Chapter II

-: Some New Selection Principles :-

In this chapter, we define some new selection principles in connection with the star operator and study some of their properties. The relation of these selection principles with other existing selection principles are also studied in this chapter.

This chapter is divided into the following sections
1. Selection Principles $\mathcal{U}_1(A, B)$ and $\mathcal{U}_{fin}(A, B)$
2. Selection Principles $\mathcal{S} \mathcal{S}^*_{\mathcal{C},1}(A, B)$ and $\mathcal{S} \mathcal{S}^*_{\mathcal{C},fin}(A, B)$
2.1 Selection Principles \( *U_1(A,B) \) and \( *U_{\text{fin}}(A,B) \)

Classical selection principles, based on the diagonalization arguments, have a long history going back to the works by Borel (1919), Menger (1924), Hurewicz (1925), Rothberger (1938), and others. In last two decades, after the paper [35] of Scheepers, which was the beginning of a systematic study in this field, selection principles took attention of a big number of mathematicians. Motivated by the recent works of Kočinac [24, 25] who initiated investigation of star selection principles, we introduce and study some new types of star-selection principles.

Let \( A \) and \( B \) be collections of families of subsets of a set \( X \), i.e. \( A, B \subseteq \mathcal{P}(\mathcal{P}(X)) \).

**Definition 2.1.1.**

\( *U_1(A,B) \) denotes the following selection principle:

For each sequence \( \{U_n : n \in \omega\} \) of elements of \( A \), there exists a sequence \( \{U_n : n \in \omega\} \) such that for each \( n \in \omega \), \( U_n \in U_n \) and \( \{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in B \).

**Definition 2.1.2.**

\( *U_{\text{fin}}(A,B) \) denotes the following selection principle:

For each sequence \( \{U_n : n \in \omega\} \) of elements of \( A \), there exists a sequence \( \{V_n : n \in \omega\} \) such that for each \( n \in \omega \), \( V_n \) is a finite subset of \( U_n \) and \( \{St(\bigcup_{i \in \omega} (\bigcup V_i), U_n) : n \in \omega\} \in B \).

**Proposition 2.1.3.** \( *U_1(A,B) \Rightarrow *U_{\text{fin}}(A,B) \).

**Proof.** The proposition follows directly from the definition 2.1.1. and definition 2.1.2. \( \square \)
**Proposition 2.1.4.** If $\mathcal{A}$ and $\mathcal{B}$ are two collections of families of subsets of an infinite set $X$ such that $\mathcal{A} \subseteq \mathcal{B}$, then

(i) $\star \mathcal{U}_1(\mathcal{B}, \mathcal{B}) \Rightarrow \star \mathcal{U}_1(\mathcal{A}, \mathcal{B})$

(ii) $\star \mathcal{U}_1(\mathcal{A}, \mathcal{A}) \Rightarrow \star \mathcal{U}_1(\mathcal{A}, \mathcal{B})$

(iii) $\star \mathcal{U}_1(\mathcal{B}, \mathcal{A}) \Rightarrow \star \mathcal{U}_1(\mathcal{A}, \mathcal{A})$

(iv) $\star \mathcal{U}_1(\mathcal{B}, \mathcal{A}) \Rightarrow \star \mathcal{U}_1(\mathcal{B}, \mathcal{B})$

**Proof.** (i) Let $\{U_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. But $\mathcal{A} \subseteq \mathcal{B}$. Therefore $\{U_n : n \in \omega\}$ is a sequence of elements of $\mathcal{B}$. Since $\star \mathcal{U}_1(\mathcal{B}, \mathcal{B})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in U_n$ for each $n \in \omega$ and $\{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in \mathcal{B}$. Therefore, $\star \mathcal{U}_1(\mathcal{A}, \mathcal{B})$ holds.

