CHAPTER 2

Fuzzy Convex Invariants

In abstract convexity theory, the classical convex invariants namely Helly number, Caratheodory number, Radon number and exchange number play a central role. Each of them is defined as a degree of independence tolerated by a convex structure. In this chapter, some basic relations between these invariants, such as inequalities of Levi, Sierksma, in a fuzzy convex structure are studied. Moreover, the behavior of these invariants under the formation of FCP, FCC images and fuzzy subspaces are also discussed.

Some of the results in this chapter were included in a paper published in the journal International Mathematical Forum, 6(20) (2011), 995–1004. [27]
Chapter 2. Fuzzy Convex Invariants

2.1 Basic definitions and properties

Definition 2.1.1 [47] Let $(X, C)$ be a fuzzy convexity space. A nonzero finite fuzzy subset $F$ of $X$ is Helly dependent (or $H$-dependent) if
\[ \bigwedge_{a \in F} \text{Co}(F \setminus a) \neq 0 \]
where $F \setminus a = F \wedge a'$. Otherwise, it is $H$-independent.

The Helly number of $X$ is the smallest ‘$n$’ such that each nonzero finite fuzzy subset $F$ of $X$, with cardinality of its support at least ‘$n + 1$’, is Helly dependent. It is denoted by $h(X)$ or by $h$.

Definition 2.1.2 [47] Let $(X, C)$ be a fuzzy convexity space. A nonzero finite fuzzy subset $F$ of $X$ is Radon dependent (or $R$-dependent) if there exists a Radon partition $\{F_1, F_2\}$ of $F$ (i.e., $F_1 \wedge F_2 = 0$; $F_1 \vee F_2 = F$) such that
\[ \text{Co}(F_1) \wedge \text{Co}(F_2) \neq 0. \]
Otherwise, it is $R$-independent.

The Radon number of $X$ is the smallest ‘$n$’ such that each nonzero finite fuzzy subset $F$ of $X$, with cardinality of its support at least ‘$n + 1$’, is Radon dependent. It is denoted by
r(X) or by r.

**Definition 2.1.3** [47] Let \((X, C)\) be a fuzzy convexity space. A nonzero finite fuzzy subset \(F\) of \(X\) is Caratheodory dependent (or \(C\)-dependent) provided

\[
Co(F) \leq \bigvee_{a_\alpha \in F} Co(F \setminus a_\alpha).
\]

Otherwise, it is \(C\)-independent.

The Caratheodory number of \(X\) is the smallest ‘n’ such that each nonzero finite fuzzy subset \(F\) of \(X\), with cardinality of its support at least ‘\(n + 1\)’, is Caratheodory dependent. It is denoted by \(c(X)\) or by \(c\).

**Definition 2.1.4** [47] Let \((X, C)\) be a fuzzy convexity space. A nonzero finite fuzzy subset \(F\) of \(X\) is exchange dependent (or \(E\)-dependent) if for each \(p_\alpha \in F\),

\[
Co(F \setminus p_\alpha) \leq \bigvee \left\{Co(F \setminus a_\beta) ; a_\beta \in F \setminus p_\alpha ; a \neq p\right\}.
\]

Otherwise, it is \(E\)-independent.

The exchange number of \(X\) is the smallest ‘n’ such that each nonzero finite fuzzy subset \(F\) of \(X\), with cardinality of its support at least ‘\(n + 1\)’, is exchange dependent. It is denoted by
e(X) or by e.

**Theorem 2.1.5** [47] For a nonzero finite fuzzy subset of a fuzzy convexity space, R-dependence implies H-dependence.

### 2.2 Relationships between fuzzy convex invariants

In this section, we establish some relationships between fuzzy convex invariants.

**Theorem 2.2.1** Let \((X, C)\) be a fuzzy convex structure and let \(n < \infty\). If each finite collection of fuzzy convex sets in \(X\) meeting \(n\) by \(n\) has a nonzero intersection then \(h \leq n\).

