CHAPTER 1
FUZZY SUBMETRIZABILITY

1.1 Introduction

Metrizability is a very nice but restrictive property for topological spaces. The notion of submetrizability (for details refer [G G]) is less restrictive but retains some of the nice properties of metrizability. A topological space \((X, T)\) is submetrizable if there exists a topology \(T' \subset T\) with \((X, T')\) metrizable. In this chapter we define analogously the concept of fuzzy metrizable spaces, fuzzy submetrizable spaces and obtain some characterizations of fuzzy submetrizable spaces. Also we study the relation between fuzzy paracompact spaces and fuzzy metrizable spaces.

1.2 Fuzzy Submetrizable Spaces

Definition 1.2.1

Let \((X, F)\) be a fuzzy topological space. Let \(\mathcal{L}(F)\) be the weakest topology on \(X\) which makes all the functions in \(F\) are lower semicontinuous. Then the fuzzy topological space \((X, F)\) is said to be fuzzy metrizable if \((X, \mathcal{L}(F))\) is metrizable.

* We have included some results of this chapter in the paper titled On Fuzzy Submetrizability in The Journal of Fuzzy Mathematics, Vol. 10 No. 2 (2002).

** Some Results mentioned in this chapter are published in the paper titled “Fuzzy Metrizability and Fuzzy Compactness” in the proceedings of The International Workshop and seminar on Transform Techniques and Their Applications held at St. Joseph’s College, Irinjalakkuda, Kerala (2001)
Definition 1.2.2

The fuzzy topological space \((X, F)\) is said to be fuzzy submetrizable if there exists \(F' \subset F\) such that \((X, F')\) is fuzzy metrizable.

Example 1.2.3 - Fuzzy metrizable spaces

1. Consider \(X = \mathbb{R}\), the real line. Let \(F\) be the fuzzy topology generated by the set \(\{ \chi_U \mid U \text{ open in the usual topology on } \mathbb{R} \}\). Then \((X, F)\) is fuzzy metrizable.

2. Let \((X, T)\) be a metrizable topological space. Then \(\omega(T) = \{ f \mid f : (X,T) \rightarrow [0,1] \text{ is lower semicontinuous} \}\) is a fuzzy topology on \(X\), called the generated fuzzy topology and the fuzzy topological space \((X, \omega(T))\) is fuzzy metrizable.

Example 1.2.4 - Fuzzy Submetrizable Spaces

1. Consider \(X = \mathbb{R}\). For intervals of the type \([a,b)\) define \(f_{(a,b)}: \mathbb{X} \rightarrow [0,1]\) by

\[
 f_{(a,b)}(x) = \begin{cases} 
 1 & \text{if } x \in (a,b) \\
 1/2 & \text{if } x = a \\
 0 & \text{if } x \notin [a,b)
\end{cases}
\]

Let \(F\) be the fuzzy topology generated by \(\{ f_{(a,b)}, \chi_{(a,b)} \mid a, b \in \mathbb{R} \}\). Now the weakest topology \(\tau(F)\), which makes all elements of \(F\) lower semicontinuous, is the lower limit topology, and \((X, \tau(F))\) is not metrizable. Therefore \((X, F)\) is not fuzzy metrizable.

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If $F'$ is the fuzzy topology generated by \( \{ \chi_{(a,b)} | a, b \in \mathbb{R} \} \), then $(X, F')$ is fuzzy metrizable, since $(X, \mathcal{L}(F'))$, where $\mathcal{L}(F')$ is the usual topology, is metrizable. Now $F' \subset F$. Therefore $(X, F)$ is fuzzy submetrizable.

2. Consider $X = \mathbb{R}$. For $G, H$ subsets of $\mathbb{R}$, $G$ open with respect to the usual topology on $\mathbb{R}$ and $H$ any subset of irrationals, define $f_{G,H}: X \to [0,1]$ by

\[
f_{G,H}(x) = \begin{cases} 
1 & \text{if } x \in G \\
1/2 & \text{if } x \notin G \text{ and } x \in H \\
0 & \text{otherwise}
\end{cases}
\]