(ii) Let $\{U_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. Since $\star \mathcal{U}_1(\mathcal{A}, \mathcal{A})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in U_n$ for each $n \in \omega$ and $\{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in \mathcal{A}$. But $\mathcal{A} \subseteq \mathcal{B}$. Thus, $\{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in \mathcal{B}$. Therefore, $\star \mathcal{U}_1(\mathcal{A}, \mathcal{B})$ holds.

(iii) Let $\{U_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. But $\mathcal{A} \subseteq \mathcal{B}$. Therefore, $\{U_n : n \in \omega\}$ is a sequence of elements of $\mathcal{B}$. Since $\star \mathcal{U}_1(\mathcal{B}, \mathcal{A})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in U_n$ for each $n \in \omega$ and $\{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in \mathcal{A}$. But $\mathcal{A} \subseteq \mathcal{B}$. Thus, $\{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in \mathcal{B}$. Therefore, $\star \mathcal{U}_1(\mathcal{A}, \mathcal{A})$ holds.

(iv) Let $\{U_n : n \in \omega\}$ be a sequence of elements of $\mathcal{B}$. Since $\star \mathcal{U}_1(\mathcal{B}, \mathcal{A})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in U_n$ for each $n \in \omega$ and $\{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in \mathcal{A}$. But $\mathcal{A} \subseteq \mathcal{B}$. Thus, $\{St(\bigcup_{i \in \omega} U_i, U_n) : n \in \omega\} \in \mathcal{B}$. Therefore, $\star \mathcal{U}_1(\mathcal{B}, \mathcal{B})$ holds.

So, we conclude that the selection principle $\star \mathcal{U}_1(\mathcal{A}, \mathcal{B})$ is monotonic in the second collection and is anti-monotonic in the first collection.

**Proposition 2.1.5.** $\star \mathcal{U}_{fin}(\mathcal{A}, \mathcal{B})$ is monotonic in the second collection and anti-monotonic in the first collection.
**Proof.** The proof of this proposition is similar to the proof of Proposition 2.1.4, so omitted.

The monotonicity of the selection principle \( \star U_1(\mathcal{A}, \mathcal{B}) \) and \( \star U_{\text{fin}}(\mathcal{A}, \mathcal{B}) \) are shown in the figure below:

\[
\begin{align*}
\star U_1(\mathcal{A}, \mathcal{A}) & \Rightarrow \star U_1(\mathcal{A}, \mathcal{B}) & \star U_{\text{fin}}(\mathcal{A}, \mathcal{A}) & \Rightarrow \star U_{\text{fin}}(\mathcal{A}, \mathcal{B}) \\
\uparrow & \quad & \uparrow & \\
\star U_1(\mathcal{B}, \mathcal{A}) & \Rightarrow \star U_1(\mathcal{B}, \mathcal{B}) & \star U_{\text{fin}}(\mathcal{B}, \mathcal{A}) & \Rightarrow \star U_{\text{fin}}(\mathcal{B}, \mathcal{B})
\end{align*}
\]

\( A \subseteq B \)

Figure 1: Monotonicity of \( \star U_1(\mathcal{A}, \mathcal{B}) \) and \( \star U_{\text{fin}}(\mathcal{A}, \mathcal{B}) \).

We emphasize on the cases where \( \mathcal{A} \) and \( \mathcal{B} \) are the classes of topologically significant open covers of a space \( X \) (i.e. \( \mathcal{O} \) - the collection of all open covers of \( X \), \( \Lambda \) - the collection of all large covers of \( X \), \( \Omega \) - the collection of all \( \omega \)-covers of \( X \), \( \Gamma \) - the collection of all \( \gamma \)-covers of \( X \), \( \mathcal{O}_{gp} \) - the collection of all groupable open-covers of \( X \) etc.).

If the covers are considered to be non-trivial then we have, \( \Gamma \subseteq \Omega \subseteq \Lambda \subseteq \mathcal{O} \).