**Proof**

Let \(F\) be a nonzero finite fuzzy subset of \(X\) with support \(\{a_1, a_2, \ldots, a_m\}\); \(m > n\). Let \(F_i = F \setminus \{a_i\}\). Consider \(\{Co(F_i) ; i = 1, 2, \ldots m\}\) which is a finite collection of fuzzy convex sets meeting \(m - 1\) by \(m - 1\) where \(m - 1 \geq n\). Then by hypothesis,

\[
\bigwedge_{i=1}^{m} Co(F_i) \neq 0.
\]

So, by definition, \(F\) is \(H\)-dependent. Hence, \(h \leq n\).

\[\square\]

The next theorem provides an alternative description of the Caratheodory number \(c\).
Theorem 2.2.2 Let \((X, C)\) be a fuzzy convexity space and \(n < \infty\). Then \(c \leq n\) iff for each nonzero fuzzy subset \(A \subseteq X\) and \(p_\alpha \in Co(A)\), there is a fuzzy subset \(F\) of \(A\) with cardinality of its support at most \(n\) and having \(p_\alpha \in Co(F)\).

Proof
Suppose \(c \leq n\). Let \(p_\alpha \in Co(A)\); \(A \subseteq X\). Since \(X\) is domain finite,

\[
Co(A) = \bigvee \{Co(F); F \subseteq A, \#Supp(F) < \infty\} \text{ for } A \subseteq X.
\]

Among all possible fuzzy subsets \(F\), choose the one with the smallest cardinality of its support and containing \(p_\alpha\). Let it be \(F^*\).

If \(\#Supp(F^*) > n\) then \(\#Supp(F^*) > c\), since \(n \geq c\).

Then \(F^*\) is \(C\)-dependent. Therefore,

\[
Co(F^*) \leq \bigvee_{a_\beta \in F^*} Co(F^* \setminus a_\beta)
\]

Since \(p_\alpha \in Co(F^*)\), \(p_\alpha \in Co(F^* \setminus a_\beta)\) for some \(a_\beta \in F^*\) which is a contradiction to our choice of \(F^*\). So our assumption is wrong. Hence

\[
\#Supp(F^*) \leq n.
\]
Conversely, if possible, let $c > n$. Then, since $A$ is $C$-independent, there is a fuzzy subset $A \subseteq X$ with cardinality of its support greater than $n$ such that

$$Co(A) \not\subseteq \bigvee_{a_\beta \in A} \{Co(A \setminus a_\beta)\}.$$ 

Hence some point of $Co(A)$ is in the hull of no proper subset of $A$. This is a contradiction to the hypothesis. Therefore, our assumption is wrong. Hence

$$c \leq n.$$ 

The next theorem shows that Levi’s inequality is true in a fuzzy convexity space.

**Theorem 2.2.3** Let $(X, C)$ be a fuzzy convexity space with Helly number $h$ and Caratheodory number $c$. Then

$$h \leq r.$$ 

**Proof**

Let $F$ be a nonzero finite fuzzy subset of $X$ which is $R$-dependent. Then by Theorem 2.1.5, it is $H$-dependent. Hence, $h \leq r$. 

□
The next theorem proves Sierksma’s inequality in the fuzzy context.

**Theorem 2.2.4** Let \((X, C)\) be a fuzzy convexity space. Then

\[
e - 1 \leq c \leq \max\{h, e - 1\}
\]

where \(e, c\) and \(h\) are the exchange number, Caratheodory number and Helly number of \(X\) respectively.

**Proof**

Assume \(c < \infty\). Let \(F\) be a nonzero finite fuzzy subset of \(X\) with cardinality of its support greater than \(c + 1\). Let \(p_\alpha \in F\). Then \(\text{Supp}(F \setminus p_\alpha)\) has more than \(c\) elements. Hence \(F \setminus p_\alpha\) is \(C\)-dependent. Then by definition,

\[
\text{Co}(F \setminus p_\alpha) \leq \lor \{\text{Co}((F \setminus p_\alpha) \setminus a_\beta; a_\beta \in F \setminus p_\alpha; a \neq p)\} \quad (2.1)
\]

But, \((F \setminus p_\alpha) \setminus a_\beta \leq F \setminus a_\beta\)

Hence, \(\text{Co}((F \setminus p_\alpha) \setminus a_\beta) \leq \text{Co}(F \setminus a_\beta)\)

