Chapter 5

Mixed Convection on a Vertical Surface in a Porous Medium Through Integral Analysis

5.1 Introduction

The phenomenon of coupled heat and mass transfer in porous media has many important applications in engineering. To name just a few, this includes the migration of moisture in fibrous insulation, the spreading of chemical pollutants in saturated soil and extraction of geothermal energy. Comparatively less work exists on the transport in porous media due to the combined buoyancy effects of thermal and species diffusion. Bejan and Khair [1] treated the most fundamental case of buoyancy induced heat and mass transfer from a vertical plate embedded in fluid saturated porous medium. Scale analysis of boundary layer equations was introduced to identify the functional relations for Nusselt and Sherwood numbers in limiting cases. The problem was re-examined along vertical surfaces with constant wall temperature and concentration and constant wall flux conditions by Lai and Kulacki[2]. Flow injection at the wall was shown to increase the boundary layer thicknesses and hence decrease in transport rates occurs. Recently Nakayama and Hossein[3] presented integral analysis for natural convection flows along semi-infinite vertical surfaces with constant wall temperature and concentration.
Mixed convection flows with coupled buoyancy effects in porous media, however, have not been exploited fully. Thus the main aim of the present study is to examine the fundamental behavior of mixed convection temperature and concentration distributions along a vertical surface in a porous medium. An integral treatment of Von-karman type is applied to present closed form analytical expressions for important physical characteristics of the boundary layer along a vertical wall with constant temperature and concentration along with constant free stream velocity. Fourth degree polynomials are used for temperature and concentration distributions. Both aiding and opposing cases are considered for a wide range of parameters which are Grashof number Gr, buoyancy ratio N and the Lewis number Le. Gr is found to be controlling parameter from free to forced convection. It is observed that for fixed N and Le the rates of heat and mass transfer increase with increasing Grashof number for aiding case while the situation is reversed for opposing case.

5.2 Basic Equations

Consider the problem of mixed convection heat and mass transfer from a non-isothermal vertical plate immersed in fluid saturated porous medium with constant temperature and concentration with constant properties of fluid and porous medium as shown in Figure 1. The fluid and the porous medium are in local thermodynamic equilibrium everywhere. After invoking Boussinesq approximation the boundary layer equations for conservation of mass, momentum, energy and concentration of Darcian fluid are:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(5.1)

\[
u = -\frac{K}{\mu} \left( \frac{\partial p}{\partial x} \pm \rho g \right)
\]

(5.2)

\[
v = -\frac{K}{\mu} \left( \frac{\partial p}{\partial y} \right)
\]

(5.3)

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\]

(5.4)
\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) \\
\rho &= \rho_\infty [1 - \beta_t(T - T_\infty) - \beta_c(C - C_\infty)]
\end{align*}
\] (5.5) (5.6)

where \(u, v\) are Darcian velocities in \(x, y\) directions respectively. \(\nu, g\) and \(K\) are the kinematic viscosity, acceleration due to gravity and permeability of the porous medium respectively. \(\alpha, D\) are the thermal and concentration diffusivities of the porous medium where \(T\) and \(C\) are the temperature and concentration inside the boundary layers, \(\rho\) is the density of fluid. The suffix \(\infty\) indicates the condition at \(\infty\). If \(T_w > T_\infty\) and the flow is vertically upwards we have the aiding case and so required the upper sign in equation (5.2) whereas when the flow is vertically downward we have the opposing case and the lower sign is required. This situation is reversed if \(T_w < T_\infty\). \(\beta_t\) and \(\beta_c\) are the thermal and concentration expansion coefficients respectively. Eliminating the pressure term from equation (5.2) and (5.3), we get

\[
\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \mp \frac{K g \partial \rho}{\mu} \frac{\partial \rho}{\partial y}
\] (5.7)

Using the density variation relation (5.6), relation (5.7) reduces to

\[
\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \pm \frac{K g}{\nu} [\beta_t \frac{\partial T}{\partial y} + \beta_c \frac{\partial C}{\partial y}]
\] (5.8)

where

\[\beta_t = -\frac{1}{\rho} \frac{\partial p}{\partial T} \text{ and } \beta_c = -\frac{1}{\rho} \frac{\partial p}{\partial C}\]

