Chapter 5

Analytic Functions with Fixed Second Coefficient

SECTION - 1

5.1 Introduction

Let $S$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

that are analytic and univalent in the unit disc $E = \{z : |z| < 1\}$.

We denote by $S^*(\alpha), K(\alpha), 0 \leq \alpha < 1$, the class of starlike functions of order $\alpha$ and convex function of order $\alpha$ respectively, where

$$S^*(\alpha) = \left\{ f \in S : \Re \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha, z \in E \right\}$$

and

$$K(\alpha) = \left\{ f \in S : \Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha, z \in E \right\}.$$ 

$S^*(0) = S^*$ and $K(0) = K$ are respectively the classes of starlike and convex functions in $S$. 
5.1 Introduction

For $1 < \beta \leq 4/3$ and $z \in E$, let

$$M(\beta) = \left\{ f \in S : \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \beta \right\}$$

and

$$L(\beta) = \left\{ f \in S : \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta \right\}.$$

Further let $P$ denote subclass of $S$ consisting of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, a_j \geq 0. \quad (5.1.1)$$

Let $P^*(\alpha) = S^*(\alpha) \cap P, P_k(\alpha) = K(\alpha) \cap P$ and $P(\beta) = M(\beta) \cap P, U(\beta) = L(\beta) \cap P$.

$P^*(0) = P^*$ and $P_k(0) = P_k$ are respectively the classes of starlike and convex functions in $P$.

Classes $P(\beta)$ and $U(\beta)$ with positive coefficients were studied recently by Uralegaddi, Ganigi and Sarangi [139]. Silverman [121] has studied the univalent functions with negative coefficients.

Dixit and Pathak [33] defined a new class $P_\lambda(\beta)$ as follows.

Let $P_\lambda(\beta)(0 \leq \lambda < 1; 1 < \beta \leq 4/3)$ be the class of functions $f$ in $P$ satisfying the inequality

$$\text{Re} \left\{ \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} \right\} < \beta, \quad (z \in E). \quad (5.1.2)$$

Here the linear operator $\Omega^\lambda$ is defined by

$$\Omega^\lambda f(z) = \Gamma(2 - \lambda) z^\lambda \mathcal{D}_z^\lambda f(z), (f \in P; 0 \leq \lambda < 1), \quad (5.1.3)$$
where $D_2^\lambda f(z)$ denote the fractional derivative of $f(z)$ of order $\lambda$, as defined below with $D_2^0 f(z) = f(z)$ and $D_2^1 f(z) = f'(z)$.

**Definition 5.1.1:** The fractional derivative of order $\lambda$ is defined, for a function $f(z)$, by

$$D_2^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\lambda}},$$

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by the requiring log $(z-\zeta)$ to be real when $(z-\zeta) > 0$.

**Definition 5.1.2:** Under the hypothesis of definition 5.1.1, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by

$$D_2^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_2^\lambda f(z),$$

where $0 \leq \lambda < 1$, and $n$ is non-negative integer. It is easily seen from (5.1.3) $\Omega^0 f(z) = f(z)$ and $\Omega^1 f(z) = zf'(z)$.

Dixit and Pathak [33] proved the following.

**Lemma 5.1.1:** A function $f(z)$ defined by (5.1.1) is in the class $P_\lambda(\beta)$ if and only if

$$\sum_{j=2}^{\infty} \phi(j)(j-\beta)a_j \leq \beta - 1, \quad a_j \geq 0 \text{ for } j = 2, 3, \ldots, \quad (5.1.4)$$

where, for convenience

$$\phi(j) = \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)}. \quad (5.1.5)$$

For a function $f$ defined by (5.1.1) and in the class $P_\lambda(\beta)$, Lemma (5.1.1)
immediately yields
\[ a_2 \leq \frac{(2 - \lambda)(\beta - 1)}{2(2 - \beta)}. \] (5.1.6)

In view of the coefficient inequality (5.1.4), it would seem to be natural to introduce and study here the class \( P_{\lambda}(\beta) \) of analytic and univalent functions where
\( P_{\lambda}'(\beta) \) denotes the subclass of \( P_{\lambda}(\beta) \) consisting of functions of the form
\[ f(z) = z + \frac{(2 - \lambda)(\beta - 1)}{2(2 - \beta)} \gamma z^2 + \sum_{j=3}^{\infty} a_j z^j, \quad (a_j \geq 0, \ 0 \leq \gamma \leq 1, \ 1 < \beta \leq 4/3). \] (5.1.7)

The present section deals with the study of univalent functions with fixed second coefficient. In this direction, the work of Silverman and Silvia [123], Ahuja and Jahangiri [2], Ahuja and Silverman [3], Al-Amiri [4], Aouf [8] and Umarangi [136] can be seen. The main object of the present section is first to derive characterization theorem for the class \( P_{\lambda}'(\beta) \). We then investigate various interesting properties and characteristics of the class \( P_{\lambda}'(\beta) \). In fact, we have investigated mainly, how a fixed second coefficient, under the certain conditions, affects radius of convexity, closure theorem etc. for the functions defined by (5.1.7).

### 5.2 Characterization Theorem for the Class \( P_{\lambda}'(\beta) \)

We first prove the following.