Therefore, (2.1) implies,

\[
\text{Co}(F \setminus p_\alpha) \leq \lor \{\text{Co}((F \setminus a_\beta); a_\beta \in F \setminus p_\alpha; a \neq p)\}
\]
Then by definition, $F$ is $E$-dependent. Hence, $e \leq c + 1$. Or,

$$e - 1 \leq c. \tag{2.2}$$

Now, to prove $c \leq \max\{h, e - 1\}$, assume $h$ and $e$ to be finite. Let $n = \max\{h, e - 1\}$. Let $F$ be a nonzero finite fuzzy subset of $X$ with cardinality of its support greater than $n$. Since $n \geq h$, $F$ is $H$-dependent. So by definition,

$$\bigwedge_{a_\alpha \in F} Co(F \setminus a_\alpha) \neq 0$$

Let $p_\beta \in \bigwedge_{a_\alpha \in F} Co(F \setminus a_\alpha)$. Consider the fuzzy bag $F \vee p_\beta$. Since

$$\# Supp(F \vee p_\beta) > n + 1 \geq e,$$

it is $E$-dependent. Then by definition,

$$Co[(F \vee p_\beta) \setminus p_\beta] \leq \lor \{Co(F \vee p_\beta \setminus a_\alpha); a_\alpha \in F \vee p_\beta; a \neq p\} \tag{2.3}$$

But $(F \vee p_\beta) \setminus p_\beta = F$. Moreover,

$$p_\beta \in \bigwedge_{a_\alpha \in F} Co(F \setminus a_\alpha) \Rightarrow p_\beta \in Co(F \setminus a_\alpha); \forall a_\alpha \in F.$$
Hence (2.3) gives

\[ \text{Co}(F) \leq \bigvee_{a \in F} \text{Co}(F\backslash a) \].

Then by definition, \( F \) is \( C \)-dependent. Hence

\[ \text{Supp} \ F \geq c \Rightarrow n \geq c \]

i.e., \( c \leq \max\{h, e - 1\} \) \hfill (2.4)

Combining (2.2) and (2.4),

\[ e - 1 \leq c \leq \max\{h, e - 1\} \].

\[ \square \]

2.3 Fuzzy convex invariants under FCP, FCC images and subspaces

In this section, the behavior of fuzzy convex invariants under the formation of FCP, FCC images and subspaces are discussed.

**Theorem 2.3.1** [43] Let \((X, C_1), (Y, C_2)\) be fuzzy convexity spaces. Then \( f : X \to Y \) is an FCP function \( \Leftrightarrow f(\text{Co}(S)) \subseteq \text{Co}(f(S)) \) for every fuzzy subset \( S \) of \( X \).
Theorem 2.3.2 [43] Let \((X, C_1), (Y, C_2)\) be fuzzy convexity spaces. Then \(f : X \rightarrow Y\) is an FCC function \(\iff\) \(\text{Co}(f(S)) \subseteq f(\text{Co}(S))\) for every fuzzy subset \(S\) of \(X\).

Theorem 2.3.3 Let \((X, C_1), (Y, C_2)\) be fuzzy convexity spaces. Then for a surjective FCP function \(f : X \rightarrow Y\), the following are true.

\[
i) \ h(X) \geq h(Y) \quad \text{ii) } \ r(X) \geq r(Y).
\]

Proof

i) Let \(F\) be a fuzzy bag on \(Y \times I\) with support \(\{a_1, a_2...a_n\}\). Since \(f\) is surjective, for each \(i\), let \(b_i \in X\) such that \(f(b_i) = a_i\). Let \(G\) be the fuzzy bag on \(X \times I\) with support \(\{b_1, b_2...b_n\}\). Clearly \(f(G) = F\) and \(f(G \setminus b_{i_a}) \subseteq F \setminus a_{i_b}\).