Under such condition, we have the following relation diagram:

![Relation Chart](attachment:relation_chart.png)

Figure 2: Relation Chart
**Proposition 2.1.6.** Every star-Lindelöf space has the property \( \ast \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \).

**Proof.** Let \( X \) be a star-Lindelöf space and \( \{ \mathcal{U}_n : n \in \omega \} \) be a sequence of open covers of \( X \). Since \( \mathcal{U}_0 \) is an open cover of \( X \) and \( X \) is star-Lindelöf, there exists a countable set, \( \{ x_0, x_1, x_2, ..., x_n, ... \} \subset X \) such that \( St(\{ x_0, x_1, x_2, ..., x_n, ... \}, \mathcal{U}_0) = X \).

For each \( n \in \omega \), we select \( U_n \in \mathcal{U}_n \) such that \( x_n \in U_n \). Clearly \( \{ x_0, x_1, x_2, ..., x_n, ... \} \subset \bigcup_{n \in \omega} U_n \). Therefore \( St(\bigcup_{n \in \omega} U_n, \mathcal{U}_0) = X \). Thus \( \{ St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) : n \in \omega \} \) is an open cover of \( X \), i.e. \( \ast \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \) holds for \( X \). Hence the theorem. \( \square \)

**Corollary 2.1.7.** Compact spaces, star-compact spaces and Lindelöf spaces have the property \( \ast \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \).

**Example 2.1.8.** \((AC_\omega)\) The converse of Corollary 2.1.7 is not necessarily true, i.e. there exists a space which has the property \( \ast \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \) but is not star-Lindelöf.

\( \mathbb{R}^+ = [0, \infty) \). We consider the space \( X = \mathbb{R}^+ \setminus (\mathbb{R}^+ \cap \mathbb{Q}) \). Let \( A = [0, 1] \setminus ([0, 1] \cap \mathbb{Q}) \). For each \( x \in A \), \( A_x = \{ (n + x) : n \in \omega \} \cup A \). Set \( Y = \{ A_x : x \in A \} \), So, \( \cup Y = X \). We define \( \tau(X) = \{ \bigcup B : B \in P(Y) \} \). \( \tau(X) \) is a topology on \( X \).

Now, let \( \{ \mathcal{U}_n : n \in \omega \} \) be a sequence of open covers of \( X \). Therefore, \( \mathcal{U}_0 \) is an open cover of \( X \). By the construction of the space, every open set other than \( \emptyset \) contains \( A \). We select \( U_0 \in \mathcal{U}_0 \) such that \( A \subset U_0 \) and for \( i \in \omega \setminus \{ 0 \} \), select any \( U_i \in \mathcal{U}_i \).

Since \( St(U_0, \mathcal{U}_0) = X \), \( St(\bigcup_{i \in \omega} U_i, \mathcal{U}_0) = X \), so that the set \( \{ St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) : n \in \omega \} \) is an open cover of \( X \). Hence, \( X \) has the property \( \ast \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \).

On the other hand, \( \mathcal{U} = \{ A_x : x \in A \} \) is an uncountable open cover of \( X \). For each \( x, y \in A \), \( x \neq y \), \( A_x \cap A_y = A \) and \( \bigcup_{x \in A} A_x = X \), but \( \bigcup_{i \in \omega} A_{x_i} \not\subset X \) for any countable set \( \{ x_i \}_{i \in \omega} \subset A \).

Let \( F = \{ y_i \}_{i \in \omega} \subset X \). For each \( y_i \in X \), there exists a \( x_i \in A \) such that \( y_i \in A_{x_i} \). Therefore, \( St(F, \mathcal{U}) = \bigcup_{i \in \omega} A_{x_i} \neq X \). We find \( X \), not star-Lindelöf even though it has the property \( \ast \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \).
Also the space is star-compact. But it is not a compact space. It is not even a Lindelöf space.