**Theorem 5.2.1:** Let the function \( f \) be defined by (5.1.7). Then \( f \) is in the class
5.3 Closure Theorem for the Class $P^\gamma_\lambda(\beta)$

$P^\gamma_\lambda(\beta)$ if and only if

$$\sum_{j=3}^{\infty} \phi(j) (j - \beta) a_j \leq (\beta - 1)(1 - \gamma). \quad (5.2.1)$$

The result is sharp for the function $f$ given by

$$f(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)} z^2 + \frac{(\beta - 1)(1 - \gamma)}{\phi(j)(j - \beta)} z^j. \quad (5.2.2)$$

Proof. By setting

$$a_2 = \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)}$$

in Lemma 5.1.1, we get

$$\frac{2(2 - \beta)}{2 - \lambda} a_2 + \sum_{j=3}^{\infty} \phi(j) (j - \beta) a_j \leq \beta - 1$$

or

$$\frac{2(2 - \beta)}{(2 - \lambda)} \cdot \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)} + \sum_{j=3}^{\infty} \phi(j)(j - \beta)a_j \leq \beta - 1$$

or

$$\sum_{j=3}^{\infty} \phi(j)(j - \beta)a_j \leq (\beta - 1)(1 - \gamma).$$

5.3 Closure Theorem for the Class $P^\gamma_\lambda(\beta)$

A closure theorem for the class $P^\gamma_\lambda(\beta)$ is given by following.

Theorem 5.3.1: Let

$$f_i(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)} z^2 + \sum_{j=3}^{\infty} a_{j,i} z^j. \quad (5.3.1)$$
If \( f_i \in P^\gamma_\lambda(\beta), \) (\( i = 1, 2, \ldots, m \)), then the function \( g \) given by

\[
g(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)} z^2 + \sum_{j=3}^{\infty} b_j z^j.
\]

with

\[
b_j = \frac{1}{m} \sum_{i=1}^{m} a_{j,i} \geq 0,
\]

is also in the class \( P^\gamma_\lambda(\beta) \).

**Proof.** Since \( f_i \in P^\gamma_\lambda(\beta) \). Then by Theorem 5.2.1, we have

\[
\sum_{j=3}^{\infty} \phi(j)(j - \beta)a_{j,i} \leq (\beta - 1)(1 - \gamma), \quad (i = 1, 2, \ldots, m).
\]

Thus by applying (5.3.4) and definition (5.3.3), we have

\[
\sum_{j=3}^{\infty} \phi(j)(j - \beta)b_j = \sum_{j=3}^{\infty} \phi(j)(j - \beta) \left( \frac{1}{m} \sum_{i=1}^{m} a_{j,i} \right) \leq (\beta - 1)(1 - \gamma),
\]

which again by virtue of Theorem 5.2.1, proves Theorem 5.3.1.

Next we prove the following.

**Theorem 5.3.2:**

\[
f_2(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)} z^2
\]

and

\[
f_j(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)} z^2 + \frac{(\beta - 1)(1 - \gamma)}{\phi(j)(j - \beta)} z^j, \quad (j = 3, 4, 5, \ldots).
\]

Then \( f \) is in the class \( P^\gamma_\lambda(\beta) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{j=2}^{\infty} c_j f_j(z), \quad (c_j \geq 0, \sum_{j=2}^{\infty} c_j = 1).
\]
5.3 Closure Theorem for the Class $P_\lambda^1(\beta)$

Proof. Suppose $f$ is given by (5.3.7), so that we find from (5.3.5) and (5.3.6) that

$$f(z) = \sum_{j=2}^{\infty} c_j f_j(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2(2 - \beta)} z^2 + \sum_{j=3}^{\infty} \frac{(\beta - 1)(1 - \gamma)}{\phi(j)(j - \beta)} c_j z^j,$$

(5.3.8)

where the coefficient $c_j$ are given with (5.3.7). Then, since

$$\sum_{j=3}^{\infty} \frac{\phi(j)(j - \beta)}{\phi(j)(j - \beta)} c_j = (\beta - 1)(1 - \gamma) \sum_{j=3}^{\infty} c_j = (\beta - 1)(1 - \gamma)(1 - c_2) \leq (\beta - 1)(1 - \gamma),$$

we conclude from Theorem 5.2.1 that $f \in P_\lambda^1(\beta)$.

Conversely, let us assume that the function $f$ defined by (5.1.7) is in the class $f \in P_\lambda^1(\beta)$. Then

$$a_j \leq \frac{(\beta - 1)(1 - \gamma)}{\phi(j)(j - \beta)},$$

(5.3.9)

which follows readily from (5.2.1).

Setting

$$c_j = \frac{\phi(j)(j - \beta)}{(\beta - 1)(1 - \gamma)} a_j$$

(5.3.10)

and

$$c_2 = 1 - \sum_{j=3}^{\infty} c_j.$$

(5.3.11)

We thus arrive at (5.3.7). This evidently complete the proof of Theorem 5.3.2. □
5.4 Radius of Convexity for the Class $P_{\lambda}^{\gamma}(\beta)$

In this section, we prove the following.