Since \(f\) is FCP, \(f(\text{Co}(G)) \subseteq \text{Co}(f(G))\). Therefore,

\[
f(\text{Co}(G \setminus b_{i_a})) \subseteq \text{Co}(f(G \setminus b_{i_a})) \subseteq \text{Co}(F \setminus a_{i_b}) \quad (2.5)
\]

Suppose \(G\) is \(H\)-dependent. Then by definition,

\[
\bigwedge_{b_{i_a} \in G} \text{Co}(G \setminus b_{i_a}) \neq 0.
\]
Hence from (2.5), we have

\[ \bigwedge_{a_{i\beta} \in F} Co\left( F \setminus a_{i\beta} \right) \neq 0. \]

Therefore, \( F \) is \( H \)-dependent.

i.e., \( F \) is \( H \)-dependent provided \( G \) is \( H \)-dependent where \( F \subseteq Y \) and \( G \subseteq X \).

Therefore,

\[ h(X) \geq h(Y). \]

ii) Take \( F \) and \( G \) as above and suppose \( G \) is \( R \)-dependent. Then there is a Radon partition \( \{G_1, G_2\} \) of \( G \) such that

\[ G_1 \lor G_2 = G, \quad G_1 \land G_2 = 0 \quad \text{and} \quad Co(G_1) \land Co(G_2) \neq 0. \]

Let \( p_\alpha \in Co(G_1) \land Co(G_2) \). Then \( p_\alpha \in Co(G_1) \) and \( p_\alpha \in Co(G_2) \).

Consider

\[ F = f(G) = f(G_1 \lor G_2) = f(G_1) \lor f(G_2) = F_1 \lor F_2 \quad \text{where} \quad F_1 = f(G_1) \subseteq F; \quad F_2 = f(G_2) \subseteq F. \]

Also,

\[ F_1 \land F_2 = f(G_1) \land f(G_2) = f(G_1 \land G_2) = f(\emptyset) = 0. \]
Hence, \( \{F_1, F_2\} \) is a partition of \( F \).

Next to show that \( \text{Co}(F_1) \wedge \text{Co}(F_2) \neq 0 \). As \( p_\alpha \in \text{Co}(G_1) \),

\[
f(p_\alpha) \in f(\text{Co}(G_1)) \subseteq \text{Co}(f(G_1)) = \text{Co}(F_1).
\]

Similarly, \( f(p_\alpha) \in \text{Co}(F_2) \Rightarrow f(p_\alpha) \in \text{Co}(F_1) \wedge \text{Co}(F_2) \). So,

\[
\text{Co}(F_1) \wedge \text{Co}(F_2) \neq 0.
\]

Hence, \( \{F_1, F_2\} \) is a Radon partition of \( F \). i.e., \( F \) is \( R \)-dependent provided \( G \) is, where \( F \subseteq Y \) and \( G \subseteq X \). Therefore,

\[
r(X) \geq r(Y).
\]

\[\square\]

**Theorem 2.3.4** Let \((X, C_1), (Y, C_2)\) be two fuzzy convexity spaces. Then for a surjective \( FCP \) and \( FCC \) function \( f : X \rightarrow Y \) the following are true.

\[i) \ c(X) \geq c(Y) \quad ii) \ e(X) \geq e(Y).\]

**Proof**

i) Let \( F \) be a fuzzy bag on \( Y \times I \) with support \( \{a_1, a_2...a_n\} \). Since \( f \) is surjective, for each i, let \( b_i \in X \) such that \( f(b_i) = a_i \). Let
\( G \) be the fuzzy bag on \( X \times I \) with support \( \{b_1, b_2...b_n\} \). Clearly, \( f(G) = F \). Suppose \( G \) is \( C \)-dependent. Then by definition,

\[
\text{Co}(G) \leq \bigvee_{i=1}^{n} \text{Co}(G \setminus b_{i_\alpha}); b_{i_\alpha} \in G.
\]

\( f \) is FCP \( \Rightarrow f(\text{Co}(G)) \subseteq \text{Co}(f(G)) \)

\( f \) is FCC \( \Rightarrow \text{Co}(f(G)) \subseteq f(\text{Co}(G)) \)

Combining, we get

\[
f(\text{Co}(G)) = \text{Co}(f(G)) = \text{Co}(F).
\]

Similarly,

\[
f(\text{Co}(G \setminus b_{i_\alpha})) = \text{Co}(f(G \setminus b_{i_\alpha}))
\]

\[
= \text{Co}(f(G) \setminus f(b_{i_\alpha}))
\]

\[
= \text{Co}(F \setminus a_{i_\beta}).
\]

