MODELLING AND ANALYSIS OF THE SPREAD OF
CARRIER DEPENDENT INFECTIOUS DISEASES:
EFFECTS OF INFRASTRUCTURES, DEPENDING
PARTIALLY ON HUMAN POPULATION DENSITY

5.1 INTRODUCTION

In chapter 4, we have proposed and analysed an SIS model for spread of carrier dependent infectious diseases by considering the effects of infrastructures the density of which is wholly dependent on human population density. In this chapter, the same problem is studied by considering the effects of infrastructures, the cumulative density of which is partially dependent on the human population density. It is assumed in this model, as in chapter 4, the density of carrier population is governed by a logistic model and growth rate and carrying capacity of which increases as the cumulative density of infrastructures increases. The growth rate of cumulative density of infrastructures is assumed to grow by a constant rate as well as by the human population density nonlinearly.

5.2 AN SIS MODEL

Let $X(t)$ and $Y(t)$ denote densities of susceptible and infective classes respectively of total human population density $N(t) = X(t) + Y(t)$, in a region under consideration. Let $C(t)$ be the carrier population density which affects all susceptibles and $I(t)$ be the cumulative density of infrastructures. By keeping in view of the above and assuming simple mass action interaction, an SIS model can be written as follows:

\[
\begin{align*}
\frac{dX}{dt} &= A - \beta XY - \lambda XC + vY - eX \\
\frac{dY}{dt} &= \beta XY + \lambda XC - (v + \alpha + e)Y \\
\frac{dC}{dt} &= sC - \frac{s_0C^2}{L} + s_1IC + s_2IC^2 \\
\frac{dI}{dt} &= Q_0 - \theta_0 I + \theta_1 N + \theta_2 NI
\end{align*}
\]

(5.2.1)

where $X + Y = N$ with initial conditions $X(0) > 0, Y(0) \geq 0, C(0) \geq 0, I(0) > 0$. 

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In the modelling process we have assumed that the birth and natural death rate of the host human population are equal. It may be noted from the third equation of the model (5.2.1) the growth rate of carrier population density and its carrying capacity are increasing functions of the cumulative density of infrastructures which is increasing with constant rate and depending on human population density nonlinearly.

**List of Coefficients in the model (5.2.1):**

- $A$: Rate of immigration of susceptibles in the host population
- $e$: Rate of emigration from the host population
- $\beta$: Transmission coefficient due to infective human population
- $\lambda$: Transmission coefficient due to carrier population
- $\alpha$: Disease related death rate
- $\nu$: Recovery rate of infectives
- $s$: Intrinsic growth rate of carrier population
- $L$: Carrying capacity of carrier population in the absence of human activities
- $s_1$: Growth rate coefficient of carrier population
- $s_2$: Growth coefficient of the carrying capacity caused by the growth of cumulative density of infrastructures.
- $Q_0$: Growth rate of cumulative density of infrastructures
- $\theta_0$: Depletion rate coefficient for infrastructures
- $\theta_1$: Growth rate coefficient of infrastructural development due to human population density related factors
- $\theta_2$: Growth rate coefficient of infrastructural development due to human population density related factors nonlinearly
All the coefficients in the model (5.2.1) are assumed to be positive and constant. If we take $Q_0 = 0$, $\theta_2 = 0$ and $\theta_1 = \theta$ in the above model, we get the same model discussed in chapter 4. In the modelling process, the following assumptions are made:

(i) The birth and natural death rates of host population are equal.

(ii) The disease spread due to direct contact between infectives and susceptibles.

(iii) There exists a carrier population in the environment which provides an additional indirect mode of transport for the spread of an infectious disease.

(iv) The growth rate of carrier population density follows logistic model, the intrinsic growth rate and carrying capacity of which increase due to human activities, assumed to be proportional to its population density.