Finally, we focus on the boundary-layer regime where the temperature and concentration gradients are steep in the vertical slender region situated near the wall. Thus invoking the boundary-layer approximation the momentum, energy and solute conservation equations become

\[
\frac{\partial u}{\partial y} = \pm \frac{K g}{\nu} [\beta_t \frac{\partial T}{\partial y} + \beta_c \frac{\partial C}{\partial y}]
\] (5.9)
5.3 Method of Solution

\[ \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial y^2} \right) \]  
(5.10)

\[ \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \left( \frac{\partial^2 C}{\partial y^2} \right) \]  
(5.11)

The boundary conditions of the problem are:
\[
\begin{align*}
  y = 0, & \quad T = T_w, C = C_w, v = 0 \\
  y \to \infty, & \quad T \to T_\infty, C \to C_\infty, u \to U_\infty
\end{align*}
\]  
(5.12)

Integrating equation (5.9) across the boundary layers, and invoking the boundary conditions (5.12) we get,

\[ u = U_\infty + \frac{Kg}{\nu} \left[ \beta_t(T - T_\infty) + \beta_c(C - C_\infty) \right] \]  
(5.13)

The governing equations (5.10),(5.11) and (5.13) with boundary conditions (5.12) are solved in closed form using the integral method of Von-Karman type.

5.3 Method of Solution

Integrating the equations (5.10) and (5.11) with respect to \( y \) from 0 to \( \infty \) and using equation (5.1), we get the following integral equations

\[ \frac{\partial}{\partial x} \int_0^\infty (u, \theta) \, dy = -\alpha \frac{\partial \theta}{\partial y} \bigg|_{y=0} \]  
(5.14)

\[ \frac{\partial}{\partial x} \int_0^\infty (u, \phi) \, dy = -D \frac{\partial \phi}{\partial y} \bigg|_{y=0} \]  
(5.15)

where \( \theta = \frac{T}{T_w - T_\infty}, \phi = \frac{C}{C_w - C_\infty} \).

The infinity is replaced by \( \delta_t \) and \( \delta_c \) the boundary layer thicknesses for temperature and concentration respectively. It is worthwhile to note that unlike the usual Karman-Pohlhausen integral procedure, in the present case we have freedom to choose only two profiles as the third profile follows immediately from equation (5.13). We treat, \( T \) and \( C \) as the primary variables. Let us assume the temperature and concentration profile as
where \( \eta = \frac{y}{L} \), \( \eta_1 = \frac{y}{D} \), \( \delta_t \) and \( \delta_c \) being the thermal and concentration boundary layer thicknesses to be determined with help of equations (5.14) and (5.15). The coefficients \( a_0, a_1, -a_4 \) and \( b_0, b_1, -b_4 \) are in general functions of \( x \) and are to be determined with help of boundary conditions (5.12) and smoothness conditions

\[
\frac{\partial \theta}{\partial y} \to 0, \quad \frac{\partial \phi}{\partial y} \to 0, \quad \frac{\partial^2 \theta}{\partial y^2} \to 0, \quad \frac{\partial^2 \phi}{\partial y^2} \to 0 \quad \text{as} \quad y \to \infty
\]

in addition to these, we use the following compatibility conditions

\[
\frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \text{at} \quad y = 0
\]

which are obtained by evaluating the equation (5.4) and (5.5) at \( y = 0 \). The temperature and concentration profiles thus take the final forms

\[
\theta = 1 - 2\eta + 2\eta^3 - \eta^4 \tag{5.18}
\]

\[
\phi = 1 - 2\eta_1 + 2\eta_1^3 - \eta_1^4 \tag{5.19}
\]

Using the equations (5.18), (5.19) and (5.12) in equation (5.14) and (5.15) and integrating we obtain a set of ordinary differential equations for unknown boundary layer thicknesses \( \delta_t \) and \( \delta_c \)

\[
\left( \frac{3}{10} + Gr \left( \frac{23}{126} + NH(\Delta) \right) \right) \frac{d\delta_t}{dx} = \frac{2\alpha}{U_\infty \delta_t} \tag{5.20}
\]

\[
\left( \frac{3}{10} + Gr \left( \Delta H(\Delta) + \frac{23}{126} N \right) \right) \frac{d\delta_c}{dx} = \frac{2\alpha}{\delta_c Le U_\infty} \tag{5.21}
\]

