Chapter 3
Uniformly convex and Starlike Functions with Negative Coefficients

3.1 Introduction

Denote by $A_k$ the class of functions of the form

$$f(z) = z + \sum_{j=k+1}^{\infty} a_j z^j \quad (k \in N = \{1, 2, 3, \ldots\}), \quad (3.1.1)$$

which are analytic in the open unit disc

$$E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let the operator

$$D^n (n \in N_0 = N \cup \{0\})$$

be defined, for a function $f \in A_k$, by

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = z f'(z),$$
and \( D^n f(z) = D(D^{n-1} f(z)) \) \((n \in N_0)\).

The operator \( D^n \) is also known as the Salagean operator of order \( n \in N_0 \) (cf. [114]; see also [129] and [130], where it was used recently in investigating several interesting results for univalence of analytic functions).

For a function \( f(z) \) given by (3.1.1), it follows from the above definition that

\[
D^n f(z) = z + \sum_{j=k+1}^{\infty} j^n a_j z^j \quad (n \in N_0).
\]

With the help of the operator \( D^n \), we say that the function \( f \in A_k \) is in the class \( \alpha - A_k(n, \lambda) \) if and only if

\[
Re(F_{n,\lambda}(z)) \geq \alpha |F_{n,\lambda}(z) - 1|, \quad z \in E \quad (0 \leq \lambda \leq 1 \text{ and } \alpha \geq 0),
\]

where, for convenience,

\[
F_{n,\lambda}(z) = \frac{(1 - \lambda)z(D^n f(z))' + \lambda z(D^{n+1} f(z))'}{(1 - \lambda)D^n f(z) + \lambda D^{n+1} f(z)} = \frac{\phi_{n,\lambda}(z)}{\psi_{n,\lambda}(z)}.
\]

The expression \( F_{n,\lambda}(z) \) was used recently by Srivastava, Patel and Sahoo [130] in determining several interesting criteria for univalence of analytic functions with negative coefficients.

We note that by specializing the parameters \( \alpha, k, n \) and \( \lambda \), the following subclasses studied by various authors can be obtained.

(i) \( \alpha - A_1(0,1) = \alpha - UCV. \)

(ii) \( \alpha - A_1(0,0) = \alpha - ST. \)

The class \( \alpha - UCV \) was introduced by Kanas and Wiśniowska [60], where its geometric definition and connection with conic domains were considered. The
class $\alpha - ST$ was investigated in [61]. In fact, it is related to the class $\alpha - UCV$ by means of the well known Alexander equivalence between the usual classes of convex and starlike functions. For further developments involving each of the class $\alpha - UCV$ and $\alpha - ST$, the work of Kanas and Srivastava [59] can also be seen. In particular, when $\alpha = 1$, we obtain

$$1 - A_1(0,1) \equiv 1 - UCV \equiv UCV \quad \text{and} \quad 1 - A_1(0,0) \equiv 1 - ST \equiv SP,$$

where $UCV$ and $SP$ are the familiar classes of uniformly convex functions and parabolic starlike functions in $E$ respectively (see, for detailed study ([48], [49], [79] and [112])). In fact, by making use of a certain fractional calculus operator, Srivastava and Mishra [129] presented a systematic and unified study of the classes $UCV$ and $SP$.

Let $T_k$ denote the subclass of $A_k$ consisting of functions of the form

$$f(z) = z - \sum_{j=k+1}^{\infty} a_j z^j \quad (a_j \geq 0; \ j = k + 1, k + 2, k + 3, \ldots)$$