In the following we analyze the model (5.2.1) by the stability theory of differential equations:

### 5.3 EQUILIBRIUM ANALYSIS

Since $X + Y = N$, the model (5.2.1) can be written as follows:

$$
\begin{align*}
\frac{dY}{dt} &= \beta(N-Y)Y + \lambda(N-Y)C - (\nu + \alpha + e)Y \\
\frac{dN}{dt} &= A - eN - \alpha Y \\
\frac{dC}{dt} &= sC - \frac{s_0 C^2}{L} + s_1 IC + s_2 C^2 I \\
\frac{dl}{dt} &= Q_0 - \theta_0 I + \theta_1 N + \theta_2 NI
\end{align*}
$$

\tag{5.3.1}

The following lemma establishes region of attraction for the system (5.3.1).

**Lemma 5.3.1:** The set

$$
\Omega = \left\{(Y, N, C, I) : 0 \leq Y \leq N \leq \frac{A}{e + \alpha}, \frac{A}{e} \leq N \leq \frac{A}{e}, 0 \leq C \leq C_m, 0 \leq I \leq I_m \right\}
$$

attracts all the solutions initiating in the positive orthant, where
\[ C_m = \frac{s + s_1 I_m}{s_0 - s_2 I_m}, \quad I_m = \frac{Q_0 + \theta_1 A}{\theta_0 - \theta_2 \frac{A}{e}} \]  \hspace{1cm} (5.3.2)

provided,

\[ \frac{s_0}{s_2 L} > I_m, \quad \theta_0 > \theta_2 \frac{A}{e} \]  \hspace{1cm} (5.3.3)

**Proof:** Here we give only a brief outline of the proof, the detail proof can be seen in Freedman and So (1985).

From the second equation of model (5.3.1), we have

\[ \frac{dN}{dt} = A - e N - \alpha Y \leq A - eN \quad \text{and} \]

\[ \frac{dN}{dt} = A - eN - \alpha Y \geq A - (e + \alpha)N \]

which give

\[ 0 \leq Y \leq N \leq \frac{A}{e}, \quad \frac{A}{e + \alpha} \leq N \leq \frac{A}{e} \]

From the last equation of model (5.3.1), we have

\[ \frac{dI}{dt} \leq Q_0 - \theta_0 I + \theta_1 \frac{A}{e} + \theta_2 \frac{A}{e} I = Q_0 + \theta_1 \frac{A}{e} - \left( \theta_0 - \theta_2 \frac{A}{e} \right) I \]

which gives \( 0 < I \leq I_m = \left( \frac{Q_0 + \theta_1 \frac{A}{e}}{\theta_0 - \theta_2 \frac{A}{e}} \right) \)

which is positive provided \( \theta_0 > \theta_2 \frac{A}{e} \).

Similarly from the equation for carrier population density in (5.3.1), we have

\[ 0 \leq C \leq C_m = \frac{(s + s_1 I_m)}{s_0 - s_2 I_m} \]

\[ \frac{s_0}{L - s_2 I_m} \]
which is positive provided \( \frac{s_0}{L} > s_2 I_m \).

We analyse the model (5.3.1), under the conditions (5.3.3).

**Theorem 5.3.1:** The system (5.3.1) has following three equilibria:

(i) \( E_0 \left( 0, \frac{A}{e}, 0, I_m \right) \): the disease free equilibrium,

\[
I_m = \frac{Q_0 + \theta_1 \frac{A}{e}}{\theta_0 - \theta_2 \frac{A}{e}} \text{ which exists if } \theta_0 > \theta_2 \frac{A}{e} \text{, as assumed in (5.3.3)}.
\]

(ii) \( E_1 \left( \overline{Y}, N, 0, \overline{I} \right) \): the carrier free equilibrium,

\[
\overline{N} = \frac{\beta A + \alpha (v + \alpha + e)}{\beta (v + \alpha + e)}, \quad \overline{Y} = \frac{\beta A - e (v + \alpha + e)}{\beta (v + \alpha + e)}, \quad \overline{I} = \frac{Q_0 + \theta_1 N}{\theta_0 - \theta_2 N}
\]

and \( \overline{Y} \) exists if \( R_0 = \frac{\beta A}{e (v + \alpha + e)} > 1 \). Here \( R_0 \) is the basic reproduction number.

(iii) \( E^* \left( Y^*, N^*, C^*, I^* \right) \): the endemic equilibrium.