Let $T$ be the topology on $\mathbb{R}$ with basic open sets as \( \{ G \cup H \mid G \text{ open with respect to usual topology, } H \text{ subset of irrationals} \} \). Consider $F = \{ f_{G,H} | G, H \subset \mathbb{R}, G \text{ open in the usual topology, } H \text{ any subset of irrationals} \} \cup \{0, 1\}$. Then the fuzzy topological space $(X, F)$ is not fuzzy metrizable, since $(X, T)$ is not metrizable and $T = \mathcal{L}(F)$. Let $F' = \{ f_G, \phi | G \text{ open in the usual topology} \} \cup \{0, 1\}$. Then the weakest topology $\mathcal{L}(F')$, which makes every member of $F'$ lower semi-continuous, is the usual topology on $\mathbb{R}$. Now $(X, \mathcal{L}(F'))$ is metrizable. Therefore $(X, F')$ is fuzzy metrizable. Also $F' \subset F$. Hence $(X, F)$ is fuzzy submetrizable.

Remark 1.2.5

The concept of fuzzy metrizability and fuzzy submetrizability that we have introduced above are 'good extensions' of the crisp metrizability and submetrizability in the sense of R. Lowen [LO1].
Definition 1.2.6

The fuzzy topological space \((X, F)\) is said to have a \(G_\delta\)-diagonal if the diagonal \(\Delta\) is a \(G_\delta\)-set in \((X^2, F_p)\) where \(F_p\) is the fuzzy product topology.

Definition 1.2.7

Let \(\mathcal{A}\) be a cover of \((X, F)\). For \(\alpha \in (0,1]\) and a fuzzy point \(x_\alpha\),

\[
\text{st}(x_\alpha, \mathcal{A}_n) = \bigvee \{B : B \in \mathcal{A} \text{ and } B(x) \geq \alpha\} \text{ and for a fuzzy set } G, \text{ st}(G, \mathcal{A}) = \bigvee \{B : B \in \mathcal{A} \text{ and } B \wedge G \neq 0\}.
\]

Theorem 1.2.8

A fuzzy topological space \((X, F)\) has a \(G_\delta\)-diagonal if and only if there exists a sequence \((\mathcal{A}_n)\) of open covers of \((X, F)\) such that for \(x, y \in X\) with \(x \neq y\), \(\alpha, \beta \in (0,1]\) there exists \(n \in \mathbb{N}\) with \(y_\beta \notin \text{st}(x_\alpha, \mathcal{A}_n)\).

Proof

First suppose that \((X, F)\) has a \(G_\delta\)-diagonal. Then \(\Delta = \bigwedge_n G_n\) where each \(G_n\) is a fuzzy open set on \((X^2, F_p)\). For each \(n \in \mathbb{N}\), \(x \in X\) and for each \(\alpha \in (0,1]\) we have \(x_\alpha \times x_\alpha \leq \Delta \leq G_n\) (here \(x_\alpha \times x_\alpha \leq \Delta\) means that \(\Delta(x,x) \geq \alpha\)). Since \(G_n \in F_p\), there exists \(H^{\alpha,x}_n \in F\) such that \(x_\alpha \leq H^{\alpha,x}_n\).
and $H_n^\alpha \times H_n^\beta \subseteq G_n$. Then $\mathcal{A}_n = \{ H_n^\alpha | x \in X, \alpha \in (0,1) \}$ forms an open cover of $(X,F)$.

For $x, y \in X$ with $x \neq y$, $\alpha, \beta \in (0,1]$, we claim that there exists $n \in \mathbb{N}$ with $y_\beta \not\in \text{st}(x_\alpha, \mathcal{A}_n)$. For otherwise suppose that $y_\beta \leq \text{st}(x_\alpha, \mathcal{A}_n)$ for all $n$. Then $y_\beta \leq H_\gamma^\alpha$ for some $\gamma \in (0,1]$ and $H_\gamma^\alpha \leq \text{st}(x_\alpha, \mathcal{A}_n)$. Also $x_\alpha \leq H_\gamma^\alpha$. Therefore $x_\alpha \times y_\beta \leq H_\gamma^\alpha \times H_\gamma^\alpha \leq G_n$ for each $n$. That is $x_\alpha \times y_\beta \leq \bigwedge_n G_n = \Delta$, which is a contradiction. Thus $(\mathcal{A}_n)$ satisfies the conclusion of the theorem.