Consider \( \text{Co}(F) = f(\text{Co}(G)) \)

\[
\leq f\left(\bigvee_{i=1}^{n} \text{Co}(G \setminus b_{i_\alpha})\right), \text{ since } G \text{ is } C-\text{dependent}
\]

\[
= \bigvee_{i=1}^{n} f(\text{Co}(G \setminus b_{i_\alpha}))
\]

\[
= \bigvee_{i=1}^{n} \left(\text{Co}(F \setminus a_{i_\beta})\right)
\]
i.e., \( \text{Co}(F) \leq \bigvee_{i=1}^{n} (\text{Co}(F \setminus a_{i\beta}); a_{i\beta} \in F) \).

Then by definition, \( F \) is \( C \)-dependent provided \( G \) is, where \( F \subseteq Y \) and \( G \subseteq X \). Hence,

\[
c(X) \geq c(Y).
\]

ii) Take \( F \) and \( G \) as above and suppose \( G \) is \( E \)-dependent. Then by definition,

\[
\text{Co}(G \setminus b_{i\alpha}) \leq \bigvee_{j=1}^{n} (\text{Co}(G \setminus b_{j\gamma}); b_{j\gamma} \in G; i \neq j).
\]

Since \( f \) is \( FCP \) and \( FCC \) we get

\[
f(\text{Co}[G \setminus b_{i\alpha}]) = \text{Co}(f[G \setminus b_{i\alpha}]) = \text{Co}(F \setminus a_{i\beta}). \quad (2.6)
\]

Now consider

\[
f(\text{Co}[G \setminus b_{i\alpha}]) \leq f \left( \bigvee_{j=1}^{n} (\text{Co}[G \setminus b_{j\gamma}]; b_{j\gamma} \in G; i \neq j) \right)
\]

\[
= \bigvee_{j=1}^{n} f \left( \text{Co}[G \setminus b_{j\gamma}]; b_{j\gamma} \in G; i \neq j \right)
\]

\[
= \bigvee_{j=1}^{n} \text{Co}(f[G \setminus b_{j\gamma}]; b_{j\gamma} \in G; i \neq j)
\]

\[
= \bigvee_{j=1}^{n} (\text{Co}[F \setminus a_{j\delta}]; a_{j\delta} \in F; i \neq j)
\]
Hence from (2.6) we get,

$$Co \left( F \setminus a_{is} \right) \leq \bigvee_{j=1}^{n} \left( Co \left[ F \setminus a_{js} \right]; a_{js} \in F; i \neq j \right).$$

Then by definition, $F$ is $E$-dependent provided $G$ is $E$-dependent where $F \subseteq Y$ and $G \subseteq X$. Hence,

$$e \left( X \right) \geq e \left( Y \right).$$

$\square$

**Theorem 2.3.5** Let $(X, C)$ be a fuzzy convexity space and $Y$ be a fuzzy subspace of $X$. Then

i) $c \left( X \right) \geq c \left( Y \right)$ \hspace{0.5cm} ii) $e \left( X \right) \geq e \left( Y \right)$.

**Proof**

i) Let $F$ be a fuzzy subset of $Y$ where $Y$ is a fuzzy subspace of $X$. Then $F$ is a fuzzy subset of $X$ also. Suppose $F$ is $C$-dependent in $X$. Then,

$$Co \left( F \right) \leq \bigvee_{a_{\alpha} \in F} Co \left( F \setminus a_{\alpha} \right).$$
Consider

\[ \text{Co}(F) \cap X \leq \left[ \bigvee_{a_{\alpha} \in F} \text{Co}(F \setminus a_{\alpha}) \right] \cap X \]

\[ = \bigvee_{a_{\alpha} \in F} \left[ \text{Co}(F \setminus a_{\alpha}) \cap X \right] \]

\( \text{i.e., } \text{Co}(F) \cap X \leq \bigvee_{a_{\alpha} \in F} \left( \text{Co}(F \setminus a_{\alpha}) \cap X \right) \)

where \( F \) is a fuzzy subset of \( Y \).