**Proof:** The existence of \( E_0 \) or \( E_1 \) is obvious. We prove the existence of \( E^* \). The equilibrium point \( E^* \) is given as the solutions of system of following equations, which are obtained after some simplification from (5.3.1) by putting left hand sides to zero:

\[
\beta Y^2 + \overline{Y} (v + \alpha + e) - \beta N + \lambda C - \lambda NC = 0 \quad (5.3.4)
\]

\[
Y = \frac{A - e N}{\alpha} \quad (5.3.5)
\]

\[
s + s_1 I = \left( \frac{s_0}{L} - s_2 I \right) C \quad (5.3.6)
\]

\[
Q_0 + \theta_1 N = (\theta_0 - \theta_2 N) I \quad (5.3.7)
\]

Now eliminating \( Y \) between equations (5.3.4) and (5.3.5) we get

\[
F(N) = \left( \frac{\beta}{\alpha^2} \right) (A - e N)^2 + \left( \frac{A - e N}{\alpha} \right) [v + \alpha + e] - \beta N + \lambda C - \lambda NC = 0 \quad (5.3.8)
\]

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where $C$ is given in terms of $N$ by using (5.3.6) and (5.3.7). From equation (5.3.8) we note the following

$$F\left(\frac{A}{\alpha + e}\right) = \frac{A}{\alpha + e}(v + \alpha + e) > 0$$

(5.3.9)

$$F\left(\frac{A}{e}\right) = -\frac{\lambda A}{e} C_m < 0$$

(5.3.10)

Thus, it is clear that there exists a root $N^*$ of $F(N) = 0$ in the interval

$$\frac{A}{e + \alpha} \leq N \leq \frac{A}{e}.$$ Further, this root will be unique if $F'(N) < 0$ for $\frac{A}{e + \alpha} \leq N \leq \frac{A}{e}$. To show this, we differentiate (5.3.8) to get

$$F'(N) = -\frac{2\beta e}{\alpha^2} (A - e N) - \frac{e}{\alpha} \left[(v + \alpha + e) - \beta N + \lambda C\right] - \frac{\beta}{\alpha} (A - eN) - \lambda C$$

\[\begin{align*}
&- \frac{\lambda}{\alpha} \left[N(\alpha + e) - A\right] \frac{dC}{dN} \\
&\text{(5.3.11)}
\end{align*}\]

Using (5.3.8), in (5.3.11), we get on simplification,

$$F'(N) = -\frac{\beta e}{\alpha^2} (A - e N) - \frac{e}{(A - eN)} \lambda NC - \frac{\beta}{\alpha} (A - eN) - \lambda C - \frac{\lambda}{\alpha} \left[N(\alpha + e) - A\right] \frac{dC}{dN}$$

(5.3.12)

Which is negative in $\frac{A}{e + \alpha} \leq N \leq \frac{A}{e}$, as $\frac{dC}{dN} = \frac{(s_1 + s_2 C)}{s_0} \frac{dI}{dN}$ and $\frac{dI}{dN} = \frac{\theta_1 + \theta_2 I}{\theta_0 - \theta_2 N}$ are positive, under the condition (5.3.3).

Now, knowing the value of $N^*$, the value of $Y^*$, $C^*$ and $I^*$ can be uniquely determined from (5.3.5), (5.3.6), (5.3.7).
Remark: Using (5.3.4), (5.3.5), (5.3.6), (5.3.7), we can check that $\frac{dY}{d\theta_0} < 0$, $\frac{dY}{d\theta_1} > 0$ and $\frac{dY}{d\theta_2} > 0$. These conditions imply that as the cumulative density of infrastructures increases (decreases), the density of infectives increases (decreases).

(i) We show that at $E^*$, $\frac{dY}{d\theta_0} < 0$.

From (5.3.5) we have

$$\frac{dN}{d\theta_0} = -\frac{\alpha}{e} \frac{dY}{d\theta_0}$$

(5.3.13)

Now on differentiating (5.3.4) w. r. t. $\theta_0$ and using (5.3.4) and (5.3.5) we get

$$\frac{dY}{d\theta_0} \left[ \beta Y + \lambda \frac{NC}{Y} + \frac{\alpha \beta}{e} Y + \frac{\lambda \alpha}{e} C \right] = \lambda (N-Y) \frac{dC}{d\theta_0}$$

(5.3.14)
Using (5.3.15) and (5.3.16) in (5.3.14), we get 
\[ E^* \frac{d\theta}{d\theta_0} < 0. \]

Thus it is seen here that as the depletion rate coefficient of cumulative density of infrastructures \( \theta_0 \) increases, the infective human population density decrease at \( E^* \).