Conversely suppose that $(\mathcal{A}_n)$ be a sequence of open covers which satisfies the conditions mentioned in the theorem. Let $G_n = \bigvee \{ A \times A | A \in \mathcal{A}_n \}$. Then for $\alpha \in (0,1]$, $x_\alpha \leq A$ for some $A \in \mathcal{A}_n$. Therefore $x_\alpha \times x_\alpha \leq A \times A$. Hence $\Delta = \bigvee (x_\alpha \times x_\alpha) \leq \bigvee \{ A \times A | A \in \mathcal{A}_n \} = G_n$ for each $n$. Therefore $\Delta \leq \bigwedge_n G_n$. If for each $\alpha, \beta \in (0,1]$ and $x \neq y$ with $x_\alpha \times y_\beta \leq \bigwedge_n G_n$, we have $x_\alpha \times y_\beta \leq G_n$ for each $n$. Hence there exists $A_n \in \mathcal{A}_n$ with $x_\alpha \times y_\beta \leq A_n \times A_n$. That is $y_\beta \leq \text{st}(x_\alpha, \mathcal{A}_n)$, which is a contradiction. Therefore $\bigwedge_n G_n = \bigvee \{ (x_\alpha \times x_\beta) | x \in X \text{ and } \alpha, \beta \in (0,1] \} = \Delta$. That is $(X,F)$ has a $G_\delta$-diagonal.
Remark

We write $\leq$ and not $>$, although they are the same here, keeping in mind the fact that the same concept of the theorem can be extended to L-fuzzy topological spaces.

Remark 1.2.9

If $(\mathcal{A}_n)$ is a sequence of fuzzy open covers with the property in theorem 1.2.8, then for $x \in X$, $\alpha \in (0, 1]$, $\bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\} = x_\alpha$.

Proof

Let $y \in \text{support of } \bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\}$. Let $\bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\}(y) = \beta$. Therefore $y_\beta \leq \bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\}$. Therefore $y_\beta \leq \bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\}$ for all $n$. Hence by the above theorem $y = x$. Therefore support of $\bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\} = \{x\}$. Let $\gamma = \bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\}(x)$ (note that $\gamma \geq \alpha > 0$). If $\gamma > \alpha$ then by passing onto refinements we can form a sequence of fuzzy open covers $(\mathcal{A}_n)$ such that $\bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\}(x) = \alpha$. Hence $\bigwedge_n \text{st}\{x_\alpha, \mathcal{A}_n\} = x_\alpha$. 

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Example 1.2.10

Consider \( X = \mathbb{R} \). For \( G, H \) subsets of \( \mathbb{R} \), \( G \) open with respect to the usual topology on \( \mathbb{R} \) and \( H \) any subset of irrationals, define \( f_{G,H} : X \to [0, 1] \) by

\[
f_{G,H}(x) = \begin{cases} 
1 & \text{if } x \in G \\
\frac{1}{2} & \text{if } x \not\in G \text{ and } x \in H \\
0 & \text{otherwise}
\end{cases}
\]

Let \( F = \{ f_{G,H} | G, H \subset \mathbb{R}, G \text{ open in the usual topology, } H \text{ any subset of irrationals} \} \cup \{ 0, 1 \} \). Consider \( (X^2, F_p) \), where \( F_p \) is the fuzzy product topology. Basic fuzzy open sets in \( (X^2, F_p) \) can be written as \( f_{G_1,H_1} \times f_{G_2,H_2} \)

where \( (f_{G_1,H_1} \times f_{G_2,H_2})(x,y) = \min\{ f_{G_1,H_1}(x), f_{G_2,H_2}(y) \} \). For each \( n \), take \( G_n = f_{\left(\frac{1}{n}, \frac{1}{n}\right), \phi} \times f_{\left(\frac{1}{n}, \frac{1}{n}\right), \phi} \). Then

\[
G_n(x,y) = \begin{cases} 
1 & \text{if } x,y \in \left(\frac{1}{n}, \frac{1}{n}\right) \\
0 & \text{otherwise}
\end{cases}
\]

Then the diagonal \( \Delta = \bigwedge G_n \). That is \( \Delta \) is a \( G_\delta \)-set. Therefore \( (X, F) \) is having a \( G_\delta \)-diagonal.