(3.1.4)

and define the class $\alpha - T_k(n, \lambda)$ by

$$\alpha - T_k(n, \lambda) = \alpha - A_k(n, \lambda) \cap T_k.$$

**Remark 3.1.1:** Two of the many interesting subclasses of the class $\alpha - T_k(n, \lambda)$ are worthy of mention here. First of all, by setting $k = 1, n = 1$ and $\lambda = 0$, the class $\alpha - T_k(n, \lambda)$ reduces essentially to the class $\alpha - UCT$ studied recently by Murugusundaramoorthy [85]. Second, if we put $k = 1, n = 0$ and $\lambda = 0$, we
obtain another interesting class studied in [85]. Furthermore, in the special case
when \( \lambda = 1, n = 1 \), the condition (3.1.3) assumes the elegant form
\[
\text{Re} \left[ \frac{z(D^2 f(z))'}{D^2 f(z)} \right] \geq \alpha \left[ \frac{z(D^2 f(z))'}{D^2 f(z)} - 1 \right], \quad z \in E \text{ and } \alpha \geq 0. \tag{3.1.5}
\]
Thus \( \alpha - T_1(1,1) \) represents the class of functions \( f(z) \in T_1 \) satisfying the
inequality (3.1.5).

For convenience of writing, we let \( \sum \) stands \( \sum_{j=k+1}^{\infty} \) in this chapter.

For the class of functions belonging to the general class \( \alpha - T_k(n,\lambda) \), we
prove a number of the various interesting and useful properties and characteristics
including for example, coefficient bounds, linear combinations and modified
Hadamard products of several functions belonging to this class \( \alpha - T_k(n,\lambda) \). Our
results involving this class provide generalization and improvements of those given
by (for example) the aforecited earlier authors.

### 3.2 Theorems on Coefficient Bounds

**Theorem 3.2.1:** A function \( f(z) \) defined by (3.1.4) is in the class \( \alpha - T_k(n,\lambda) \)
if and only if
\[
\sum (1 - \lambda + \lambda j)[j(\alpha + 1) - \alpha] j^n a_j \leq 1. \tag{3.2.1}
\]

**Proof.** Assume that \( f(z) \) belonging to the class \( \alpha - T_k(n,\lambda) \). Then by definition,
we have
\[
\text{Re}(F_{n,\lambda}(z)) \geq \alpha \left| F_{n,\lambda}(z) - 1 \right|, \quad z \in E.
\]
Or, equivalently,

\[
Re \left[ \frac{1 - \sum (1 - \lambda + \lambda j) j^{n+1} a_j z^{j-1}}{1 - \sum (1 - \lambda + \lambda j) j^n a_j z^j} \right] \geq \alpha \left| \frac{1 - \sum (1 - \lambda + \lambda j) j^{n+1} a_j z^{j-1}}{1 - \sum (1 - \lambda + \lambda j) j^n a_j z^j} - 1 \right|.
\]

\[\text{(3.2.2)}\]

Choosing values of \( z \) on the real axes so that the left side of (3.2.2) is real and letting \( z \to 1 \), we get

\[
1 - \sum (1 - \lambda + \lambda j) j^{n+1} a_j \geq \alpha \sum (1 - \lambda + \lambda j)(j - 1) j^n a_j,
\]

which yields

\[
\sum (1 - \lambda + \lambda j)[j(\alpha + 1) - \alpha] j^n a_j \leq 1.
\]

Conversely, suppose that (3.2.1) is true for \( z \in E \). Then

\[
Re[F_{n,\lambda}(z)] - \alpha |F_{n,\lambda}(z) - 1| \geq 0,
\]

if

\[
\frac{1 - \sum (1 - \lambda + \lambda j) j^{n+1} a_j |z|^{j-1}}{1 - \sum (1 - \lambda + \lambda j) j^n a_j |z|^j} - \alpha \sum (1 - \lambda + \lambda j)(j^{n+1} - j^n) a_j |z|^{j-1} \geq 0,
\]

that is, if

\[
\sum (1 - \lambda + \lambda j)[j(\alpha + 1) - \alpha] j^n a_j \leq 1 \quad (|z| \to 1).
\]

This completes the proof. \( \Box \)