We have the following diagram of implications:

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compact → Lindelöf
↓      ↓
Star-compact → Star-Lindelöf
↓                  ↓
*U_1(O,O)
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Figure 3: Relation Chart

**Proposition 2.1.9.** \( S_1^*(\mathcal{A}, \mathcal{O}) \Rightarrow *U_1(\mathcal{A}, \mathcal{O}) \).

**Proof.** Let \( \{U_n : n \in \omega\} \) be a sequence of elements of \( \mathcal{A} \). Since \( S_1^*(\mathcal{A}, \mathcal{O}) \) holds, there exists a sequence \( \{U_n : n \in \omega\} \) such that for each \( n \in \omega \), \( U_n \in \mathcal{U}_n \) and \( \{St(U_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{O} \). Hence, \( \{St(U_n, \mathcal{U}_n) : n \in \omega\} \) is an open cover for \( X \).

We have, for each \( n \in \omega \), \( St(U_n, \mathcal{U}_n) \subseteq St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) \). Therefore, \( \{St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) : n \in \omega\} \) is also an open cover for \( X \). Thus \( *U_1(\mathcal{A}, \mathcal{O}) \) holds. \qed

But \( *U_1(\mathcal{O}, \mathcal{O}) \not\Rightarrow S_1^*(\mathcal{O}, \mathcal{O}) \) in general. This follows from the example given below.

**Example 2.1.10.** (\( AC_\omega \)) Let \( X = (0, 3) \subset \mathbb{R} \). We consider the topology \( \tau(X) = \{(x, y] : x, y \in [0, 3) \text{ and } x < y\} \cup \{\emptyset, X\} \), the upper limit topology on \( X \) induced from the upper limit topology of \( \mathbb{R} \).
We construct a sequence of open covers of $X$ as follows:

$$U_0 = \{(0,1], (1,2], (2,3]\},$$

$$U_1 = \left\{\left(0, \frac{1}{2}\right], \left(\frac{1}{2}, \frac{2}{2}\right], \left(\frac{2}{2}, \frac{3}{2}\right], \left(\frac{3}{2}, \frac{4}{2}\right], \left(\frac{4}{2}, \frac{5}{2}\right], \left(\frac{5}{2}, \frac{6}{2}\right]\right\},$$

$$U_2 = \left\{\left(0, \frac{1}{2^2}\right], \left(\frac{1}{2^2}, \frac{2}{2^2}\right], \left(\frac{2}{2^2}, \frac{3}{2^2}\right], \cdots, \left(\frac{11}{2^2}, \frac{12}{2^2}\right]\right\},$$

$$\cdots$$

$$U_n = \left\{\left(0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \left(\frac{2}{2^n}, \frac{3}{2^n}\right], \cdots, \left(\frac{3.2^n - 1}{2^n}, \frac{3.2^n}{2^n}\right]\right\},$$

$$\cdots$$

$$\cdots$$

For each $n \in \omega$, length of each interval contained in $U_n$ is $\frac{1}{2^n}$. Also, for each $n \in \omega$, $U_n$ is a pairwise disjoint collection of open sets. For any, $U_n \in U_n$, we have $St(U_n, U_n) = U_n$.

So, length of $St(U_n, U_n) = \frac{1}{2^n}$, for each $n \in \omega$. If $St(U_n, U_n)$ covers different portions of $X$ for each $n \in \omega$, it will cover a length of $X$. The maximum length of the subset of $X$ covered by $\{St(U_n, U_n) : n \in \omega\}$ is

$$\sum_{n \in \omega} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$

$$= \left(1 - \frac{1}{2}\right)^{-1}$$

$$= 2.$$
But length of $X$ is 3. Hence, $\{St(u_n, U_n) : n \in \omega\}$ can not be a cover of $X$. So, it is not possible to find a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in U_n$ and $\{St(U_n, U_n) : n \in \omega\}$ is an open cover for $X$. This implies that $X$ does not have the property $S_1^*(\mathcal{O}, \mathcal{O})$.