Definition 1.2.11

A sequence \( (\mathcal{A}_n) \) of fuzzy open covers of \( (X,F) \) is called a \( G_\delta \)-diagonal sequence, if for each \( x, y \in X \) with \( x \neq y \), \( \alpha, \beta \in (0, 1] \), there exists
n \in \mathbb{N} \text{ with } y_\beta \not\in \text{st}(x_\alpha \mathcal{A}_n). \text{ That is } \bigwedge_{n} \text{st}(x_\alpha, \mathcal{A}_n) = x_\alpha. \text{ A space } (X, F) \text{ has a } G_\delta \text{-diagonal if there exists a } G_\delta \text{-diagonal sequence.}

**Definition 1.2.12**

A cover \( \mathcal{A} \) of \((X, F)\) is called a star refinement of the cover \( \mathcal{B} \) if

\[ \{ \text{st}(G, \mathcal{A}) : G \in \mathcal{A} \} \text{ refines } \mathcal{B}. \text{ That is for each } G \in \mathcal{A}, \text{ there exists } H \in \mathcal{B} \text{ such that } \text{st}(G, \mathcal{A}) \subseteq H. \]

**Definition 1.2.13**

Let \((X, F)\) be a fuzzy topological space. A fuzzy covering \( \mathcal{A}_{n+1} \) is a regular refinement of the covering \( \mathcal{A}_n \) if for \( G, H \in \mathcal{A}_{n+1} \) with \( G \wedge H \neq \emptyset, G \vee H \leq B_n \) for some \( B_n \in \mathcal{A}_n \).

**Theorem 1.2.14**

The following are equivalent for an induced fuzzy topological space \((X, F)\).

(a) \((X, F)\) is fuzzy submetrizable

(b) \((X, F)\) has a \( G_\delta \)-diagonal sequence \( (\mathcal{A}_n) \) such that \( \mathcal{A}_{n+1} \) star refines \( \mathcal{A}_n \) for each \( n \)

(c) \((X, F)\) has a \( G_\delta \)-diagonal sequence \( (\mathcal{A}_n) \) such that \( \mathcal{A}_{n+1} \) is a regular refinement of \( \mathcal{A}_n \) for each \( n \).
Proof

(a) \implies (b)

Assume that \((X, F)\) is fuzzy submetrizable. Then there exists \(F^1 \subset F\) such that \((X, F^1)\) is fuzzy metrizable. Therefore \((X, \lambda(F^1))\) is metrizable. Hence \((X, \lambda(F^1))\) is paracompact. Therefore each open cover of \((X, \lambda(F^1))\) has an open star refinement (see [WI]). By metrizability we can form a \(G_\delta\)-diagonal sequence \((G_n)\) for \((X, \lambda(F^1))\). For each \(G_n \in G_n\), let \(A_{G_n}\) be the characteristic function of \(G_n\). Each \(A_{G_n}\) is lower semicontinuous and hence belongs to \(F^1\). We claim that \((A_{G_n})\) where \(A_{G_n} = \{ A_{G_n} \mid G_n \in G_n \}\) forms a \(G_\delta\)-diagonal sequence for \((X, F^1)\).

Consider \(x, y \in X\) with \(x \neq y\), \(\alpha, \beta \in (0, 1]\). Since \((G_n)\) forms a \(G_\delta\)-diagonal sequence for \((X, \lambda(F^1))\), there exists \(n \in \mathbb{N}\) such that \(y \notin \text{st}(x, G_n)\).

Now \(x_\alpha \neq y_\beta\) and \(\text{st}(x_\alpha, A_{G_n}) = \bigvee \{ B_{G_n} \mid B_{G_n} \in A_{G_n} \text{ and } B_{G_n}(x) \geq \alpha \}\).

Therefore \(\text{st}(x_\alpha, A_{G_n})(y) = 0\) where as \(y_\beta(y) = \beta\). Therefore \(y_\beta \notin \text{st}(x_\alpha, A_{G_n})\).

