Chapter 4

Subclasses of Analytic and Univalent Functions with Negative Coefficients

SECTION - 1

4.1 Introduction

Let $S$ denote the class of all analytic and univalent functions in the unit disc $E = \{z : |z| < 1\}$, with Taylor expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. A function $f \in S$ is said to be starlike of order $\alpha$, $0 \leq \alpha < 1$, written as $f \in S^*(\alpha)$, iff $\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$ and $f$ is said to be convex of order $\alpha$ written as $f \in K(\alpha)$, iff $zf'(z) \in S^*(\alpha)$ i.e. $\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha$ for $z \in E$. Clearly $S^*(0) = S^*$, the class of starlike univalent in $E$ and $K(0) = K$, the class of convex univalent in $E$. Also let $UCV$ be the class of uniformly convex univalent functions in $E$ and $S_p$ be the class of starlike functions corresponding to uniformly convex functions.

Further, denoted by $UCV(\alpha)$, where $0 \leq \alpha < \infty$, the class of $\alpha$-uniformly convex univalent functions in $E$ introduced and investigated by Kanas and Yaguchi [62]
and independently by Murugusundaramoorthy [85]. Recall here the definition.

**Definition 4.1.1:** Let $\alpha \in [0, \infty)$. A function $f \in S$ is said to be $\alpha$-uniformly convex in $E$, if the image of every circular area $\gamma$ contained in $E$, with centre $\zeta$, where $|\zeta| \leq \alpha$ is convex.

An analytical and more applicable one variable characterization of the class $\alpha - UCV$ and the related class $S_p(\alpha)$ due to Kanas and Yaguchi [62] {also Murugusundaramoorthy [85]} are stated below.

**Theorem 4.1.1:** Let $f \in S$ and $\alpha \in [0, \infty)$. Then $f \in \alpha - UCV$ iff

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in E,$$

**Theorem 4.1.2:** Let $f \in S$ and $\alpha \in [0, \infty)$. Then $f \in S_p(\alpha)$ iff

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in E.$$

It is clearly seen that $f \in UCV(\alpha)$, iff $zf' \in S_p(\alpha)$ and observed that the family of $\alpha$-uniformly convex functions and [$S_p(\alpha)$ respectively] is characterized by the property that the expression $1 + \frac{zf''(z)}{f'(z)}, z \in E$ (and $\frac{zf'(z)}{f(z)}$ respectively) lies in an ellipse, a parabola or a hyperbola, depending on the values of the parameter $\alpha (0 \leq \alpha < \infty)$. Besides, this family generalizes the idea of convexity in the sense that when $\alpha = 0$, the class $UCV(\alpha)$ reduces to class $K$ and the case $\alpha = 1$ corresponds to the class $UCV$, introduced by Goodman [48] and studied extensively by Ronning [112] and independently by Ma and Minda ([79], [80]). Furthermore, a function $f \in S$ is said to be in the class $T$, if it is expressed in
the form
\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad z \in E. \] (4.1.1)

Let \( S^*(\alpha) \cap T = T^*(\alpha), K(\alpha) \cap T = C(\alpha) \), where \( 0 \leq \alpha < 1 \).

Clearly \( C(0) = C \), the class of convex functions with negative coefficients and \( T^*(0) = T^* \), the class of starlike functions with negative coefficients. Also \( UCV(\alpha) \cap T = UCT(\alpha) \) and

\[ S_p(\alpha) \cap T = S_pT(\alpha). \]

Clearly \( S_pT(1) = S_pT \) and \( S_pT(0) = T^*(0) = T^* \). It may be noted that \( UCT(\alpha) \subseteq UCT \) for \( \alpha \geq 1 \).

\( UCT \subseteq UCT(\alpha) \) for \( 0 \leq \alpha < 1 \), and \( S_pT(\alpha) \subseteq S_pT \) for \( \alpha \geq 1 \), \( S_pT \subseteq S_p(\alpha) \) for \( 0 \leq \alpha < 1 \).