We have $\mathbb{R}$ with the upper limit topology is hereditarily Lindelöf, hence $X$ is a Lindelöf space. Thus, by Corollary 2.1.7, $X$ has the property $^*U_1(\mathcal{O}, \mathcal{O})$.

**Proposition 2.1.11.** $U_{fin}'(\mathcal{A}, \mathcal{O}) \Rightarrow ^*U_{fin}(\mathcal{A}, \mathcal{O})$.

**Proof.** The proof is similar to that of Proposition 2.1.9, so omitted \qed

In view of the above results, we have the following relation diagram:

![Relation Chart](image)

**Theorem 2.1.12.** If a space $X$ is compact, then it has the property $^*U_{fin}(\mathcal{O}, \mathcal{O})$.

**Proof.** Let $\{U_n : n \in \omega\}$ be a sequence of open covers for $X$. Since $X$ is compact, there exists $A_n \in [U_n]^{<\omega}$ for each $n \in \omega$, such that $A_n$ is a cover for $X$. Let $x \in X$ be an arbitrary point. For each $n \in \omega$, there exists $A_{n_x} \subseteq A_n \subseteq U_n$ such that $x \in A_{n_x} \subseteq U_n$. So, $x \in A_{n_x} \subseteq \bigcup A_n \Rightarrow A_{n_x} \cap (\bigcup A_n) \neq \emptyset$, for each $n \in \omega$.

So, $x \in A_{n_x} \subseteq St(\bigcup A_n, U_n)$, for each $n \in \omega$, i.e. $x \in St(\bigcup_{i \in \omega}(\bigcup A_i), U_n)$, for each $n \in \omega$. Therefore, $\{St(\bigcup_{i \in \omega}(\bigcup A_i), U_n) : n \in \omega\}$ is an open cover for $X$. Hence $X$ has the property $^*U_{fin}(\mathcal{O}, \mathcal{O})$. \qed
Theorem 2.1.13. If \( f : X \to Y \) is a continuous surjection and \( X \) has the property *\( \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \), then \( Y \) also has the property *\( \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \).

Proof. Let \( \{ \mathcal{V}_n : n \in \omega \} \) be a sequence of open covers for \( Y \). For each \( n \in \omega \), let \( \mathcal{U}_n = \{ f^{-1}(V) : V \in \mathcal{V}_n \} \) be a sequence of open covers of \( X \). Since, \( X \) has the property *\( \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \), there exists a sequence \( \{ f^{-1}(V_n) : n \in \omega \} \) where \( V_n \in \mathcal{V}_n \) for all \( n \in \omega \) such that \( \{ St(\bigcup_{n \in \omega} f^{-1}(V_n), \mathcal{U}_n) : n \in \omega \} \) is an open cover of \( X \). Therefore, there exists \( x \in X \) such that \( x \notin St(\bigcup_{n \in \omega} f^{-1}(V_n, \mathcal{U}_n)) \) for each \( n \in \omega \).

Let \( y \in Y \) be an arbitrary point. Then, there exists \( x \in X \) such that \( f(x) = y \). Thus, \( x \in St(\bigcup_{i \in \omega} f^{-1}(V_i), \mathcal{U}_m) \) for some \( m \in \omega \). Therefore, there exists a \( f^{-1}(V_y) \in \mathcal{U}_m \) such that \( x \in f^{-1}(V_y) \) and \( f^{-1}(V_y) \cap (\bigcup_{i \in \omega} f^{-1}(V_i)) \neq \emptyset \). So, \( y \in V_y \in \mathcal{V}_m \) and \( f^{-1}(V_y) \cap f^{-1}(V_i) \neq \emptyset \) for some \( i \in \omega \). i.e. \( V_y \cap V_i \neq \emptyset \).