That is \((A_{G_n})\) forms a \(G_\delta\)-diagonal sequence for \((X, F^1)\). Since \(G_{n+1}\) star refines \(G_n\), it follows that \(A_{G_{n+1}}\) star refines \(A_{G_n}\).
(b) $\Rightarrow$ (c)

Let $(\mathcal{A}_n)$ be a $G_δ$-diagonal sequence for $(X, F)$ such that $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$ for each $n$. Let $G, H \in \mathcal{A}_{n+1}$ be such that $G \wedge H \neq 0$. Since $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$, there exists $A_n \in \mathcal{A}_n$ such that $G \leq \text{st}(G, \mathcal{A}_{n+1}) \leq A_n$. Since $G \wedge H \neq 0$, $H \leq \text{st}(G, \mathcal{A}_{n+1})$. Therefore $G \vee H \leq \text{st}(G, \mathcal{A}_{n+1}) \leq A_n \in \mathcal{A}_n$. Hence $\mathcal{A}_{n+1}$ is a regular refinement of $\mathcal{A}_n$ for each $n$.

(c) $\Rightarrow$ (a)

Let $(X, F)$ has a $G_δ$-diagonal sequence $(\mathcal{A}_n)$ such that $\mathcal{A}_{n+1}$ is a regular refinement of $\mathcal{A}_n$ for each $n$. For each $A_n \in \mathcal{A}_n$, let $G_n = \bigcup_{\alpha \in (0,1]} A_n^{-1}(\alpha, 1]$.

Then $G_n = \{G_n\}$ forms $G_δ$-diagonal sequence for $(X, \mathcal{L}(F))$. Since $F$ is an induced fuzzy topological space $[F] = \mathcal{L}(F)$. Take $G_{n+1}$, $G'_{n+1} \in \mathcal{G}_{n+1}$ with $G'_{n+1} \cap G_{n+1} \neq \emptyset$. Then there exists $A_{n+1}, A'_{n+1} \in \mathcal{A}_{n+1}$ with $A_{n+1} \wedge A'_{n+1} \neq 0$

and $G_{n+1} = \bigcup_{\alpha \in (0,1]} A_{n+1}^{-1}(\alpha, 1]$, $G'_{n+1} = \bigcup_{\alpha \in (0,1]} A'_{n+1}^{-1}(\alpha, 1]$.
By assumption \( A_{n+1} \lor A'_{n+1} \leq A_n \) for some \( A_n \in \mathcal{A}_n \). Therefore

\[
\bigcup_{\alpha \in (0,1)} \{ A_{n+1}^{-1}(\alpha,1] \lor A'_{n+1}^{-1}(\alpha,1] \} \subset \bigcup_{\alpha \in (0,1)} A_n^{-1}(\alpha,1] .
\]

That is \( G_{n+1} \lor G'_{n+1} \subset G_n \).

Thus the sequence \( (G_n) \) of open covers are such that \( G_{n+1} \) is a regular refinement of \( G_n \). Therefore by the Lemma 0.5.4, there exists a pseudometric \( \rho \) on \( X \) such that \( U \) is open in the topology generated by \( \rho \) if and only if for each \( x \in U \), there exists \( n \in \mathbb{N} \) such that \( st(x, G_n) \subset U \). Now \( \{x\} = \bigcap_{n} st(x, G_n) \). Therefore \( \rho \) is metric on \( X \) [by part(i) of Lemma 0.5.4]. Also by part (ii) of the Lemma 0.5.4, the topology generated by \( \rho \), say \( T' \), is contained in the topology \( \tau(F) \). Therefore \( (X, \tau(F)) \) is submetrizable. Let \( \omega(T') \) be the collection of all lower semicontinuous mappings from \( (X, T') \rightarrow [0,1] \). Now \( T' \subset \tau(F) \) and since \( F \) is induced \( \omega(T') \subset F \). Therefore \( (X, \omega(T')) \) is fuzzy metrizable. Hence \( (X, F) \) is fuzzy submetrizable.

### 1.3 Fuzzy Compactness, Fuzzy Paracompactness and Fuzzy Metrizability

In this section we connect fuzzy paracompact spaces with fuzzy submetrizable spaces. Also we prove some relationship between fuzzy metrizable spaces and fuzzy compact spaces.
Lemma 1.3.1

If \((X, F)\) is an induced fuzzy paracompact space, then the generated topological space \((X, \mathcal{L}(F))\) is paracompact.

Proof

Since \((X, F)\) is induced, \([F] = \mathcal{L}(F)\). Also as \((X, F)\) is a fuzzy paracompact space, \((X, [F])\) is paracompact [see theorem 0.4.10]. That is \((X, \mathcal{L}(F))\) is paracompact.