Libera \[76\] showed that, if \( f(z) \in S^* \), then so does the function \( F(z) \) defined by

\[ F(z) = \frac{2}{z} \int_{0}^{z} f(t)dt. \] (4.1.2)

Subsequently, Kumar and Shukla \[73\] studied the Libera integral operator for certain univalent functions. One is tempted to study Libera integral operator for uniformly convex functions and related starlike functions. Therefore, an attempt has been made to study Libera integral operator for these classes having negative coefficients only. The results obtained are sharp. In fact our basic tools are the following theorems due to Murugusundaramoorthy \[85\].
Theorem 4.1.3: $f(z) \in UCT(\alpha)$ iff

$$\sum_{n=2}^{\infty} n[n(\alpha + 1) - \alpha]a_n \leq 1. \quad (4.1.3)$$

The result is sharp for functions

$$f_n(z) = z - \frac{z^n}{n[n(\alpha + 1) - \alpha]}.$$ 

Theorem 4.1.4: $f \in S_pT(\alpha)$ iff

$$\sum_{n=2}^{\infty} [n(\alpha + 1) - \alpha]a_n \leq 1. \quad (4.1.4)$$

The result is sharp for functions

$$f_n(z) = z - \frac{z^n}{n(\alpha + 1) - \alpha}.$$ 

4.2 Main Results

We now state our first characterization theorem involving Libera integral operator.

Theorem 4.2.1: If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in UCT(\alpha)$, then the function $F(z)$ defined by (4.1.2) belongs to $UCT(\rho)$, where $\rho = \frac{3\alpha + 2}{2}$.

Further the result is sharp.

Proof. Since $f(z) \in UCT(\alpha)$, Theorem 4.1.3 ensures that

$$\sum_{n=2}^{\infty} n[n(\alpha + 1) - \alpha]a_n \leq 1.$$
4.2 Main Results

Also from (4.1.2), we have

\[ F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \text{ where } b_n = \frac{2}{n + 1} a_n. \]

Let \( F(z) \in UCT(\beta) \), then by Theorem 4.1.3, it holds if and only if

\[ \sum_{n=2}^{\infty} n[n(\beta + 1) - \beta]b_n \leq 1. \]

Thus, we have to find the largest value of \( \beta \), so that the above inequality holds. Now this inequality holds if

\[ \sum_{n=2}^{\infty} n[n(\beta + 1) - \beta]b_n \leq \sum_{n=2}^{\infty} n[n(\alpha + 1) - \alpha]a_n \]

or, if

\[ [n(\beta + 1) - \beta]b_n \leq [n(\alpha + 1) - \alpha]a_n \]

or, if

\[ [n(\beta + 1) - \beta] \frac{2}{n + 1} a_n \leq [n(\alpha + 1) - \alpha]a_n \]

which is equivalent to

\[ \beta \leq \frac{\alpha(n + 1) + n}{2} = \rho_n, \text{ say, } (n = 2, 3, \ldots). \]

Clearly \( \rho_n \) is an increasing function of \( n \).

Therefore, \( \rho = \inf_{n \geq 2} \rho_n = \rho_2 \) and hence

\[ \rho = \frac{3\alpha + 2}{2}. \]

To show the sharpness we take the function \( f(z) \) given by

\[ f(z) = z - \frac{1}{2(\alpha + 2)} z^2. \]
4.2 Main Results

We now show that the converse of the theorem need not be true. To this end, we consider the function \( F(z) = z - \frac{1}{3(2\rho + 3)} z^3 \).

Theorem 4.1.3 guarantees that \( F(z) \in UCT(\rho) \), but the corresponding functions \( f(z) = z - \frac{2}{3(2\rho + 3)} z^3 \) does not belong to \( UCT(\rho) \). We now state corollaries of Theorem 4.2.1 for the class of \( \alpha \)-uniformly convex functions having negative coefficients.

**Corollary 4.2.1:** Let \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), if \( f(z) \in UCT \), then the function \( F(z) \) defined by (4.1.2) belongs to \( UCT(5/2) \). The result is sharp. The converse need not be true.

**Corollary 4.2.2:** let \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), if \( f(z) \in UCT(0) = C \), then the function \( F(z) \) defined by (4.1.2) belongs to \( UCT(1) = UCT \), the class of uniformly convex function with negative coefficients.

**Theorem 4.2.2:** Let \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), \( (a_n \geq 0) \in S_p T(\alpha) \). Then the function \( F(z) \) defined by (4.1.2) belongs to \( S_p T(\beta) \) where \( \beta = \frac{3\alpha + 2}{2} \).

Further, the result is sharp.