where \( \Delta = \frac{\delta_t}{\delta_c} \) is the ratio of boundary layer thicknesses, \( Gr = \frac{Kg_\infty \theta_\infty}{\nu U_\infty} \) is th Grashof number, \( Le = \alpha/D \), is Lewis number, \( N = \frac{\beta_c \phi \theta_\infty}{\beta_t \theta_\infty} \) is the buoyancy ratio. the solution of which is easily obtained as
\[ \delta_t = \delta_t^* \sqrt{\frac{\alpha_x}{U_\infty}} \]
\[ \delta_c = \delta_c^* \sqrt{\frac{\alpha_x}{U_\infty}} \]

where the unknowns \( \delta_t^* \) and \( \delta_c^* \) are given by

\[ \left( \frac{3}{10} + Gr \left( \frac{23}{126} + NH(\Delta) \right) \right) \delta_t^* = 4 \]  \hspace{1cm} (5.22)

\[ \left( \frac{3}{10} + Gr(\Delta H(\Delta) + \frac{23}{126} N) \right) \delta_c^* = \frac{4}{Le} \]  \hspace{1cm} (5.23)

where \( H(\Delta) = \frac{3}{10\Delta} - \frac{2}{15\Delta^2} + \frac{3}{140\Delta^4} - \frac{1}{180\Delta^6} \) for \( \Delta > 1 \)

\[ H(\Delta) = \frac{3}{10} - \frac{2\Delta}{15} + \frac{3\Delta^2}{140} - \frac{\Delta^4}{180} \]  \hspace{1cm} for \( \Delta < 1 \)

### 5.4 Results and discussions

The physical characteristics of importance are rates of heat and mass transfer from the wall to the medium.

The heat and mass transfer coefficients in terms of Nusselt and Sherwood numbers are

\[ Nu_x = \frac{2}{\delta_t^*} \sqrt{Pec_x} \]
\[ Sh_x = \frac{2}{\delta_c^*} \sqrt{Pec_x} \]

where \( Pec_x = \frac{U_\infty x}{\alpha} \) is the local peclet number.

The values of \( Nu/Pec_x^{1/2} \) and \( Sh/Pec_x^{1/2} \) are computed for various values of Grashof number, buoyancy ratio and Lewis number and are presented through tables and graphs. To check the accuracy of present approximate solution, the Nusselt and Sherwood numbers for aiding and opposing cases are compared with those obtained by similarity solution in table 1(a) and 1(b). It is found that the two results are in good agreement.
Aiding Case:

Figures 2 and 3 show how thermal and concentration boundary layers react to change in the Grashof number Gr, for various values of buoyancy ratio and Lewis number. It is observed that the rates of heat and mass transfer increase as Grashof number increases. It is also found that Nusselt and Sherwood numbers decrease as N increases for fixed Grashof number and fixed Le. For fixed N and Gr, the rate of heat transfer decreases with increasing Le while mass transfer coefficient increases.

Boundary layer thicknesses ratio verses Lewis number for Gr = 10 is plotted in Figure 4 and it is found that $\frac{Nu_x}{Sh_x}$ decreases with increasing N for $Le > 1$ and increases for $Le < 1$.

Opposing Case:

From Figures 5 and 6, it is obvious that the rates of heat and mass transfer decrease as Grashof number increases. These physical quantities are also found to decrease with increase of N while these properties are found to increase with Lewis number. Figure 7 shows the ratio of Nusselt and Sherwood numbers increase with N for $Le > 1$ and it decreases for $Le < 1$. 
Bibliography


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Fig. 1 The regime in which the phenomenon of mixed convection heat and mass transfer exist.
Fig. 2 Heat Transfer Coefficient as a Function of the Grashof number for Aiding case.
Fig. 3 Mass Transfer Coefficient as a Function of Grashof number for Aiding case.
Fig. 4 Ratio of Nusselt and Sherwood numbers as a Function of Lewis Number for Aiding Case.
Fig. 5 Heat Transfer Coefficient as a Function of Grashof Number for Opposing Case.
Fig. 6 Mass Transfer Coefficient as a Function of Grashof Number for Opposing case.
Fig. 7 Ratio of Nusselt and Sherwood numbers as a Function of Lewis number for Opposing Case.