(ii) For \( E^* \), \( \frac{dY}{d\theta_1} > 0 \).

In the similar manner as in (i), we can show that \( \frac{dY}{d\theta_1} > 0 \).

Thus it is seen here that as the growth rate coefficient of infrastructural development due to human population density related factor \( \theta_1 \) increases, the infective human population density increases at the equilibrium point \( E^* \).

(iii) Again for \( E^* \), we get, \( \frac{dY}{d\theta_2} > 0 \).

Thus it is seen here that as the growth rate coefficient caused by the bilinear interaction of human population density \( \theta_2 \) increases, the infective human population density increases at the equilibrium point \( E^* \).

From the above discussion it may be concluded that the spread of the carrier dependent infectious disease increases as cumulative density of infrastructures increase in a habitat.
5.4 STABILITY ANALYSIS

Now we shall study the stability behaviour of above equilibria. The local stability result of equilibria $E_0$, $E_1$ and $E^*$ are given in the following theorem:

**Theorem 5.4.1:** The equilibria $E_0$ and $E_1$ are locally unstable and the equilibrium $E^*$ is locally asymptotically stable provided the following conditions are satisfied,

\[
\alpha \lambda^2 C^* < e \beta^2 Y^*^2 \\
2\alpha \lambda^2 (N^* - Y^*)^2 \left( s_1 + s_2 C^* \right)^2 \left( \theta_1 + \theta_2 I^* \right)^2 < e \beta^2 Y^*^2 \left( \frac{S_0}{L} - s_2 I^* \right)^2 \left( \theta_0 - \theta_2 N^* \right)^2
\]  

(5.4.1)  

(5.4.2)

**Proof:** The local stability behaviour of each of the two equilibria $E_0$ or $E_1$ is studied by computing corresponding variational matrices for system (5.3.1) and for the nontrivial equilibrium point $E^*$ it is studied by using Lyapunov’s theory.

The variational matrix $M_i$ corresponding to equilibrium points is given by:

\[
M_i = \begin{bmatrix}
\beta N - 2\beta Y - \lambda C - (v + \alpha + e) & \beta Y + \lambda C & \lambda(N - Y) & 0 \\
-\alpha & -e & 0 & 0 \\
0 & 0 & s - \frac{2S_0}{L} C + s_1 I + 2s_2 CI & s_1 C + s_2 C^2 \\
0 & \theta_1 + \theta_2 I & 0 & -\theta_0 + \theta_2 N
\end{bmatrix}
\]

**Local Stability Behaviour of $E_0(0,A/e,0,\bar{I}_m)$:**

The variational matrix corresponding to equilibrium point $E_0$ is given by:

\[
M_0 = \begin{bmatrix}
\beta \frac{A}{e} - (v + \alpha + e) & 0 & \lambda \frac{A}{e} & 0 \\
-\alpha & -e & 0 & 0 \\
0 & 0 & s + s_1 I & 0 \\
0 & \theta_1 + \theta_2 I & 0 & -\theta_0 + \theta_2 \frac{A}{e}
\end{bmatrix}
\]

Here one of the eigen value $s + s_1 I_m$ is positive and hence $E_0$, if exists, is unstable.

**Local Stability Behaviour of $E_1(\bar{Y},\bar{N},0,\bar{I})$:** In this case the variational matrix will be
This variational matrix has a positive eigen value $s + s_i \bar{I}$ and hence $E_1$, if exists, is unstable.

**Local Stability Behaviour of $E^* (Y^*, N^*, C^*, I^*)$:** We study the stability behaviour of $E^*$ by Lyapunov’s method. For this we linearize the system (5.3.1) by using following transformations

\[ Y = Y^* + y, \quad N = N^* + n, \quad C = C^* + c, \quad I = I^* + i \]

and use following positive definite function to find the sufficient condition for stability

\[ V = \frac{1}{2} y^2 + k_1 n^2 + k_2 c^2 + k_3 i^2 \]  

(5.4.3)

(where $k_1$, $k_2$ and $k_3$ are positive constants to be chosen appropriately).