By using Theorem 4.1.4, the proof of this theorem follows on the lines of Theorem 4.2.1 above. Equality holds for the function \( f(z) \) defined by \( f(z) = z - \frac{1}{\alpha + 2} z^2 \).

Further, converse of this theorem need not be true. For example, we consider the function

\[ F(z) = z - \frac{1}{2\beta + 3} z^3. \]
Clearly $F(z) \in S_p T(\beta)$. But the corresponding function $f(z) = z - \frac{2}{2\beta + 3}z^3$ does not belong to $S_p T(\alpha)$.

**Corollary 4.2.3:** Let $f(z) \in S_p T$, then the function $F(z)$ defined by (4.1.2) belong to $S_p T(5/2) = T^*(5/7)$. The result is sharp.

**Corollary 4.2.4:** Let $f(z) \in S_p T'(0) = T^*(0) = T^*$, the class of analytic and univalent function in the unit disc $E$, having negative coefficients. Then the function $F(z)$ defined by (4.1.2) belong to $S_p T(1) = S_p T$, the class of starlike functions corresponding to uniformly convex functions having negative coefficients, which is equivalent to $T^*(1/2)$, that is the class of starlike functions of order $1/2$ having negative coefficients only.

**Theorem 4.2.3:** If $F(z) = z - \sum_{n=2}^{\infty} a_n z^n \in UCT(\alpha), (0 \leq \alpha \leq 1)$, then the function $f(z)$ defined by (4.1.2) belongs to $K(\rho)(0 \leq \rho < 1)$ in $|z| < r^*(\rho, \alpha)$, where

$$r^*(\rho, \alpha) = \inf_{n \geq 2} \left[ \frac{2\{n(\alpha + 1) - \alpha\}(1 - \rho)}{(n + 1)(n - \rho)} \right]\frac{1}{n-1}. \tag{4.2.1}$$

The result is sharp.

**Proof.** Since $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, it follows from (4.1.2) that

$$f(z) = z - \sum_{n=2}^{\infty} \left( n + \frac{1}{2} \right) a_n z^n.$$

In order to establish the required result it suffices to show that:

$$\left| \frac{zf''(z)}{f'(z)} \right| < (1 - \rho) \text{ in } |z| < r^*(\rho, \alpha).$$
4.2 Main Results

Now

\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq \sum_{n=2}^{\infty} \frac{n(n-1)(n+1)}{2} a_n |z|^{n-1}.
\]

The right hand side of this inequality is less than \((1 - \rho)\) iff

\[
1 - \sum_{n=2}^{\infty} \frac{n(n+1)}{2} a_n |z|^{n-1} < 1,
\]

but, for \(F(z) \in UCT(\alpha)\), Theorem 4.1.3 ensures that

\[
\sum_{n=2}^{\infty} n[n(\alpha + 1) - \alpha]a_n \leq 1.
\]

Therefore, the inequality (4.2.2) holds if

\[
\frac{n(n+1)(n-\rho)a_n z^{n-1}}{2(1-\rho)} < n[n(\alpha + 1) - \alpha]a_n,
\]

for each \(n = 2, 3, \ldots\)

or, if

\[
|z| < \left[ \frac{2\{n(\alpha + 1) - \alpha\}(1-\rho)}{(n+1)(n-\rho)} \right]^{1/n-1}.
\]

Hence \(f(z) \in K(\rho)\) in \(|z| < r^*(\rho, \alpha)\). Sharpness follows if we take the function

\(F(z)\) given by

\[F(z) = z - \frac{1}{n[n(\alpha + 1) - \alpha]} z^n, \quad n = 2, 3, \ldots.\]

This completes the proof of theorem.

Since \(r^*(0, 0) = 2/3\), we have the following corollary as an immediate consequence of the theorem.

\(\square\)
Corollary 4.2.5: Let $F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, if $F(z) \in UCT(0) = K$, the class of convex functions having negative coefficients, then the function $f(z)$ defined by (4.1.2) belongs to $K(0) = K$, in $|z| < 2/3$. The result is sharp with extremal function $F(z) = z - \frac{z^2}{4}$.

Also, since $r^*(0,1) = 1$, we have the following interesting corollary.