**Theorem 5.4.1:** Let the function $f$ be in the class $P_{\lambda}^{\gamma}(\beta)$. Then $f$ is a univalently convex function of order $\delta(0 \leq \delta < 1)$ in $|z| < r_1 = r_1(\lambda, \delta, \gamma)$, where $r_1(\lambda, \delta, \gamma)$ is the largest value of $r$ for which

$$
\frac{(2 - \lambda)(\beta - 1)(2 - \delta)\gamma}{(2 - \beta)} r + \frac{j(\beta - 1)(j + \delta)(1 - \gamma)}{\phi(j)(j - \beta)} r^{j-1} \leq 1 - \delta. \tag{5.4.1}
$$

The result is sharp for the function $f_j$ given by (5.3.6).

**Proof.** It is sufficient to show that for $f \in P_{\lambda}^{\gamma}(\beta)$ that

$$
\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta, \quad (|z| < r_1(\lambda, \delta, \gamma)), \tag{5.4.2}
$$

where $r_1(\lambda, \delta, \gamma)$ is the largest value of $r$ for which the inequality (5.4.1) holds true. Observe that, if $f \in P_{\lambda}^{\gamma}(\beta)$ is given by (5.1.7), we have

$$
\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{(2 - \lambda)(\beta - 1)\gamma}{(2 - \beta)} r + \sum_{j=3}^{\infty} j(j - 1) a_j r^{j-1} \leq 1 - \delta, \quad (|z| < r, \quad 0 \leq \delta < 1),
$$

if and only if

$$
\frac{(2 - \lambda)(\beta - 1)(2 - \delta)\gamma}{(2 - \beta)} r + \sum_{j=3}^{\infty} j(j - \delta) a_j r^{j-1} \leq 1 - \delta, \quad (0 \leq \delta < 1). \tag{5.4.3}
$$

Since $f \in P_{\lambda}^{\gamma}(\beta)$, in view of Theorem 5.2.1, we may set

$$
a_j = \frac{(\beta - 1)(1 - \gamma)}{\phi(j)(j - \beta)} c_j, \quad (c_j \geq 0, \sum_{j=3}^{\infty} c_j \leq 1). \tag{5.4.4}
$$
Now, for each fixed \( r \), we choose a positive integer \( j_0 = j_0(r) \) for which

\[
\frac{j(j - \delta)}{\phi(j)(j - \beta)} r^{j-1}
\]

is maximal. Then

\[
\sum_{j=3}^{\infty} j(j - \delta) a_j r^{j-1} \leq \frac{j_0(\beta - 1)(j_0 - \delta)(1 - \gamma)}{\phi(j_0)(j_0 - \beta)} r^{j_0-1}.
\]

Consequently, the function \( f \) is univalently convex of order \( \delta (0 \leq \delta < 1) \) in \(|z| < r_1(\lambda, \delta, \gamma)\) provided that

\[
\frac{(2 - \lambda)(\beta - 1)(2 - \delta)}{(2 - \beta)} r_1 + \frac{j_0(\beta - 1)(j_0 - \delta)(1 - \gamma)}{\phi(j_0)(j_0 - \beta)} r_1^{j_0-1} \leq 1 - \delta, \quad (0 \leq \delta < 1).
\]

(5.4.5)

We find the value \( r_1 = r_1(\lambda, \delta, \gamma) \) and the corresponding integer \( j_0(r_1) \) so that

\[
\frac{(2 - \lambda)(\beta - 1)(2 - \delta)}{(2 - \beta)} r_1 + \frac{j_0(\beta - 1)(j_0 - \delta)(1 - \gamma)}{\phi(j_0)(j_0 - \beta)} r_1^{j_0-1} = 1 - \delta, \quad (0 \leq \delta < 1).
\]

(5.4.6)

Then this value \( r_1 \) is the radius of univalent convexity of order \( \delta \) for function \( f \in P_\lambda^\gamma(\beta) \).

\[ \square \]

SECTION - 2

5.5 Introduction

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{j=2}^{\infty} a_j z^j
\]

that are analytic in the unit disc \( E = \{z : |z| < 1\} \).
Let $S$ be the subclass of $A$ consisting of analytic and univalent functions $f(z)$ in the unit disc $E$. Further $T$ denote subclass of $S$ consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j.$$  

(5.5.1)

We denote by $S^*(\alpha), K(\alpha), 0 \leq \alpha < 1$, the class of starlike functions of order $\alpha$, and the class of convex functions of order $\alpha$ respectively, where

$$S^*(\alpha) = \left\{ f \in S : \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, z \in E \right\}$$

and

$$K(\alpha) = \left\{ f \in S : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, z \in E \right\}.$$

Further $T^*(\alpha) = S^*(\alpha) \cap T$ and $C(\alpha) = K(\alpha) \cap T$.

Silverman [121] proved some results for the subclass $S^*(\alpha), T^*(\alpha), K(\alpha)$ and $C(\alpha)$. For $f(z)$ belonging to $S$, Salagean [114] has introduced the following operator called salagean operator:

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = zf'(z)$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots, \}).$$

Note that

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j (n \in N_0 = N \cup \{0\}).$$

A function $f(z) \in S$ is said to belong to the class $S_n(\alpha)$ if it satisfies

$$\Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \alpha \quad (z \in E).$$
for some $\alpha (0 \leq \alpha < 1)$ and $n \in N_0$.

This class $S_n(\alpha)$ was investigated by Owa et al. ([78], [95]).

Note that $S_0(\alpha) = S^*(\alpha)$ and $S_1(\alpha) = K(\alpha)$.