Differentiating (5.4.3) w.r.t. ‘$t$’ and using the linearized version of (5.3.1), \( \frac{dV}{dt} \) can be written as:

\[
\frac{dV}{dt} = -\left( \beta Y^* + \lambda \frac{N^* C^*}{Y^*} \right) y^2 - (k_1 e) n^2 - k_2 C^* \left( \frac{s_0}{L} - s_i I^* \right) c^2 - k_3 \left( \theta_0 - \theta_2 N^* \right) i^2
\]
\[ + \left[ \beta Y^* + \lambda C^* - k_1 \alpha \right] y n + \lambda (N^* - Y^*) y c + k_3 \left( \theta_1 + \theta_2 I^* \right) n i + k_2 C^* (s_i + s_2 C^*) c i \]
\[ = \left( \beta Y^* - k_1 \alpha \right) y n - \lambda \frac{N^* C^*}{Y^*} y^2 - \left[ \left( \frac{\beta}{2} Y^* \right) y^2 - (\lambda C^*) y n + \frac{k_1 e n^2}{2} \right] \]
\[ - \left[ \left( \frac{\beta}{2} Y^* \right) y^2 - \lambda (N^* - Y^*) y c + k_2 C^* \left( \frac{s_0}{L} - s_i I^* \right) c^2 \right] \]
\[ - \left[ k_2 C^* \left( \frac{s_0}{L} - s_i I^* \right) c^2 - k_2 C^* (s_i + s_2 C^*) c i + k_3 \left( \theta_0 - \theta_2 N^* \right) i^2 \right] \]
Choosing $k_1 = \frac{\beta Y}{\alpha}$, the conditions for $\frac{dV}{dt}$ to be negative definite can be written as follows:

$$\alpha \lambda^2 C^2 < e \beta^2 Y^2$$  \hspace{1cm} (5.4.4)

$$k_2 > \frac{\lambda^2 (N^* - Y^*)^2}{\beta Y C^* \left( \frac{s_0}{L} - s_2 I^* \right)}$$  \hspace{1cm} (5.4.5)

$$k_2 < \frac{\left( \frac{s_0}{L} - s_2 I^* \right) (\theta_0 - \theta_2 N^*)}{C^* \left( s_1 + s_2 C^* \right)^2} k_3$$  \hspace{1cm} (5.4.6)

$$k_3 < \frac{\beta Y}{\alpha} \frac{\left( \theta_0 - \theta_2 N^* \right) e}{\left( \theta_1 + \theta_2 I^* \right)^2}$$  \hspace{1cm} (5.4.7)

Now if we choose $k_3 = \frac{\beta Y}{2\alpha} \frac{\left( \theta_0 - \theta_2 N^* \right) e}{\left( \theta_1 + \theta_2 I^* \right)^2}$, then inequality (5.4.7) will satisfy automatically. Now we can choose $k_2$ satisfying inequality (5.4.5) and (5.4.6) provided

$$\frac{\lambda^2 (N^* - Y^*)^2}{\beta Y C^* \left( \frac{s_0}{L} - s_2 I^* \right)} < \frac{\left( \frac{s_0}{L} - s_2 I^* \right) (\theta_0 - \theta_2 N^*)}{C^* \left( s_1 + s_2 C^* \right)^2} \frac{\beta Y}{2\alpha} \frac{\left( \theta_0 - \theta_2 N^* \right) e}{\left( \theta_1 + \theta_2 I^* \right)^2}$$

or

$$2\alpha \lambda^2 (N^* - Y^*)^2 \left( s_1 + s_2 C^* \right)^2 (\theta_1 + \theta_2 I^*)^2 < e \beta^2 Y^2 \left( \frac{s_0}{L} - s_2 I^* \right)^2 \left( \theta_0 - \theta_2 N^* \right)^2$$  \hspace{1cm} (5.4.8)

