{i-N} e = 1
\]

Now by using the second equation in (5.22) we get

\[
\delta = [1 + y(0) \sum_{i=0}^{N-1} \prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1} e]^{-1}
\]

(5.23)

5.3 Other system characteristics

1. Probability mass function of the number of customers in the orbit.

\( Pr \{ i \text{ customers in the orbit} \} \) is given by

\[
a_i = x(i)e
\]
\[
= \begin{cases} 
\delta y(0) \prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1} e, & 0 \leq i \leq N - 1 \\
\delta y(0) R^{i-N} e, & i \geq N 
\end{cases}
\]

2. Expected number of customers in the orbit is given by

\[
EN = \sum_{i=1}^{\infty} i x(i)e
\]
\[
= \delta y(0) \{ \sum_{i=0}^{N-1} i \prod_{j=i}^{N} A_{2j}(-A'_{j-1}) e + R(I - R)^{-2} e + N(I - R)^{-1} e \}
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3. Probability mass function of the server state

\[ P_0 = \Pr[\text{server is idle}] = \sum_{i=0}^{\infty} x_i(0) \]
\[ = \sum_{i=0}^{N-1} x_i(0) + x_{N-1}(1)(I - R_1)^{-1}R_0 \]

\[ P_1 = \Pr[\text{server is busy}] = \sum_{i=0}^{\infty} x_i(1)e \]
\[ = \sum_{i=0}^{N-1} x_i(1)e + x_{N-1}(1)(I - R_1)^{-1}R_1e \]

4. The mean time spent by an arbitrary customer in the orbit \( W_q = \frac{1}{\lambda} EN \).

5. The overall rate of retrials

\[ \mu_1^* = \sum_{i=1}^{\infty} i \mu x(i)e = \mu \cdot EN \]

6. The successful rate of retrials

\[ \mu_2^* = \sum_{i=1}^{\infty} i \mu x_i(0) \]
\[ = \mu \{ \sum_{i=1}^{N-1} ix_i(0) + x_{N-1}(1)(R_1(I - R_1)^{-2} + N(I - R_1)^{-1}R_0) \} \]

7. Expected number of busy servers

\[ EC = p_1 \]
8. Expected number of customers in the system

\[ ES = EN + EC \]

9. The mean time spent by an arbitrary customer in the system

\[ W_s = \frac{1}{\lambda} ES. \]