**Corollary 3.2.1:** Let the function \( f(z) \) defined by (3.1.4) be in the class \( \alpha - T_k(n, \lambda) \), then

\[
0 \leq a_j \leq \frac{1}{(1 - \lambda + \lambda j)[j(\alpha + 1) - \alpha] j^n}.
\]
3.2 Theorems on Coefficient Bounds

The result is sharp for the function

\[ f(z) = z - \frac{z^j}{(1 - \lambda + \lambda^j)[j(\alpha + 1) - \alpha]j^n}. \tag{3.2.3} \]

Remark 3.2.1: Since

\[(1 - \lambda + \lambda^j) \leq (1 - \mu + \mu^j) \quad (0 \leq \lambda \leq \mu \leq 1),\]

we have the inclusion property:

\[ \alpha - T_k(n, \mu) \subseteq \alpha - T_k(n, \lambda). \]

Furthermore, for \(0 \leq \alpha_1 \leq \alpha_2\), it is easily verified that

\[(1 - \lambda + \lambda^j)[j(\alpha_1 + 1) - \alpha_1]j^n a_j \leq (1 - \lambda + \lambda^j)[j(\alpha_2 + 1) - \alpha_2]j^n a_j,\]

so that, with the aid of Theorem 3.2.1, we derive the inclusion property:

\[ \alpha_2 - T_k(n, \lambda) \subseteq \alpha_1 - T_k(n, \lambda). \]

Theorem 3.2.2: For each \(n \in N_0\),

\[ \alpha - T_k(n + 1, \lambda) \subset \beta - T_k(n, \lambda), \]

where

\[ \beta = (1 + \alpha)(k + 1). \]

The result is sharp.

Proof. Suppose that the function \(f\) defined by (3.1.4) belongs to the class \(\alpha - T_k(n + 1, \lambda)\). Then, by Theorem 3.2.1,

\[ \sum (1 - \lambda + \lambda^j)[j(\alpha + 1) - \alpha]j^{n+1} a_j \leq 1. \tag{3.2.4} \]
3.2 Theorems on Coefficient Bounds

To prove that $f \in \beta - T_k(n, \lambda)$, it is sufficient to find the largest $\beta$ such that

$$
\sum (1 - \lambda + \lambda j) [j(\beta + 1) - \beta] j^n a_j \leq 1. \quad (3.2.5)
$$

In view of (3.2.4), (3.2.5) will hold true if

$$(1 - \lambda + \lambda j)[j(\beta + 1) - \beta] j^n a_j \leq (1 - \lambda + \lambda j)[j(1 + \alpha) - \alpha] j^{n+1} a_j$$

$$(j \geq k + 1; k \in N)$$

that is, if

$$\beta \leq (1 + \alpha) j \quad (j \geq k + 1; k \in N). \quad (3.2.6)$$

Since the right-hand side of (3.2.6) is an increasing function of $j$, letting $j = k + 1$ in (3.2.6) we obtain

$$\beta \leq (1 + \alpha)(k + 1),$$

which proves the main assertion of Theorem 3.2.2.

Finally, by taking the function $f$ given by

$$f(z) = z - \frac{1}{(1 + \lambda z)[(k + 1)(\alpha + 1) - \alpha](k + 1)^n} z^{k+1}, \quad (k \in N), \quad (3.2.7)$$

we can see that result of Theorem 3.2.2 is sharp. \qed

Remark 3.2.2: Since $\beta > \alpha$, it follows from remark 3.2.1 that

$$\beta - T_k(n, \lambda) \subset \alpha - T_k(n, \lambda) \quad (n \in N_0)$$

and hence that

$$\alpha - T_k(n + 1, \lambda) \subset \beta - T_k(n, \lambda) \subset \alpha - T_k(n, \lambda) \quad (n \in N_0),$$

where $\beta$ is defined with Theorem 3.2.2.
3.3 Inclusion Properties Associated with Modified Hadamard Product

Let \( f(z) \) be defined by (3.1.4) and let

\[
g(z) = z - \sum b_j z^j (b_j \geq 0; j = k + 1, k + 2, k + 3, \ldots; k \in N).
\]

(3.3.1)

Then the modified Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is defined here by

\[
(f * g)(z) = z - \sum a_j b_j z^j
\]

(3.3.2)

\[
(a_j \geq 0; b_j \geq 0; j = k + 1, k + 2, k + 3, \ldots; k \in N).
\]

Making use of coefficient inequality (3.2.1) in conjunction with Cauchy-Schwarz inequality, we can easily prove the following.