Ekrem Kadioglu defined $T_n(\alpha)$ by

$$T_n(\alpha) = \left\{ f \in T : \text{Re} \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \alpha, \quad z \in E \right\}.$$  

Clearly $T_0(\alpha) = T^*(\alpha)$ and $T_1(\alpha) = C(\alpha)$.

Note that $T_n(\alpha) \subset S_n(\alpha)$.

Ekrem Kadioglu [57] proved the following:

**Lemma 5.5.1:** A function $f(z) = z - \sum_{j=2}^{\infty} |a_j|z^j$ is in $T_n(\alpha)$ if and only if

$$\sum_{j=2}^{\infty} (j^{n+1} - \alpha j^n)a_j \leq 1 - \alpha, \quad a_j \geq 0 \quad \text{for} \quad j = 2, 3, \ldots. \quad (5.5.2)$$

For a function $f$ defined by (5.5.1) and in the class $T_n(\alpha)$, Lemma 5.5.1 immediately yields

$$|a_2| \leq \frac{1 - \alpha}{2^{n+1} - \alpha \cdot 2^n}. \quad (5.5.3)$$

In view of the coefficient inequality (5.5.2), it would seem to be natural to introduce and study here the class $T_n^\lambda(\alpha)$ of analytic and univalent functions where $T_n^\lambda(\alpha)$ denotes the subclass of $T_n(\alpha)$ consisting of functions of the form

$$f(z) = z - \frac{(1 - \alpha)\lambda}{2^{n+1} - \alpha \cdot 2^n}z^2 - \sum_{j=3}^{\infty} a_jz^j, \quad (a_j \geq 0, \quad 0 \leq \alpha < 1, \quad 0 \leq \lambda \leq 1). \quad (5.5.4)$$

The main object of the present section is first to derive characterization theorem for the class $T_n^\lambda(\alpha)$. We then investigate various interesting properties
and characteristics of the class $T_n^\lambda(\alpha)$. In fact, we have investigated mainly how a fixed second coefficient under certain conditions, affects radius of convexity, closure theorem etc. for the functions defined above.

## 5.6 Characterization Theorem for the Class $T_n^\lambda(\alpha)$

We first prove the following.

**Theorem 5.6.1:** Let the function $f$ be defined by (5.5.4). Then $f$ is in class $T_n^\lambda(\alpha)$ if and only if

$$\sum_{j=3}^{\infty} (j^{n+1} - \alpha j^n) a_j \leq (1 - \alpha)(1 - \lambda).$$  \hspace{1cm} (5.6.1)

The result is sharp for the function $f$ given by

$$f(z) = z - \frac{(1 - \alpha)\lambda}{2^{n+1} - \alpha \cdot 2^n} z^2 - \frac{(1 - \alpha)(1 - \lambda)}{j^{n+1} - 2 \cdot j^n} z^n.$$  \hspace{1cm} (5.6.2)

**Proof.** By setting

$$\frac{(1 - \alpha)\lambda}{2^{n+1} - \alpha \cdot 2^n} \quad (0 \leq \lambda \leq 1)$$

in Lemma 5.5.1, and simplifying the inequality resulting from (5.5.2), we are led easily to assertion (5.6.1) of Theorem 5.6.1. \hfill \square

## 5.7 Closure Theorem for the Class $T_n^\lambda(\alpha)$

A closure theorem for the class $T_n^\lambda(\alpha)$ is given by following.

**Theorem 5.7.1:** Let

$$f_i(z) = z - \frac{(1 - \alpha)\lambda}{2^{n+1} - \alpha \cdot 2^n} z^2 - \sum_{j=3}^{\infty} a_{j,i} z^j (a_{j,i} \geq 0, i = 1, 2, 3, \ldots, m).$$  \hspace{1cm} (5.7.1)
If \( f_i \in T^\lambda_n(\alpha) \) (\( i = 1, 2, 3, \ldots, m \)), then the function \( g \) given by

\[
g(z) = z - \frac{(1 - \alpha)\lambda}{2^{n+1} - \alpha \cdot 2^n} z^2 - \sum_{j=3}^{\infty} b_j z^j,
\]

with

\[
b_j = \frac{1}{m} \sum_{i=1}^{m} a_{j,i} \geq 0,
\]

is also in the class \( T^\lambda_n(\alpha) \).

**Proof.** Since \( f_i \in T^\lambda_n(\alpha) \). Then by Theorem 5.6.1, we have

\[
\sum_{j=3}^{\infty} (j^{n+1} - \alpha j^n) a_{j,i} \leq (1 - \alpha)(1 - \lambda), \quad (i = 1, 2, 3, \ldots, m).
\]

Thus by applying (5.7.4) and definition (5.7.3), we have

\[
\sum_{j=3}^{\infty} (j^{n+1} - \alpha j^n) b_j \leq (1 - \alpha)(1 - \lambda),
\]

which, again by virtue of Theorem 5.6.1, proves Theorem 5.7.1.

Next we prove the following.