Then by definition,

\[ \text{Co}_{Y}(F) \leq \bigvee_{a_{\alpha} \in F} \text{Co}_{Y}(F \setminus a_{\alpha}) \]

i.e., \( F \) is \( C \)-dependent in \( Y \) provided it is \( C \)-dependent in \( X \).

Therefore,

\[ c(X) \geq c(Y). \]

ii) Let \( F \) be a fuzzy subset of the fuzzy subspace \( Y \) of \( X \). Then it is a fuzzy subset of \( X \) also. Suppose \( F \) is \( E \)-dependent in \( X \).

Then by definition, for each \( p_{\alpha} \in F \),

\[ \text{Co}(F \setminus p_{\alpha}) \leq \bigvee \{ \text{Co}(F \setminus a_{\beta}); a_{\beta} \in F; a \neq p \}. \]
Consider

\[
Co (F \setminus p_\alpha) \cap X \leq \vee \{Co (F \setminus a_\beta); a_\beta \in F; a \neq p\} \cap X
= \vee \{Co (F \setminus a_\beta) \cap X; a_\beta \in F; a \neq p\}
\]

i.e., \(Co (F \setminus p_\alpha) \cap X \leq \vee \{Co (F \setminus a_\beta) \cap X; a_\beta \in F; a \neq p\}\)
for fuzzy subsets \(F\) of \(Y\).

i.e., \(Co_Y (F \setminus p_\alpha) \leq \vee \{Co_Y (F \setminus a_\beta); a_\beta \in F; a \neq p\}\). Hence,

\(F\) is \(E\)-dependent in \(Y\) provided it is \(E\)-dependent in \(X\). So,

\[e (X) \geq e (Y) .\]

\[\square\]
H-dependent in $X$. Then,

$$\wedge_{a_\alpha \in F} Co(F \setminus a_\alpha) \neq 0.$$  

Consider

$$Co_Y (F \setminus a_\alpha) = Co(F \setminus a_\alpha) \cap X \text{ where } F \setminus a_\alpha \text{ is a fuzzy subset of } Y.$$  

But,

$$F \setminus a_\alpha \leq Y \Rightarrow Co(F \setminus a_\alpha) \leq Co(Y) = Y, \text{ since } Y \text{ is convex.}$$

i.e., $Co(F \setminus a_\alpha) \leq Y \subseteq X$. Hence,

$$Co(F \setminus a_\alpha) \cap X = Co(F \setminus a_\alpha)$$

i.e., $Co_Y(F \setminus a_\alpha) = Co(F \setminus a_\alpha)$

Therefore,

$$\wedge_{a_\alpha \in F} Co_Y (F \setminus a_\alpha) = \wedge_{a_\alpha \in F} Co(F \setminus a_\alpha) \neq 0.$$  

So, $F$ is $H$- dependent in $Y$ provided it is $H$-dependent in $X$.

Therefore,

$$h(X) \geq h(Y).$$
ii) Let $F$ be a fuzzy subset of $Y$ where $Y$, a fuzzy convex subspace of $X$. Then $F$ is a fuzzy subset of $X$. Suppose $F$ is $R$-dependent in $X$. Then by definition, there exists a Radon partition $\{F_1, F_2\}$ of $F$ such that

$$F_1 \lor F_2 = F; F_1 \land F_2 = 0 \text{ and } Co(F_1) \land Co(F_2) \neq 0.$$ 

Consider

$$Co_Y(F_1) \land Co_Y(F_2) = (Co(F_1) \cap X) \land (Co(F_2) \cap X)$$

where $F_1, F_2$ are fuzzy subsets of $Y$. Since $F_1, F_2$ are fuzzy subsets of $Y$, $Co(F_1) \leq Co(Y) = Y$ and $Co(F_2) \leq Y$ where $Y \subseteq X$. Hence

$$Co(F_1) \cap X = Co(F_1) \text{ and } Co(F_2) \cap X = Co(F_2).$$

Thus we get,

$$Co_Y(F_1) \land Co_Y(F_2) = Co(F_1) \land Co(F_2) \neq 0.$$ 

Hence, $F$ is $R$-dependent in $Y$ provided it is $R$-dependent in $X$. Therefore,

$$r(X) \geq r(Y).$$