Theorem 3.3.1: Let the function \( f \) defined by (3.1.4) and the function \( g \) defined by (3.3.1) belong to the class \( \alpha - T_k(n, \lambda) \). Then the modified Hadamard product \( f * g \) defined by (3.3.2) belongs to the class \( \eta - T_k(n, \lambda) \), where

\[
\eta = \frac{(1 + \lambda k)[k(\alpha + 1) + 1]^2(k + 1)^n - (k + 1)}{k}.
\]

The result is sharp for the functions given by

\[
f(z) = g(z) = z - \frac{1}{(1 + \lambda k)[k(\alpha + 1) + 1](k + 1)^n} z^{k+1} \quad (k \in N).
\]

(3.3.3)

Theorem 3.3.2: Let the function \( f \) defined by (3.1.4) and the function \( g \) defined by (3.3.1) be in the same class \( \alpha - T_k(n, \lambda) \). Then the function \( h(z) \) defined by

\[
h(z) = z - \sum (a_j^2 + b_j^2) z^j
\]
3.3 Inclusion Properties Associated with Modified Hadamard...

belongs to the class \( \sigma - T_k(n, \lambda) \), where

\[
\sigma = \frac{(1 + \lambda k)[k(\alpha + 1) + 1]^2(k + 1)^n - (k + 1)}{2k}.
\]

The result is sharp for the functions \( f(z) \) and \( g(z) \) defined by (3.3.3).

Proof. We have

\[
\sum \{(1 - \lambda + \lambda j)[(\alpha + 1) - \alpha]j^n\}^2a_j^2
\leq \{\sum(1 - \lambda + \lambda j)[(\alpha + 1) - \alpha]j^na_j\}^2 \leq 1. \tag{3.3.4}
\]

Similarly, we have

\[
\sum \{(1 - \lambda + \lambda j)[(\alpha + 1) - \alpha]j^n\}^2b_j^2 \leq 1. \tag{3.3.5}
\]

It follows from (3.3.4) and (3.3.5) that

\[
\sum \frac{1}{2}\{(1 - \lambda + \lambda j)[(\alpha + 1) - \alpha]j^n\}^2(a_j^2 + b_j^2) \leq 1.
\]

Therefore, we need to find the largest \( \sigma \) such that

\[
(1 - \lambda + \lambda j)[j(\sigma + 1) - \sigma]j^n \leq \frac{1}{2}\{(1 - \lambda + \lambda j)[(\alpha + 1) - \alpha]j^n\}^2, \quad (j \geq k + 1; k \in N),
\]
that is,

\[
\sigma \leq \frac{(1 - \lambda + \lambda j)[(\alpha + 1) - \alpha]j^n - j}{2(j - 1)} \quad (j \geq k + 1). \tag{3.3.6}
\]

Since the right-hand side of (3.3.6) is an increasing function of \( j \), we readily have

\[
\sigma \leq \frac{(1 + \lambda k)[(k + 1)(\alpha + 1) - \alpha]^2(k + 1)^n - (k + 1)}{2k},
\]
or,

\[ \sigma \leq \frac{(1 + \lambda k)[k(\alpha + 1) + 1]^2(k + 1)^n - (k + 1)}{2k}, \quad (3.3.7) \]

and Theorem 3.3.2 follows at once. \qed

### 3.4 A Family of Integral Operators

**Theorem 3.4.1:** Let the function \( f \) defined by (3.1.4) be in the class \( \alpha - T_k(n, \lambda) \), and let \( c > -1 \). Then the function \( F(z) \) defined by

\[ F(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1; f \in A_k) \quad (3.4.1) \]

belongs to the class \( \delta - T_k(n, \lambda) \), where

\[ \delta = \frac{[k(\alpha + 1) + 1](c + k + 1) - (c + 1)(k + 1)}{(c + 1)k}. \]

The result is sharp for the function \( f(z) \) defined by (3.2.7).