**Theorem 5.7.2:** Let

\[
f_2(z) = z - \frac{(1 - \alpha)\lambda}{2^{n+1} - \alpha \cdot 2^n} z^2
\]

and

\[
f_j(z) = z - \frac{(1 - \alpha)\lambda}{2^{n+1} - \alpha \cdot 2^n} z^2 - \frac{(1 - \alpha)(1 - \lambda)}{j^{n+1} - \alpha \cdot j^n} z^j, \quad (j = 3, 4, \ldots).
\]

Then \( f \) is in the class \( T^\lambda_n(\alpha) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{j=2}^{\infty} c_j f_j, \quad (c_j \geq 0, \sum_{j=2}^{\infty} c_j = 1).
\]
5.8 The Radius of Convexity for the Class $T_n^\lambda(\alpha)$

Proof. Suppose that $f$ is given (5.7.7), so that we find from (5.7.5) and (5.7.6) that

$$f(z) = z - \frac{(1-\alpha)\lambda}{2n+1-\alpha \cdot 2^n} z^2 - \sum_{j=3}^{\infty} \frac{(1-\alpha)(1-\lambda)}{j^{n+1}-\alpha \cdot j^n} c_j z^j,$$

(5.7.8)

where the coefficient $c_j$ are given with (5.7.7). Then, since

$$\sum_{j=3}^{\infty} (j^{n+1} - \alpha j^n) \frac{(1-\alpha)(1-\lambda)}{j^{n+1} - \alpha \cdot j^n} c_j = (1-\alpha)(1-\lambda) \sum_{j=3}^{\infty} c_j$$

$$= (1-\alpha)(1-\lambda)(1-c_2),$$

$$\leq (1-\alpha)(1-\lambda),$$

we conclude from Theorem 5.6.1 that $f \in T_n^\lambda(\alpha)$.

Conversely, let us assume that the function $f$ defined by (5.5.4), is in class $T_n^\lambda(\alpha)$, then

$$a_j \leq \frac{(1-\alpha)(1-\lambda)}{j^{n+1} - \alpha \cdot j^n},$$

which follows readily from (5.6.1).

Setting

$$c_j = \frac{j^{n+1} - \alpha \cdot j^n}{(1-\alpha)(1-\lambda)} a_j$$

and

$$c_2 = 1 - \sum_{j=3}^{\infty} c_j.$$

We thus arrive at (5.7.7). This evidently completes the proof of Theorem 5.7.2. $\square$

5.8 The Radius of Convexity for the Class $T_n^\lambda(\alpha)$

In this section, we prove the following.
Theorem 5.8.1: let the function $f$ be in the class $T^\lambda_n(\alpha)$. Then $f$ is a univalent convex function of order $\beta(0 \leq \beta < 1)$ in

$$|z| < r_1 = r_1(\alpha, \beta, \lambda),$$

where $r_1(\alpha, \beta, \lambda)$ is the largest value of $r$ for which

$$\frac{2\lambda(1 - \alpha)(2 - \beta)}{2^{n+1} - \alpha \cdot 2^n} r + \sum_{j=3}^{\infty} \frac{j(j - \beta)(1 - \lambda)}{j^{n+1} - \alpha \cdot j^n} r^{j-1} \leq 1 - \beta.$$  \hspace{1cm} (5.8.1)

The result is sharp for the function $f_j$ given by (5.7.6).

Proof. It is sufficient to show that for $f \in T^\lambda_n(\alpha)$ that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \beta, \quad (|z| < r_1(\alpha, \beta, \lambda), \ 0 \leq \beta < 1),$$

where $r_1(\alpha, \beta, \lambda)$ is largest value $r$ for which the inequality (5.8.1) hold true.

Observe that if $f \in T^\lambda_n(\alpha)$ is given by (5.8.1), we have

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{(1 - \alpha) \cdot \lambda \cdot 2}{2^{n+1} - \alpha \cdot 2^n} r + \sum_{j=3}^{\infty} a_j \cdot \frac{j \cdot (j - 1)}{j^{n+1} - \alpha \cdot j^n} r^{j-1}$$

$$\leq 1 - \beta, (|z| < r, \ 0 \leq \beta < 1),$$

if and only if

$$\frac{2\lambda(1 - \alpha)(2 - \beta)}{2^{n+1} - \alpha \cdot 2^n} r + \sum_{j=3}^{\infty} j(j - \beta)a_j r^{j-1} \leq 1 - \beta, \ (0 \leq \beta < 1).$$  \hspace{1cm} (5.8.3)

Since $f \in T^\lambda_n(\alpha)$, in view of Theorem 5.6.1, we may set

$$|a_j| = \frac{(1 - \alpha)(1 - \lambda)}{j^{n+1} - \alpha \cdot j^n} c_j (c_j \geq 0; \sum_{j=3}^{\infty} c_j \leq 1).$$  \hspace{1cm} (5.8.4)
Now, for each fixed $r$, we choose a positive integer $j_0 = j_0(r)$ for which
\[
\frac{j(j - \beta)}{j^{n+1} - \alpha \cdot j^n} r^{j-1}
\]
is maximal. Then
\[
\sum_{j=3}^{\infty} j(j - \beta) a_j r^{j-1} \leq \frac{j_0(1 - \alpha)(j_0 - \beta)(1 - \lambda)}{j_0^{n+1} - \alpha \cdot j_0^n} r^{j_0-1}.
\]
Consequently, the function $f$ is univalently convex of order $\beta(0 \leq \beta < 1)$ in $|z| < r_1(\alpha, \beta, \lambda)$ provided that
\[
\frac{2\lambda(1 - \alpha)(2 - \beta)}{2^{n+1} - \alpha \cdot 2^n} r_1 + \frac{j_0(1 - \alpha)(j_0 - \beta)(1 - \lambda)}{j_0^{n+1} - \alpha \cdot j_0^n} r_1^{j_0-1} \leq 1 - \beta.
\]
(5.8.5)