Corollary 4.2.6: Let $F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, if $F(z) \in UCT(1) = UCT$, the class of uniformly convex function having negative coefficients, then the function $f(z)$ defined by (4.1.2) belongs to $K(0) = K$, in $|z| < 1$. The result is sharp with extremal function $F(z) = z - \frac{z^2}{6}$.

Theorem 4.2.4: Let $F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$. If $F(z) \in S_p T(\alpha)(0 \leq \alpha \leq 1)$, then the function $f(z)$ defined by (4.1.2) belongs to $S^*(\rho)(0 \leq \rho < 1)$ in $|z| < r^{**}(\rho, \alpha)$, where

$$r^{**}(\rho, \alpha) = \inf_{n \geq 2} \left[ \frac{2(1 - \rho)\{n(\alpha + 1) - \alpha\}}{(n + 1)(n - \rho)} \right]^{1/n-1}.$$ 

Proof. Since $F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, it follows from (4.1.2) that

$$f(z) = z - \sum_{n=2}^{\infty} \left( \frac{n + 1}{2} \right) a_n z^n.$$ 

In order to establish required result it suffices to prove that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1 - \rho) \text{ in } |z| < r^{**}(\rho, \alpha).$$
Now
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| -\sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{2} a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{2} a_n |z|^{n-1} \leq 1 - \sum_{n=2}^{\infty} \frac{(n+1)}{2} a_n |z|^{n-1} < 1 - \rho,
\]
provided
\[
\sum_{n=2}^{\infty} \left( \frac{n-\rho}{1-\rho} \right) \left( \frac{n+1}{2} \right) a_n |z|^{n-1} < 1. \tag{4.2.3}
\]

But, for \( F(z) \in S_p T(\alpha) \), Theorem 4.1.4 ensures that
\[
\sum_{n=2}^{\infty} [n(\alpha + 1) - \alpha] a_n \leq 1.
\]

Therefore, the inequality (4.2.3) holds if
\[
\left( \frac{n-\rho}{1-\rho} \right) \left( \frac{n+1}{2} \right) a_n |z|^{n-1} < [n(\alpha + 1) - \alpha] a_n, \text{ for each } n = 2, 3, \ldots,
\]
or, if
\[
|z| < \left[ \frac{\{n(\alpha + 1) - \alpha\}}{2(\alpha + 1)(n-\rho)} \right]^{1/\alpha-1} \text{ for each } n = 2, 3, \ldots.
\]

Hence \( f(z) \in S^*(\rho) \) in \( |z| < r^{**}(\rho, \alpha) \). Sharpness follows if we take the function \( F(z) \) given by
\[
F(z) = z - \frac{1}{[n(\alpha + 1) - \alpha]} z^n, \quad n = 2, 3, \ldots.
\]

This completes the proof of theorem, since \( r^{**}(0, 0) = 2/3 \), we have following corollary of the Theorem 4.2.4.
4.2 Main Results

Corollary 4.2.7: Let

\[ F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \]

If \( F(z) \in S_p T(0) \), then the function \( f(z) \) defined by (4.1.2) belongs to \( S^*(0) = S^* \) in \( |z| < \frac{2}{3} \). The result is sharp with the extremal function \( F(z) = z - \frac{z^2}{2} \).

Further, since \( r^*(0,1) = 1 \), we have the following corollary as an immediate consequence of Theorem 4.2.4.

Corollary 4.2.8: Let

\[ F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \]

If \( F(z) \in S_p T(1) \), then the function \( f(z) \) defined by (4.1.2) belongs to \( S^*(0) \) in \( |z| < 1 \). The result is sharp with the extremal function \( F(z) = z - \frac{z^2}{3} \).

Theorem 4.2.5: Let \( F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \) If \( F(z) \in S_p T(\alpha) \), then the function \( f(z) \) defined by (4.1.2) belongs to \( K(\rho) \) in \( |z| < r(\rho, \alpha) \), where

\[
r(\rho, \alpha) = \inf_{n \geq 2} \left[ \frac{2(1 - \rho)(n(\alpha + 1) - \alpha)}{n(n+1)(n-\rho)} \right]^{1/n-1}. \tag{4.2.4}
\]

The result is sharp.

The proof of this theorem is similar to that of Theorem 4.2.3, so we omit the details involved.