**Proof.** From the representation (3.4.1) of \( F(z) \), it follows that

\[ F(z) = z - \sum \left( \frac{c + 1}{c + j} \right) a_j z^j. \]

We need to find the largest \( \delta \) such that

\[ [j(\delta + 1) - \delta] \left( \frac{c + 1}{c + j} \right) \leq j(\alpha + 1) - \alpha \quad (j \geq k + 1; k \in N) \]

or, equivalently,

\[ \delta \leq \frac{[j(\alpha + 1) - \alpha](c + j) - (c + 1)j}{(c + 1)(j - 1)} \quad (j = k + 1; k \in N). \quad (3.4.2) \]
The right-hand side of (3.4.2) being an increasing function of \( j \), setting \( j = k + 1 \) in (3.4.2), we obtain

\[
\delta \leq \frac{[k(\alpha + 1) + 1](c + k + 1) - (c + 1)(k + 1)}{(c + 1)k},
\]

which completes the proof of Theorem 3.4.1.

\[
\text{Theorem 3.4.2: Let the function } F(z) \text{ given by}
\]

\[
F(z) = z - \sum_{j} d_j z^j \quad (d_j \geq 0; j = k + 1, k + 2, k + 3, \ldots, k \in N)
\]

be in the class \( \alpha - T_k(n, \lambda) \), and let \( c \) be real number such that \( c > -1 \). Then the function \( f(z) \) defined by (3.4.1) is univalent in \( |z| = R \), where

\[
R = \inf_{j \geq k+1} \left[ \frac{(c + 1)(1 - \lambda + \lambda j)(j(\alpha + 1) - \alpha)j^{n-1}}{(c + j)} \right]^{\frac{1}{j+1}}. \tag{3.4.3}
\]

The result is sharp.

**Proof.** We find from (3.4.1) that

\[
f(z) = z^{1-c}(z^c F(z))' = z - \sum \left( \frac{c + j}{c + 1} \right) d_j z^j.
\]

In order to obtain the desired result, it suffices to show that

\[|f'(z) - 1| < 1 \text{ whenever } |z| < R,\]

where \( R \) is given by (3.4.3). Now

\[|f'(z) - 1| \leq \sum \frac{j(c + j)}{(c + 1)} d_j |z|^{j-1}.\]
3.5 Closure Theorems for the class $\alpha - T_k(n,\lambda)$

Thus we have $|f'(z) - 1| < 1$ if

$$\sum \frac{j(c+j)}{(c+1)}d_j|z|^{j-1} < 1. \quad (3.4.4)$$

But, by Theorem 3.2.1, we know that

$$\sum (1 - \lambda + \lambda j) [j(\alpha + 1) - \alpha] j^n d_j \leq 1.$$ 

Hence (3.4.4) will be satisfied if

$$\frac{j(c+j)}{(c+1)} |z|^{j-1} < (1 - \lambda + \lambda j) [j(\alpha + 1) - \alpha] j^n,$$

that is, if

$$|z| < \left[ \frac{(c+1)(1 - \lambda + \lambda j) \{j(\alpha + 1) - \alpha\} j^{n-1}}{(c+j)} \right]^{\frac{1}{j-1}} \quad (3.4.5)$$

$$(j \geq k + 1; k \in N).$$

Therefore, the function $f(z)$ given by (3.4.1) is univalent in $|z| < R$, where $R$ is defined by (3.4.3). The sharpness of the result follows if we take

$$f(z) = z - \frac{(c+j)}{(1 - \lambda + \lambda j)[j(\alpha + 1) - \alpha] (c+1) j^n} z^j \quad (j \geq k + 1; k \in N). \quad (3.4.6)$$

$\square$

3.5 Closure Theorems for the class $\alpha - T_k(n,\lambda)$

The following inclusion properties are an easy consequences of Theorem 3.2.1.