We find the value of
\[
r_1 = r_1(\alpha, \beta, \lambda)
\]
and corresponding integer $j_0(r_1)$ so that
\[
\frac{2\lambda(1 - \alpha)(2 - \beta)}{2^{n+1} - \alpha \cdot 2^n} r_1 + \frac{j_0(1 - \alpha)(j_0 - \beta)(1 - \lambda)}{j_0^{n+1} - \alpha \cdot j_0^n} r_1^{j_0-1} = 1 - \beta.
\]
(5.8.6)

Then this value $r_1$ is the radius of univalent convexity of order $\beta$ for functions $f \in T_{n}^\lambda(\alpha)$.

\[\square\]

SECTION - 3

5.9 Introduction

Let $A$ denote the class of functions of the form $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ that are analytic in the unit disc $E = \{z : |z| < 1\}$. Let $S$ be the subclass of $A$ consisting
of univalent functions in $E$. Let $ST$ be the subclass of $S$, the members of which are starlike (with respect to origin) in $E$. A function $f \in S$ is said to be close-to-convex of order $\alpha$, denoted by $f \in C(\alpha)$, $0 \leq \alpha < 1$ if there exist a function $g \in ST$ such that

$$\text{Re}\frac{zf'(z)}{g(z)} > \alpha, \text{ for } z \in E.$$ 

$C(0) = C$ is the subclass of close-to-convex functions. Next we denote by $V$, the subclass of $S$, consisting of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geq 0. \quad (5.9.1)$$

Further let $P(\alpha)$ be the subclass of $S$ consisting of the functions that satisfy

$$\text{Re}\ f'(z) > \alpha, 0 \leq \alpha < 1, z \in E.$$ 

Such functions are close-to-convex of order $\alpha$ with respect to identity function $z$. Thus $P(\alpha) \subset C(\alpha), P(0) = P$ is the subclass of close-to-convex functions in $V$. For $1 < \beta \leq \frac{3}{2}$ and $z \in E$, let

$$U(\beta) = \left\{ f \in V : \text{Re}\left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < \beta \right\}$$

and for $1 < \beta \leq 2$ and $z \in E$, let

$$R(\beta) = \{ f \in V : \text{Re}\ f'(z) < \beta \}. \quad (5.9.2)$$

Uralegaddi, Ganigi and Sarangi [139] have studied the univalent functions with positive coefficients. Subsequently, Sarangi and Uralegaddi [115] and H.S. Al-Amiri [5] have studied the functions with negative coefficients that satisfy

$$\text{Re}\ f'(z) > \alpha, 0 \leq \alpha < 1 \text{ for } z \in E.$$ 

In [99] Ozaki has determined that if $f \in A$, 


satisfies
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{3}{2},
\]
then \( f \) is univalent. And in [124] R. Singh and S. Singh have shown that such functions are close-to-convex. Coefficient inequalities, distortion theorems and radius of convexity have been determined by Uralegaddi, Ganigi and Sarangi [140] for the class \( R(\beta) \). Dixit and Pathak [34] introduced an interesting class \( R_\lambda(\beta) \) of analytic function \( f(z) \) belonging to the class \( A \) and satisfying the condition
\[
\text{Re} \{ \Gamma(2 - \lambda)z^{\lambda-1}D_z^\lambda f(z) \} < \beta \quad (z \in E) \quad (5.9.3)
\]
for \( 0 \leq \lambda \leq 1, 1 < \beta \leq 2 \), here \( D_z^\lambda f(z) \) denotes the fractional derivative of \( f(z) \) of order \( \lambda \), as defined below, with
\[
D_z^0 f(z) = f(z) \quad \text{and} \quad D_z^1 f(z) = f'(z) \quad (5.9.4)
\]
**Definition 5.9.1:** The fractional derivative of order \( \lambda \) is defined, for a function \( f(z) \), by
\[
D_z^\lambda f(z) = \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)d\zeta}{(z - \zeta)^{1-\lambda}},
\]
where \( 0 \leq \lambda < 1 \), \( f(z) \) is analytic function in a simply connected region of the \( z \)-plane containing the origin and the multiplicity of \( (z - \zeta)^{\lambda-1} \) is removed by the requiring \( \log (z - \zeta) \) to be real when \( (z - \zeta) > 0 \).

**Definition 5.9.2:** Under the hypothesis of definition 5.9.1, the fractional derivative of order \( n + \lambda \) is defined, for a function \( f(z) \), by
\[
D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z),
\]
5.10 Characterization Theorem for the Class $R^2_\lambda(\beta)$

where $0 \leq \lambda < 1$, and $n$ is non-negative integer.

Dixit and Pathak [34] proved the following.