Sharpness follows, if take the function \( F(z) \) given by

\[ F(z) = z - \frac{1}{[n(\alpha + 1) - \alpha]} z^n, n = 2, 3, \ldots \]

Since \( r(0,0) = \frac{1}{3} \), we have the following corollary.
Corollary 4.2.9: Let \( F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \). If \( F(z) \in S_p T(0) = T^* \), then the functions \( f(z) \) defined by (4.1.2) belongs to \( K \) in \( |z| < 1/3 \). The result is sharp with extremal function \( F(z) = z - \frac{z^2}{2} \).

On the other hand, we have \( r(0, 1) = 1/2 \), we have another interesting corollary.

Corollary 4.2.10: Let \( F(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \). If \( F(z) \in S_p T(1) = S_p(T) \), then the functions \( f(z) \) defined by (4.1.2) belongs to \( K \) in \( |z| < 1/2 \). The result is sharp with extremal function \( F(z) = z - \frac{z^2}{3} \).

SECTION - 2

4.3 Introduction

Let \( E \) denote the unit disc \( \{ z : |z| < 1 \} \). A function \( f \) analytic and univalent in \( E, f(0) = 0 \), is said to be starlike univalent, written as \( f \in S^* \), if and only if

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0
\]

and \( f \) is said to be convex univalent, written as \( f \in K \), if and only if \( zf'(z) \in S^* \) i.e.

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0,
\]

for \( z \in E \). Further, a function \( f \) analytic and univalent in \( E, f(0) = 0 \), is said to be uniformly convex, written as \( f \in UCV \), if and only if

\[
1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right|
\]
and $f$ is said to be starlike functions corresponding to uniformly convex functions, written as $f \in S_p$, if and only if

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \Re \left\{ \frac{zf'(z)}{f(z)} \right\}.$$

Class $UCV$ of uniformly convex functions was introduced by Goodman [48], and class $S_p$ of starlike functions corresponding to uniformly convex functions was introduced by Ma and Minda ([79], [80]) and Ronning [112]. It can be easily seen that a function $g$ is said to be in the class $S_p$ if $g \in S^*$ and $g(z) = zf'(z)$ for some $f \in UCV$.

Throughout this section, let the functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, a_1 > 0, z \in E), \quad (4.3.1)$$

$$f_i(z) = a_{1,i} z - \sum_{n=2}^{\infty} a_{n,i} z^n, \quad (a_{1,i} > 0, a_{n,i} \geq 0), \quad (4.3.2)$$

$$g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n, \quad (b_1 > 0, b_n \geq 0), \quad (4.3.3)$$

$$g_j(z) = b_{1,j} z - \sum_{n=2}^{\infty} b_{n,j} z^n, \quad (b_{1,j} > 0, b_{n,j} \geq 0) \quad (4.3.4)$$

be analytic and univalent in $E$.

Let $S_pT_1$ denote the class of functions $f(z) \in S_p$ of the form (4.3.1) for all $z \in E$. We note that function $f$ of the form (4.3.1) belongs to the class $UCT_1$ if and only if $zf'(z)$ belongs to the class $S_pT_1$.

One can easily prove on the lines of Theorem 2.2.2 and 3.2.1 of the Thesis of
Murugusundaramoorthy [85] that \( f(z) \in UCT_1 \) if and only if

\[
\sum_{n=2}^{\infty} n(2n-1)a_n \leq a_1,
\]

and \( f(z) \in S_pT_1 \) if and only if

\[
\sum_{n=2}^{\infty} (2n-1)a_n \leq a_1.
\]

We now introduce the following class of analytic and univalent functions, which plays an important role in the discussion that follows.

A function \( f(z) \), defined by (4.3.1) belongs to the class \( S_pT_1(k) \) if and only if

\[
\sum_{n=2}^{\infty} [n^k(2n-1)a_n] \leq a_1,
\]

where \( k \) is any fixed real number.

It is evident \( S_pT_1(1) = UCT_1 \) and \( S_pT_1(0) = S_pT_1 \). Further if \( k > h \geq 0 \), then \( S_pT_1(k) \subset S_pT_1(h) \), the containment being proper. Moreover, for any positive integer \( k \), we have the following inclusive relation

\[
S_pT_1(k) \subset S_pT_1(k-1) \subset \ldots \subset S_pT_1(2) \subset S_pT_1(1) = UCT_1 = \subset S_pT_1(0) = S_pT_1.
\]

However, for \( k < 0 \), \( S_pT_1(k) \) contains non-univalent functions as well.