**Theorem 3.5.1:** Let

$$f_i(z) = z - \sum a_{j,i} z^j \quad (a_{j,i} \geq 0), \quad (3.5.1)$$
be in the class $\alpha - T_k(n, \lambda)$ for every $i = 1, 2, \ldots m$. Then the function $g(z)$ defined by

$$g(z) = \sum_{i=1}^{m} d_i f_i(z) \quad (3.5.2)$$

is also in the class $\alpha - T_k(n, \lambda)$, where

$$\sum_{i=1}^{m} d_i = 1. \quad (3.5.3)$$

**Theorem 3.5.2:** Let the function $f_i(z)$ defined by (3.5.1) be in the class $\alpha - T_k(n, \lambda)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum b_j z^j \quad (b_j \geq 0)$$

is also in the same class $\alpha - T_k(n, \lambda)$, where

$$b_j = \frac{1}{m} \sum_{i=1}^{m} a_{j,i}. \quad (3.5.4)$$

**Theorem 3.5.3:** Let

$$f_1(z) = z \quad (3.5.5)$$

and

$$f_j(z) = z - \frac{1}{(1 - \lambda + \lambda_j)[j(\alpha + 1) - \alpha]n^j} z^j \quad (j \geq k + 1). \quad (3.5.6)$$

Then the function $f(z)$ is in the class $\alpha - T_k(n, \lambda)$ if and only if it can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum \lambda_j f_j(z), \quad (\lambda_j \geq 0; j \geq k + 1; \lambda_1 + \sum \lambda_j = 1).$$

**Corollary 3.5.1:** The extreme points of the class $\alpha - T_k(n, \lambda)$ are functions $f_1(z)$ and $f_j(z)$ $(j \geq k + 1)$ given by (3.5.5) and (3.5.6).
3.6 Applications of Fractional Calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivative and fractional integrals) have been given in the literature (cf; e.g., [103] and [138]). We find it to be convenient to recall here the following definitions which were used earlier by Owa [91] and Kumar, Dixit and Nishimoto [72] (and, more recently, by Srivastava and Aouf [127]; see also Aouf [10]).

**Definition 3.6.1:** The fractional integral of order \( \mu \) is defined, for a function \( f(z) \), by

\[
\mathcal{D}_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\mu}} \, d\zeta \quad (\mu > 0),
\]

(3.6.1)

where \( f(z) \) is analytic function in a simply-connected region of the complex \( z \)-plane containing the origin, and the multiplicity of \( (z - \zeta)^{-1} \) is removed by requiring \( \log (z - \zeta) \) to be real when \( (z - \zeta) > 0 \).

**Definition 3.6.2:** The fractional derivative of order \( \mu \) is defined, for a function \( f(z) \), by

\[
\mathcal{D}_z^\mu f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{\mu}} \, d\zeta \quad (0 \leq \mu < 1),
\]

(3.6.2)

where \( f(z) \) is constrained, and the multiplicity of \( (z - \zeta)^{-\mu} \) is removed, as in Definition 3.6.1.