**Lemma 5.9.1**: A function $f(z)$ defined by (5.9.1) is in the class $R_\lambda(\beta)$ if and only if

$$
\sum_{j=2}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} a_j \leq \beta - 1.
$$

(5.9.5)

For a function $f$ defined by (5.9.1) and in the class $R_\lambda(\beta)$, Lemma 5.9.1 immediately yields

$$
a_2 \leq \frac{(\beta - 1)\Gamma(3-\lambda)}{3\Gamma(2-\lambda)} = \frac{(\beta - 1)(2-\lambda)}{2}
$$

(5.9.6)

In view of the coefficient inequality (5.9.5) it would seem to be natural to introduce and study here the class $R^2_\lambda(\beta)$ of analytic and univalent functions where $R^2_\lambda(\beta)$ denote the subclass of $R_\lambda(\beta)$ consisting of functions of the form

$$
f(z) = z + \frac{(\beta - 1)(2-\lambda)\gamma}{2} z^2 + \sum_{j=3}^{\infty} a_j z^j, \quad (a_j \geq 0, \quad 0 \leq \gamma \leq 1, \quad 1 < \beta \leq 2).
$$

(5.9.7)

In the present section we have derived coefficient inequality, closure theorems and radius of convexity.

### 5.10 Characterization Theorem for the Class $R^2_\lambda(\beta)$

We first prove the following.

**Theorem 5.10.1**: Let the function $f$ be defined by (5.9.7). Then $f$ is in the class
$R^3_\lambda(\beta)$ if and only if

$$\sum_{j=3}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} a_j \leq (\beta-1)(1-\gamma). \quad (5.10.1)$$

The result is sharp for the function $f$ given by

$$f(z) = z + \frac{(2-\lambda)(\beta-1)\gamma}{2} z^2 + \frac{(\beta-1)(1-\gamma)}{\Gamma(j+1)\Gamma(2-\lambda)} \Gamma(j+1-\lambda) z^j. \quad (5.10.2)$$

Proof. By setting $a_2 = \frac{(2-\lambda)(\beta-1)\gamma}{2}$ in lemma 5.9.1, we get

$$\frac{\Gamma(2-\lambda)}{\Gamma(3-\lambda)} a_2 + \sum_{j=3}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} a_j \leq \beta-1,$$

or

$$\sum_{j=3}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} a_j \leq (\beta-1)(1-\gamma).$$

\hspace{1cm} \square

### 5.11 Closure Theorem for the Class $R^\gamma_\lambda(\beta)$

A closure theorem for the class $R^3_\lambda(\beta)$ is given by following.

**Theorem 5.11.1:** Let

$$f_i(z) = z + \frac{(2-\lambda)(\beta-1)\gamma}{2} z^2 + \sum_{j=3}^{\infty} a_{j,i} z^j. \quad (5.11.1)$$

If $f_i \in R^\gamma_\lambda(\beta)$, $(i = 1, 2, \ldots, m)$, then the function $g$ given by

$$g(z) = z + \frac{(2-\lambda)(\beta-1)\gamma}{2} z^2 + \sum_{j=3}^{\infty} b_j z^j, \quad (5.11.2)$$

with

$$b_j = \frac{1}{m} \sum_{i=1}^{m} a_{j,i} \geq 0, \quad (5.11.3)$$

is also in the class $R^3_\lambda(\beta)$. 
5.11 Closure Theorem for the Class $R_\lambda^\gamma(\beta)$

Proof. Since $f_i \in R_\lambda^\gamma(\beta)$. Then by Theorem 5.10.1, we have

$$\sum_{j=3}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} a_{j,i} \leq (\beta - 1) (1 - \gamma), \quad (i = 1, 2, \ldots, m). \quad (5.11.4)$$

Thus by applying (5.11.4) and definition (5.11.3), we have

$$\sum_{j=3}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} b_j = \sum_{j=3}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} \left( \frac{1}{m} \sum_{j=1}^{m} a_{j,i} \right) \leq (\beta - 1) (1 - \gamma),$$

which again by virtue of Theorem 5.10.1, proves Theorem 5.11.1.

Next we prove the following.

□

Theorem 5.11.2: Let

$$f_2(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2} z^2 \quad (5.11.5)$$

and

$$f_j(z) = z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2} z^2 + \frac{(\beta - 1)(1 - \gamma)\Gamma(j+1-\lambda)}{\Gamma(j+1)\Gamma(2-\lambda)} z^j, \quad (j = 3, 4, 5, \ldots). \quad (5.11.6)$$

Then $f$ is in the class $R_\lambda^\gamma(\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=2}^{\infty} c_j f_j(z), \quad (c_j \geq 0, \sum_{j=2}^{\infty} c_j = 1). \quad (5.11.7)$$

Proof. Suppose $f$ is given by (5.11.7), so that we find from (5.11.5) and (5.11.6) that

$$f(z) = \sum_{j=2}^{\infty} c_j f_j(z)$$

$$= z + \frac{(2 - \lambda)(\beta - 1)\gamma}{2} z^2 + \sum_{j=3}^{\infty} \frac{(\beta - 1)(1 - \gamma)\Gamma(j+1-\lambda)}{\Gamma(j+1)\Gamma(2-\lambda)} c_j z^j, \quad (5.11.8)$$
5.12 The Radius of Convexity for the Class $R_\lambda^3(\beta)$

where the coefficient $c_j$ are given with (5.11.7).

Then, since

$$\sum_{j=3}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+1-\lambda)} \frac{(\beta - 1)(1 - \gamma)\Gamma(j + 1 - \lambda)}{\Gamma(j + 1)\Gamma(2 - \lambda)} c_j = (\beta - 1)(1 - \gamma) \sum_{j=3}^{\infty} c_j$$

$$= (\beta - 1)(1 - \gamma)(1 - c_2)$$

$$\leq (\beta - 1)(1 - \gamma),$$

we conclude from Theorem 5.10.1 that $f \in R_\lambda^3(\beta)$.