Hence $\frac{dV}{dt}$ is negative definite if (5.4.4) and (5.4.8) are satisfied. Thus, $E^*$ is locally stable if (5.4.1) and (5.4.2) are satisfied.
The nonlinear stability results for \( E^* \) are given by the following theorem:

**Theorem 5.4.2:** The equilibrium point \( E^* \) is nonlinearly asymptotically stable in \( \Omega \) provided the following inequalities are satisfied:

\[
\alpha^2 C_m^2 < e \beta^2 Y^2 \tag{5.4.9}
\]

\[
\alpha^2 (N^* - Y^*)^2 (s_i + s_3 C_m)^2 (\theta_1 + \theta_2 I_m)^2 < e \beta^2 Y^2 \left( \frac{s_0}{L} - s_2 I \right)^2 \left( \theta_0 - \theta_2 N^* \right)^2 \tag{5.4.10}
\]

**Proof:** We prove the theorem by using the following positive definite function:

\[
V = \left( Y - Y^* - Y^* \ln \frac{Y}{Y^*} \right) + m_1 \left( N - N^* \right)^2 + m_2 \left( C - C^* - C^* \ln \frac{C}{C^*} \right) + \frac{m_3}{2} \left( I - I^* \right)^2 \tag{5.4.11}
\]

(where \( m_1, m_2 \) and \( m_3 \) are positive constants to be chosen appropriately).

Differentiating (5.4.11) w.r.t. \( 't' \) and using (5.3.1), we get

\[
\frac{dV}{dt} = \left( \frac{Y - Y^*}{Y} \right) \frac{dY}{dt} + m_1 \left( N - N^* \right) \frac{dN}{dt} + m_2 \left( C - C^* \right) \frac{dC}{dt} + m_3 \left( I - I^* \right) \frac{dI}{dt}
\]

\[
= -\left[ \beta + \frac{NC}{YY^*} \right] (Y - Y^*)^2 - m_1 e (N - N^*)^2 - m_2 \left[ \frac{s_0}{L} - s_2 I^* \right] (C - C^*)^2
\]

\[
-m_3 \left( \theta_0 - \theta_2 N^* \right) \left( I - I^* \right)^2 + \left[ \beta - m_1 \alpha + \lambda \frac{C}{Y^*} \right] (Y - Y^*) (N - N^*)
\]

\[
+ \lambda \left[ \frac{N^*}{Y^*} - 1 \right] (Y - Y^*) (C - C^*) + m_2 \left[ s_i + s_j C \right] (C - C^*) (I - I^*)
\]

\[
+ m_3 \left[ \theta_1 + \theta_2 I \right] (I - I^*) (N - N^*)
\]

Taking \( m_1 = \frac{\beta}{\alpha} \),

\[
\frac{dV}{dt} = -\frac{NC}{YY^*} (Y - Y^*)^2 - \left[ \frac{\beta}{2} (Y - Y^*)^2 - \lambda \frac{C}{Y^*} (Y - Y^*) (N - N^*) + \frac{m_3 e}{2} (N - N^*)^2 \right]
\]

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\[-\left[ \frac{\beta}{2}(Y-Y'^2) - \lambda \left( \frac{N^*}{Y^*} - 1 \right)(Y-Y'^2)(C-C^*) + \frac{m_2}{2} \left( \frac{s_0}{L} - s_2 I^* \right)(C-C^*)^2 \right] \]

\[-\frac{m_2}{2} \left( \frac{s_0}{L} - s_2 I^* \right)(C-C^*)^2 - m_2 (s_1 + s_2 C)(C-C^*)(I-I^*) + \frac{m_3}{2} (\theta_0 - \theta_2 N^* )(I-I^*)^2 \]

\[-\frac{m_3}{2} (N-N^*)^2 - m_3 (\theta_1 + \theta_2 I)(I-I^*)(N-N^*) + \frac{m_3}{2} (\theta_0 - \theta_2 N^* )(I-I^*)^2 \]