**Definition 3.6.3:** Under the hypothesis of Definition 3.6.2, the fractional derivative of order \( n + \mu \) is defined, for a function \( f(z) \), by

\[
\mathcal{D}_z^{n+\mu} f(z) = \frac{d^n}{dz^n} \{\mathcal{D}_z^\mu f(z)\} \quad (0 \leq \mu < 1; n \in N_0).
\]

(3.6.3)
Theorem 3.6.1: Let the function $f$ defined by (3.1.4) be in the class $\alpha - T_k(n, \lambda)$. Then

$$\left|D_z^{-\mu}(D^i f(z))\right| \geq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \frac{\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)[k(\alpha+1)+1]\Gamma(k+1+\mu)} r^k\right)$$

(3.6.4)

$$(|z| = r < 1; \mu > 0; i \in \{0, 1, \ldots, n\})$$

and

$$\left|D_z^{-\mu}(D^i f(z))\right| \leq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)[k(\alpha+1)+1]\Gamma(k+1+\mu)} r^k\right)$$

(3.6.5)

$$(|z| = r < 1; \mu > 0; i \in \{0, 1, \ldots, n\})$$

Each of the assertions (3.6.4) and (3.6.5) is sharp.

Proof. We observe that

$$f(z) \in \alpha - T_k(n, \lambda) \Leftrightarrow D^i f(z) \in \alpha - T_k(n-i, \lambda)$$

and that (cf. Equation (3.1.2))

$$D^i f(z) = z - \sum j^i a_j z^j \quad (i \in N_0).$$

In view of Theorem 3.2.1, we have

$$(k+1)^{n-i}(1+\lambda k)[k(\alpha+1)+1] \sum j^i a_j$$

$$\leq \sum (1-\lambda+\lambda j)[j(\alpha+1) - \alpha] j^n a_j \leq 1,$$
so that

\[ \sum j^i a_j \leq \frac{1}{(k+1)^{n-i} (1 + \lambda k)[k(\alpha + 1) + 1]} \tag{3.6.6} \]

Consider the function \( G(z) \) defined by

\[ G(z) = \Gamma(2 + \mu) z^{-\mu} D_z^{-\mu} (D_i f(z)) \]

\[ = z - \sum \frac{\Gamma(j + 1) \Gamma(2 + \mu)}{\Gamma(j + 1 + \mu)} j^i a_j z^j \]

\[ = z - \sum \phi(j) j^i a_j z^j, \]

where

\[ \phi(j) = \frac{\Gamma(j + 1) \Gamma(2 + \mu)}{\Gamma(j + 1 + \mu)} \quad (j \geq k + 1; k \in N; \mu > 0). \]

Since \( \phi(j) \) is a decreasing function of \( j \), we get

\[ 0 < \phi(j) \leq \phi(k + 1) = \frac{\Gamma(k + 2) \Gamma(2 + \mu)}{\Gamma(k + 2 + \mu)} \quad (j \geq k + 1; k \in N; \mu > 0). \tag{3.6.7} \]

Thus, by using (3.6.6) and (3.6.7), we see that

\[ |G(z)| \geq r - \phi(k + 1) r^{k+1} \sum j^i a_j \]

\[ \geq r - \frac{\Gamma(k + 2) \Gamma(2 + \mu)}{(k + 1)^{n-i} (1 + \lambda k)[k(\alpha + 1) + 1]\Gamma(k + 2 + \mu)} r^{k+1} \]

\[ (|z| = r < 1; \mu > 0; i \in \{0, 1, \ldots, n\}) \]

and

\[ |G(z)| \leq r + \phi(k + 1) r^{k+1} \sum j^i a_j \]

\[ \leq r + \frac{\Gamma(k + 2) \Gamma(2 + \mu)}{(k + 1)^{n-i} (1 + \lambda k)[k(\alpha + 1) + 1]\Gamma(k + 2 + \mu)} r^{k+1} \]

\[ (|z| = r < 1; \mu > 0; i \in \{0, 1, \ldots, n\}), \]
which prove the inequalities (3.6.4) and (3.6.5) of Theorem 3.6.1.

The equalities in (3.6.4) and (3.6.5) are attained for the function $f(z)$ given by

$$D^i f(z) = z - \frac{1}{(k + 1)^{n-i}(1 + \lambda k)[k(\alpha + 1) + 1]} z^{k+1} \quad (k \in N). \quad (3.6.8)$$

This completes the proof of theorem 3.6.1.