Conversely, let us assume that the function $f$ defined by (5.9.7) is in the class $R_\lambda^3(\beta)$. Then

$$a_j \leq \frac{(\beta - 1)(1 - \gamma)\Gamma(j + 1 - \lambda)}{\Gamma(j + 1)\Gamma(2 - \lambda)}, \quad (5.11.9)$$

which follows readily from (5.10.1).

Setting

$$c_j = \frac{\Gamma(j + 1)\Gamma(2 - \lambda)}{(\beta - 1)(1 - \gamma)\Gamma(j + 1 - \lambda)} a_j \quad (5.11.10)$$

and

$$c_2 = 1 - \sum_{j=3}^{\infty} c_j, \quad (5.11.11)$$

we thus arrive at (5.11.7). This evidently complete the proof of Theorem 5.11.2.

\[\square\]
Theorem 5.12.1: Let the function $f$ be in the class $R_1^\lambda(\beta)$. Then $f$ is a univalently convex function of order $\delta(0 \leq \delta < 1)$ in $|z| < r_1 = r_1(\lambda, \delta, \gamma)$, where $r_1(\lambda, \delta, \gamma)$ is the largest value of $r$ for which
\[
(2 - \lambda)(\beta - 1)\gamma \delta r + \frac{j(j + \delta - 2)(\beta - 1)(1 - \gamma)\Gamma(j + 1 - \lambda)}{\Gamma(j + 1)\Gamma(2 - \lambda)} r^{j-1} \leq 1 - \delta. \quad (5.12.1)
\]
The result is sharp for the function $f_j$ given by (5.11.6).

Proof. It is sufficient to show that for $f \in R_1^\lambda(\beta)$ that
\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta, \quad (|z| < r_1(\lambda, \delta, \gamma), 0 \leq \delta < 1), \quad (5.12.2)
\]
where $r_1(\lambda, \delta, \gamma)$ is the largest value of $r$ for which inequality (5.12.1) holds true.

Observe that, if $f \in R_1^\lambda(\beta)$ is given by (5.9.7), we have
\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{(\beta - 1)(2 - \lambda)\gamma r + \sum_{j=3}^{\infty} a_j \cdot j(j - 1)r^{j-1}}{1 + (\beta - 1)(2 - \lambda)\gamma r + \sum_{j=3}^{\infty} a_j \cdot j \cdot r^{j-1}}
\leq 1 - \delta, \quad (|z| < r, \ 0 \leq \delta < 1),
\]
if and only if
\[
(2 - \lambda)(\beta - 1)\gamma \delta r + \sum_{j=3}^{\infty} j(j + \delta - 2)a_j r^{j-1} \leq 1 - \delta, \ (0 \leq \delta < 1). \quad (5.12.3)
\]
Since $f \in R_1^\lambda(\beta)$, in view of Theorem 5.10.1, we may set
\[
a_j = \frac{(\beta - 1)(1 - \gamma)\Gamma(j + 1 - \lambda)}{\Gamma(j + 1)\Gamma(2 - \lambda)} c_j, \ (c_j \geq 0, \sum_{j=3}^{\infty} c_j \leq 1) \quad (5.12.4)
\]
Now, for each fixed $r$, we choose a positive integer $j_0 = j_0(r)$ for which
\[
\frac{j(j + \delta - 2)\Gamma(j + 1 - \lambda)}{\Gamma(j + 1)\Gamma(2 - \lambda)} r^{j-1}
\]
is maximal. Then

\[ \sum_{j=3}^{\infty} j(j + \delta - 2) a_j r^{j-1} \leq \frac{j_0(j_0 + \delta - 2)(\beta - 1)(1 - \gamma)\Gamma(j_0 + 1 - \lambda)}{\Gamma(j_0 + 1)\Gamma(2 - \lambda)} r^{j_0 - 1}. \]

Consequently, the function \( f \) is univalently convex of order \( \delta (0 \leq \delta < 1) \) in \(|z| < r_1(\lambda, \delta, \gamma)\) provided that

\[ (2 - \lambda)(\beta - 1)\gamma \delta r_1 + \frac{j_0(j_0 + \delta - 2)(\beta - 1)(1 - \gamma)\Gamma(j_0 + 1 - \lambda)}{\Gamma(j_0 + 1)\Gamma(2 - \lambda)} r_1^{j_0 - 1} \leq 1 - \delta, (0 \leq \delta < 1). \] (5.12.5)

We find the value \( r_1 = r_1(\lambda, \delta, \gamma) \) and the corresponding integer \( j_0(\lambda, \delta, \gamma) \) so that

\[ (2 - \lambda)(\beta - 1)\gamma \delta r_1 + \frac{j_0(j_0 + \delta - 2)(\beta - 1)(1 - \gamma)\Gamma(j_0 + 1 - \lambda)}{\Gamma(j_0 + 1)\Gamma(2 - \lambda)} r_1^{j_0 - 1} = 1 - \delta, (0 \leq \delta < 1). \] (5.12.6)

Then this value \( r_1 \) is the radius of univalent convexity of order \( \delta \) for function \( f \in R^2_\lambda(\beta) \).