Now \( \frac{dV}{dt} \) will be negative definite if following conditions holds:

\[ \alpha \lambda^2 C^2 < e \beta^2 Y'^2 \]

\[ \frac{\lambda^2 \left( \frac{N^*}{Y^*} - 1 \right)^2}{\beta \left( \frac{s_0}{L} - s_2 I^* \right)} \]

\[ m_2 > \frac{\lambda^2 \left( \frac{N^*}{Y^*} - 1 \right)^2}{\beta Y'^2 \left( \frac{s_0}{L} - s_2 I^* \right)} \]  

\[ m_2 < \frac{\left( \frac{s_0}{L} - s_2 I^* \right) (\theta_0 - \theta_2 N^*)}{(s_1 + s_2 C)^2} m_3 \]

\[ m_3 < \frac{(\beta/\alpha) e (\theta_0 - \theta_2 N^*)}{(\theta_1 + \theta_2 I_m)^2} \]

Now on maximizing the left hand sides and minimizing right hand side of the above inequalities, we get

\[ \alpha \lambda^2 C_m^2 < e \beta^2 Y'^2 \]

\[ m_2 > \frac{\lambda^2 \left( \frac{N^*}{Y^*} - 1 \right)^2}{\beta Y'^2 \left( \frac{s_0}{L} - s_2 I^* \right)} \]  

\[ m_2 < \frac{\left( \frac{s_0}{L} - s_2 I^* \right) (\theta_0 - \theta_2 N^*)}{(s_1 + s_2 C_m)^2} m_3 \]

\[ m_3 < \frac{(\beta/\alpha) e (\theta_0 - \theta_2 N^*)}{(\theta_1 + \theta_2 I_m)^2} \]
If we choose \( m_3 = \frac{1}{2} \frac{(\beta / \alpha) e (\theta_0 - \theta_2 N'^*)}{(\theta_1 + \theta_2 I_m)^2} \), the inequality (5.4.15) will satisfy automatically and we can choose \( m_2 \) satisfying inequality (5.4.13) and (5.4.14) provided

\[
\frac{\lambda^2 \left( N^* - Y^* \right)^2}{\beta \left( \frac{s_0}{L} - s_2 I' \right)} \left( \frac{s_0}{L} - s_2 I' \right) \left( \frac{s_0}{L} - s_2 I' \right) < \frac{1}{2} \frac{(\beta / \alpha) e (\theta_0 - \theta_2 N'^*)}{(\theta_1 + \theta_2 I_m)^2}
\]

or

\[
\alpha \frac{\lambda^2 (N^* - Y^*)^2 (s_1 + s_2 C_m)^2 (\theta_1 + \theta_2 I_m)^2}{\beta^2 Y'^2} \left( \frac{s_0}{L} - s_2 I' \right) < e \beta^2 Y'^2 \left( \frac{s_0}{L} - s_2 I' \right)
\]

(5.4.16)

Hence \( \frac{dV}{dt} \) is negative definite if (5.4.12) and (5.4.16) are satisfied. Thus \( E' \) is nonlinearly asymptotically stable if (5.4.9) and (5.4.10) are satisfied, as stated in theorem 5.4.2.

**Remark:** It is noted here that if \( \lambda = 0 \) or \( \alpha = 0 \), the above inequalities are satisfied automatically, which shows that \( \lambda \) and \( \alpha \) have destabilizing effects on the system.

### 5.5 NUMERICAL SIMULATION

Here we discuss the existence and stability of the nontrivial equilibrium point \( E' \) by taking the following set of parameter values and using MAPLE.

\[ A = 500, \ e = 0.02, \ \alpha = 0.03, \ \beta = 0.000005, \ \lambda = 0.000001, \ \nu = 0.05, \ s = 0.899, \ s_0 = 0.9, \]

\[ L = 100000, \ s_1 = 0.002, \ s_2 = 0.00000001, \ Q_0 = 1, \theta_0 = 0.1, \theta_1 = 0.002, \theta_2 = 0.00000001. \]

For these values of parameters, the value of nontrivial equilibrium point \( E' \) corresponding to (5.3.1) is obtained as follows

\[ Y' = 7978, \ N' = 13032, \ C' = 117966, \ I' = 271 \]

The variational matrix corresponding to the equilibrium point \( E' \) is obtained as:
The eigenvalues of this matrix are:

- 0.2179, - 0.0674 + 0.0213 i, - 0.0674 - 0.0213 i, - 0.7415

All of which have negative real part. Hence $E^\ast(Y^\ast, N^\ast, C^\ast, I^\ast)$ is locally stable.