Setting $i = 0$ in Theorem 3.6.1, we obtain:

□

**Corollary 3.6.1:** Let the function $f$ defined by (3.1.4) be in the class $\alpha - T_k(n, \lambda)$.
Then

$$|D_z^{-\mu} f(z)| \geq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \frac{\Gamma(k + 2)\Gamma(2 + \mu)}{(k + 1)^{n}(1 + \lambda k)[k(\alpha + 1) + 1]\Gamma(k + 2 + \mu)} r^{k}\right) \quad (3.6.9)$$

$$(|z| = r < 1; \mu > 0)$$

and

$$|D_z^{-\mu} f(z)| \leq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{\Gamma(k + 2)\Gamma(2 + \mu)}{(k + 1)^{n}(1 + \lambda k)[k(\alpha + 1) + 1]\Gamma(k + 2 + \mu)} r^{k}\right) \quad (3.6.10)$$

$$(|z| = r < 1; \mu > 0).$$

The estimates in (3.6.9) and (3.6.10) are sharp for the function $f(z)$ given by (3.6.8) with $i = 0$. 
Theorem 3.6.2: Let the function \( f \) defined by (3.1.4) be in the class \( \alpha-T_k(n, \lambda) \).

Then

\[
\left| D_z^\mu (D^i f(z)) \right| \leq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left( 1 - \frac{\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^{\alpha+i}(1+\lambda k)[\alpha+1+i]k(k+2-\mu)} r^k \right) \quad (3.6.11)
\]

\[
(|z| = r < 1; 0 \leq \mu < 1; i \in \{0, 1, \ldots, n\})
\]

and

\[
\left| D_z^\mu (D^i f(z)) \right| \geq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left( 1 + \frac{\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^{\alpha+i}(1+\lambda k)[\alpha+1+i]k(k+2-\mu)} r^k \right) \quad (3.6.12)
\]

\[
(|z| = r < 1; 0 \leq \mu < 1; i \in \{0, 1, \ldots, n\}).
\]

Each of the assertions (3.6.11) and (3.6.12) is sharp.

Proof. Consider the function \( H(z) \) defined by

\[
H(z) = \Gamma(2-\mu)z^\mu D_z^\mu (D^i f(z)) = z - \sum \frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+1-\mu)} j^i a_j z^j
\]

\[
= z - \sum \psi(j) j^i a_j z^j,
\]

where

\[
\psi(j) = \frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+1-\mu)} \quad (j \geq k + 1; k \in N; 0 \leq \mu < 1). \quad (3.6.13)
\]

It is easily seen from (3.6.13) that

\[
0 < \psi(j) \leq \psi(k+1) = \frac{\Gamma(k+2)\Gamma(2-\mu)}{\Gamma(k+2-\mu)} \quad (j \geq k + 1; k \in N; 0 \leq \mu < 1). \quad (3.6.14)
\]
Consequently, with the aid of (3.6.6) and (3.6.14), by simple computation, we can arrive at the assertions (3.6.11) and (3.6.12).

Each of these estimates is sharp for the function \( f(z) \) given by (3.6.8).

Letting \( i = 0 \) in Theorem 3.6.2, we have:

Corollary 3.6.2: Let the function \( f \) defined by (3.1.4) be in the class \( \alpha - T_k(n, \lambda) \) then

\[
|D_2^\mu f(z)| \geq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left( 1 - \frac{\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^n(1+\lambda k)[k(\alpha+1)+1]\Gamma(k+2-\mu)} r^k \right) \quad (3.6.15)
\]

\[|D_2^\mu f(z)| \leq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left( 1 + \frac{\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^n(1+\lambda k)[k(\alpha+1)+1]\Gamma(k+2-\mu)} r^k \right) \quad (3.6.16)
\]

\(|z| = r < 1; 0 \leq \mu < 1\)

The estimates in (3.6.15) and (3.6.16) are sharp for the function \( f(z) \) given by (3.6.8) with \( i = 0 \).