The quasi-Hadamard product of two or more functions has been defined and used by Kumar ([69], [70], [71]), Owa ([92], [93]), Mishra [84] and others. Accordingly, the quasi-Hadamard product of two functions \( f(z) \) and \( g(z) \) defined by

\[
f \ast g(z) = a_1b_1z - \sum_{n=2}^{\infty} a_nb_nz^n.
\]
4.4 Main Results

Similarly, the quasi-Hadamard product of more than two functions can also be defined.

Since to a certain extent the work in the uniformly convex case has paralleled that of starlike case, one is tempted to search results analogous to Kumar ([69], [70], [71]), Owa ([92], [93]), Mishra [84] for uniformly convex functions and starlike functions corresponding to uniformly convex functions. Thus, we establish certain interesting results concerning the quasi-Hadamard product of functions in the class $S_p T_1$ and $UCT_1$ analogous to the results due to Kumar ([69], [70], [71]).

4.4 Main Results

Theorem 4.4.1: Let function $f_i(z)$ defined by (4.3.2) be in the class $UCT_1$ for every $i = 1, 2, 3, \ldots, m$; and let the function $g_j(z)$ defined by (4.3.4) be in the class $S_p T_1$ for every $j = 1, 2, 3, \ldots, q$. Then the quasi-Hadamard product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_q(z)$ belongs to the class $S_p T_1(2m + q - 1)$.

Proof. Let $h(z) = f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_q(z)$.

Clearly,

$$h(z) = \left\{ \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q} b_{1,j} \right\} z - \sum_{n=2}^{\infty} \left\{ \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right\} z^n. \quad (4.4.1)$$

To prove the Theorem, we need to show that

$$\sum_{n=2}^{\infty} n^{2m+q-1}(2n-1) \left\{ \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right\} \leq \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q} b_{1,j}. \quad (4.4.2)$$
Since \( f_i(z) \in UCT_1 \) by (4.3.5)

\[
\sum_{n=2}^{\infty} [n(2n-1)] a_{n,i} \leq a_{1,i},
\]

(4.4.3)

for every \( i = 1, 2, \ldots, m \).

Therefore, \( n(2n-1)a_{n,i} \leq a_{1,i} \)

or

\[
a_{n,i} \leq \frac{a_{1,i}}{n(2n-1)}, \quad \text{for every } i = 1, 2, \ldots, m.
\]

The right hand expression of this inequality is not greater than \( n^{-2}a_{1,i} \). Hence,

\[
a_{n,i} \leq n^{-2}a_{1,i},
\]

(4.4.4)

for every \( i = 1, 2, \ldots, m \).

Similarly, for \( g_j(z) \in S_pT_1 \), by (4.3.6),

\[
\sum_{n=2}^{\infty} (2n-1)b_{n,j} \leq b_{1,j},
\]

(4.4.5)

for every \( i = 1, 2, \ldots, q \). Whence we obtain

\[
b_{n,j} \leq n^{-1}b_{1,j},
\]

(4.4.6)

for every \( i = 1, 2, \ldots, q \).

Using (4.4.4) for \( i = 1, 2, 3, \ldots, m \), (4.4.6) \( i = 1, 2, 3, \ldots, q-1 \), and (4.4.5) for \( j = q \), we get

\[
\sum_{n=2}^{\infty} n^{2m+q-1}(2n-1) \left\{ \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right\} \\
\leq \sum_{n=2}^{\infty} \left[ n^{2m+q-1}(2n-1) \left\{ n^{-2m} n^{-(q-1)} \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q-1} b_{i,j} \right\} b_{n,q} \right]
\]
\[
\leq \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q} b_{1,j}.
\]

Hence \( h(z) \in S_{p}T_{1}(2m + q - 1) \). This completes the proof.

It should be noted that the required estimates can also be obtained by using (4.4.4) for \( i = 1, 2, 3, \ldots, m - 1 \), (4.4.6) for \( j = 1, 2, 3, \ldots, q \), and (4.4.3) for \( i = m \).