Lemma 1.3.2

Let \((X, F)\) be an induced fuzzy topological space. If \((X, F)\) is fuzzy paracompact with a \(G_δ\)-diagonal then it is fuzzy submetrizable.

Proof

Let \((\mathcal{A}_n)\) be a \(G_δ\)-diagonal sequence for the induced fuzzy paracompact space \((X, F)\). Then by lemma 1.3.1 \((X, \mathcal{L}(F))\) is paracompact. Therefore every open cover of \((X, \mathcal{L}(F))\) has an open star refinement [see Theorem 0.5.10].

Consider \(\mathcal{G}_n = \{G_n \mid G_n = \bigcup_{\alpha \in (0,1]} A_n^{-1}(\alpha,1], A_n \in \mathcal{A}_n\}\). Then \((\mathcal{G}_n)\) forms a \(G_δ\)-diagonal sequence for \((X, \mathcal{L}(F))\). For if \(x, y \in X, x \neq y\) and there exists no \(n \in \mathbb{N}\) with \(y \not\in \text{st}(x, \mathcal{G}_n)\), then we have \(y \in \text{st}(x, \mathcal{G}_n)\) for all \(n\).
For each \( n \), \( y \in \text{st}(x, G_n) \Rightarrow x, y \in G_n \), for some \( G_n \in \mathcal{G}_n \).

\[
y \in A_n^{-1}(\beta, 1], x \in A_n^{-1}(\alpha, 1] \text{ for some } \alpha, \beta \in (0, 1]
\]

\[
\Rightarrow A_n(y) > \beta, A_n(x) > \alpha
\]

\[
\Rightarrow y_\beta < A_n \text{ and } x_\alpha < A_n \text{ for some } A_n \in \mathcal{A}_n.
\]

\[
\Rightarrow y_\beta < \text{st}(x_\alpha, \mathcal{A}_n).
\]

This is a contradiction as \( (\mathcal{A}_n) \) is a \( G_\delta \)-diagonal sequence for \( (X, F) \).

As \( (X, T) \) is paracompact, there exists a \( G_\delta \)-diagonal sequence say \( (\mathcal{G}_n^\prime) \) such that, \( \mathcal{G}_{n+1}^\prime \) star refines \( \mathcal{G}_n^\prime \) (see [GG]). Corresponding to each \( \mathcal{G}_n^\prime \), form \( \mathcal{A}_n^\prime \) where \( \mathcal{A}_n^\prime = \{ \chi_{G_{n}^\prime}, G_{n}^\prime \in \mathcal{G}_n^\prime \} \). Then \( (\mathcal{A}_n^\prime) \) forms a \( G_\delta \)-diagonal sequence for \( (X,F) \) such that \( \mathcal{A}_{n+1}^\prime \) star refines \( \mathcal{A}_n^\prime \). Therefore by theorem 1.2.14(b), \( (X, F) \) is fuzzy submetrizable.

**Theorem 1.3.3**

Let \( (X,F) \) be an induced Hausdorff fuzzy topological space. If \( (X, F) \) is a fuzzy compact space with a \( G_\delta \)-diagonal, then it is fuzzy metrizable.
Proof

Let \((X, F)\) be an induced Hausdorff fuzzy compact space with a \(G_\delta\)-diagonal. Then \((X, F)\) is a fuzzy paracompact space [see Theorem 0.4.11 and Theorem 0.4.12]. Since \((X, F)\) is induced, by theorem 1.3.2, it follows that \((X, F)\) is fuzzy submetrizable. Therefore there exists \(F' \subseteq F\) such that \((X, F')\) is fuzzy metrizable. Now, as \((X, F)\) is induced, \(\mathcal{L}(F) = [F]\). Then \((X, \mathcal{L}(F))\) is a compact Hausdorff space [see Theorem 0.3.7 and Remark 0.4.13]. Also, \((X, \mathcal{L}(F')) \subseteq (X, \mathcal{L}(F))\) and \((X, \mathcal{L}(F'))\) is metrizable. But a topology which is strictly weaker than a compact Hausdorff topology cannot be Hausdorff. Therefore \(\mathcal{L}(F') = \mathcal{L}(F)\) so that \(F' = F\). Hence \((X, F)\) is fuzzy metrizable.