Now numerical simulation is performed for $N$ vs. $Y$ for the different initial starts and displayed in the figure 5.2 which indicates nonlinear stability of the point $(N^\ast, Y^\ast)$ in $N - Y$ plane.

Figure 5.2
Phase plot between $N$ and $Y$
The model (5.3.1) has also been solved by using MAPLE and the graphs of the variable \(Y\) with respect to \(t\) for various values of different parameters have been plotted in figure 5.3 – figure 5.9 and following observations have been drawn

(i) From figure 5.3, it is noted that \(Y(t)\) increases as intrinsic growth rate of carriers in absence of human activities \(s\) increases.

(ii) From figure 5.4, it is seen that \(Y(t)\) increases as growth rate coefficient of carrier population \(s_1\) increases.

(iii) From figure 5.5, we note that \(Y(t)\) increases as growth rate coefficient of carrying capacity of carriers \(s_2\) increases.

The above results are expected, as the carrier population increases with the parameters \(s, s_1, s_2\). Further, it can also be noted that

(iv) From figure 5.6, it is seen that \(Y(t)\) increases as growth rate of cumulative density of infrastructures \(Q_0\) increases.

(v) From figure 5.7, we note that \(Y(t)\) increases as depletion rate coefficient for infrastructures \(\theta_0\) decreases.

(vi) From figure 5.8, it is seen that \(Y(t)\) increases as growth rate of infrastructures due to human population \(\theta_1\) increases.

(vii) From figure 5.9, it is seen that \(Y(t)\) increases as growth rate of infrastructures due to human activities non-linearly \(\theta_2\) increases.

These results are again expected as increase (decrease) in the cumulative density of infrastructure causes increase (decrease) in the density of carrier population, resulting increase (decrease) of the density of infectives.
Figure 5.3
Plot between $Y$ and $t$ for different values of intrinsic growth rate of carriers in absence of human activities $s$

Figure 5.4
Plot between $Y$ and $t$ for different values of growth rate coefficient of carrier population $s_i$
Figure 5.5
Plot between $Y$ and $t$ for different values of growth rate coefficient of carrying capacity of carriers $s_2$

Figure 5.6
Plot between $Y$ and $t$ for different values of growth rate of cumulative density of infrastructures $Q_0$
Figure 5.7
Plot between $Y$ and $t$ for different values of depletion rate coefficient for infrastructures $\theta_0$

Figure 5.8
Plot between $Y$ and $t$ for different values of growth rate of infrastructures due to human population $\theta_1$
Figure 5.9

Plot between $Y$ and $t$ for different values of growth rate of infrastructures due to human activities non-linearly $\theta_2$

5.6 CONCLUSIONS

In this chapter, a four dimensional SIS non-linear model with immigration has been proposed and analyzed to study the spread of carrier dependent infectious diseases. The density of carriers has been assumed to be governed by a logistic model, the growth rate and carrying capacity of which depend on the cumulative density of infrastructures which depends upon human population density partially.

In the modelling process, the cumulative density of infrastructure has been assumed to grow with a constant rate, and it is depleted with a rate, which is proportional to cumulative density of infrastructures. In a realistic situation, this cumulative density must depend upon human population density in the habitat. Therefore, the growth rate of cumulative density increases due to non-linear interaction with human population density has been considered in the model. The model has been analyzed by stability
theory of differential equations as well as by computer simulation. The following main conclusions have been drawn from the analysis:

(i) When $Q_0 = 0$, $\theta_2 = 0$ and $\theta_1 = \theta$, we get the same results as in chapter 4.

(ii) The density of infectives increases as the parameters, related to the increase in the density of carrier population, caused by infrastructures increase.

(iii) The density of carrier population increases, as the parameters related to increase in infrastructural development due to human population density related factors, increases.

(iv) As the cumulative density of infrastructure increases ($Q_0 \neq 0$), the spread of carrier dependent infectious disease increases much more in the comparison to the case when $Q_0 = 0$.

It may then be concluded that the spread of carrier dependent infectious diseases increases due to increase in infrastructures in the habitat.