Now we discuss applications of Theorem 4.4.1. Taking into account the quasi-Hadamard product of the function \( f_1(z), f_2(z), \ldots, f_m(z) \) only, in the proof of Theorem (4.4.1), and using (4.4.4) for \( i = 1, 2, 3, \ldots, m - 1 \), and (4.4.3) for \( i = m \), we are led to

\[
\square
\]

**Corollary 4.4.1:** Let the function \( f_i(z) \) defined by (4.3.2) belongs to the class \( U_{CT_1} \) for every \( i = 1, 2, 3, \ldots, m \). Then the quasi-Hadamard product \( f_1 * f_2 * \ldots * f_m(z) \) belongs to the class \( S_{p}T_{1}(2m - 1) \).

Next, taking into account the quasi-Hadamard product of the functions \( g_1(z), g_2(z), \ldots, g_q(z) \) only, in the proof of Theorem 4.4.4, and using (4.4.6) for \( j = 1, 2, 3, \ldots, q - 1 \) and (4.4.5) for \( j = q \), we are led to

**Corollary 4.4.2:** Let the function \( g_j(z) \) defined by (4.3.4) belongs to the class \( S_{p}T_{1} \) for every \( j = 1, 2, 3, \ldots, q \). Then the quasi-Hadamard product \( g_1 * g_2 * \ldots * g_q(z) \) belongs to the class \( S_{p}T_{1}(q - 1) \).

**Theorem 4.4.2:** For each \( i = 1, 2, 3, \ldots, m \), let \( f_i(z) \) belong to the class \( S_{p}T_{1}(k_i) \) respectively. Then the quasi-Hadamard product \( f_1 * f_2 * \ldots * f_m(z) \) belongs to the
class $S_p T_1(0) = S_p T_1$, provided

$$\sum_{i=1}^{m} k_i \geq -(m - 1).$$

**Proof.** We need to show that

$$\sum_{n=2}^{\infty} \left[(2n-1) \prod_{i=1}^{m} a_{n,i}\right] \leq \prod_{i=1}^{m} a_{1,i}.$$

Since $f_i(z) \in S_p T_1(k_i)$, we have

$$\sum_{n=2}^{\infty} n^{k_i}(2n-1)a_{n,i} \leq a_{1,i}. \quad (4.4.7)$$

Therefore,

$$a_{n,i} \leq \frac{n^{-k_i}}{2n-1}a_{1,i},$$

which implies that,

$$a_{n,i} \leq n^{-(1+k_i)}a_{1,i}. \quad (4.4.8)$$

Using (4.4.8) for $i = 1, 2, 3, \ldots, m - 1$, and (4.4.7) for $i = m$, we have

$$\sum_{n=2}^{\infty} \left[(2n-1) \prod_{i=1}^{m} a_{n,i}\right] = \sum_{n=2}^{\infty} \left[(2n-1) \prod_{i=1}^{m-1} a_{n,i}\right] a_{n,m}$$

$$\leq \sum_{n=2}^{\infty} (2n-1) \prod_{i=1}^{m-1} n^{-(1+k_i)}a_{1,i} \quad a_{n,m}$$

$$\leq \prod_{i=1}^{m-1} a_{1,i} \sum_{n=2}^{\infty} [n^k(2n-1)] a_{n,m}, \quad \text{provided} \quad \sum_{i=1}^{m} k_i \geq -(m - 1)$$

$$\leq \prod_{i=1}^{m} a_{1,i}.$$

Hence, $f_1 * f_2 * \ldots * f_m(z) \in S_p T_1(0) = S_p T_1$, provided $\sum_{i=1}^{m} k_i \geq -(m - 1) \quad \square$
Remark 4.4.1: We can suitably choose some or all \(k_i\)'s to be negative. This leads to the important and new information that the quasi-Hadamard product of non-univalent functions is starlike function corresponding to uniformly convex function.

**Theorem 4.4.3:** For each \(i = 1, 2, 3, \ldots, m\) let \(f_i(z)\) belong to the class \(S_pT_1(k_i)\) respectively, then the quasi-Hadamard product \(f_1 \ast f_2 \ast \ldots \ast f_m(z)\) belongs to the class \(S_pT_1(1) = UCT_1\), provided \(\prod_{i=1}^{m} k_i \geq -(m - 2)\).

*Proof.* The proof of the Theorem follows on the lines of Theorem 4.4.2 above. \(\square\)

Remark 4.4.2: We can suitably choose some or all \(k_i\)'s to be negative. This also leads to the important and new information that the quasi-Hadamard product of non-univalent functions is uniformly convex univalent.