Thus, \( V_y \cap (\bigcup_{i \in \omega} V_i) \neq \emptyset \). So, \( y \in St(\bigcup_{i \in \omega} V_i, \mathcal{V}_m) \). Hence \( \{ St(\bigcup_{i \in \omega} V_i, \mathcal{V}_n) : n \in \omega \} \) is an open cover for \( Y \). This completes the proof of the theorem. □

Theorem 2.1.14. If \( f : X \to Y \) is a continuous surjection and \( X \) has the property *\( \mathcal{U}_{fin}(\mathcal{O}, \mathcal{O}) \), then \( Y \) also has the property *\( \mathcal{U}_{fin}(\mathcal{O}, \mathcal{O}) \).

Proof. The proof is similar to that of Theorem 2.1.13, so omitted. □

Theorem 2.1.15. If the product of two spaces belongs to the class *\( \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \), then each of them belongs to the class *\( \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \).

Proof. Let \( \{ \mathcal{U}_n : n \in \omega \} \) be a sequence of open covers of \( X \) and \( \{ \mathcal{V}_n : n \in \omega \} \) be a sequence of open covers of \( Y \). Suppose for each \( n \in \omega \), \( \mathcal{W}_n = \{ U \times V : U \in \mathcal{U}_n, V \in \mathcal{V}_n \} \). Then \( \{ \mathcal{W}_n : n \in \omega \} \) is a sequence of open cover of \( X \times Y \). Since \( X \times Y \) belongs to the class *\( \mathcal{U}_1(\mathcal{O}, \mathcal{O}) \), there exists a sequence \( \{ \mathcal{W}_n : n \in \omega \} \) such that \( \mathcal{W}_n \in \mathcal{W}_n \) for each \( n \in \omega \) and \( \{ St(\bigcup_{i \in \omega} \mathcal{W}_i, \mathcal{W}_n) : n \in \omega \} \) is an open cover of \( X \times Y \). Let \( \mathcal{W}_n = U_n \times V_n \) where \( U_n \in \mathcal{U}_n \) and \( V_n \in \mathcal{V}_n \), \( n \in \omega \). So, \( \{ St(\bigcup_{i \in \omega} (U_i \times V_i), \mathcal{W}_n) : n \in \omega \} \) is an open cover of \( X \times Y \), i.e. \( \{ St(\bigcup_{i \in \omega} U_i \times \bigcup_{i \in \omega} V_i, \mathcal{W}_n) : n \in \omega \} \) is an open cover of \( X \times Y \).

Suppose, on the contrary, \( \{ St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) : n \in \omega \} \) is not an open cover of \( X \). Therefore, there exists \( x \in X \) such that \( x \notin St(\bigcup_{n \in \omega} U_i, \mathcal{U}_n) \) for each \( n \in \omega \).
Hence, for each $n \in \omega$, if $U_{nx} \in \mathcal{U}_n$ such that $x \in U_{nx}$, then $(\bigcup_{i \in \omega} U_i) \cap U_{nx} = \emptyset$. 

Now, for each $n \in \omega$ and $y \in Y$, there exists some $V_{ny} \in \mathcal{V}_n$ such that $y \in V_{ny}$, i.e. for each $n \in \omega$, $U_{nx} \times V_{ny} \in \mathcal{W}_n$ such that $(x, y) \in U_{nx} \times V_{ny}$, then $(\bigcup_{i \in \omega} U_i \times \bigcup_{i \in \omega} V_i) \cap (U_{nx} \times V_{ny}) = \emptyset$. 

So, $\text{St}(\bigcup_{i \in \omega} V_i, \mathcal{W}_n) : n \in \omega$ is not an open cover of $X \times Y$, a contradiction. Thus $\{St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) : n \in \omega\}$ is an open cover of $X$. Thus $X$ has the property $^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$. Similarly $Y$ has the property $^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$. Hence the theorem. 

\hspace{1cm} \Box

**Theorem 2.1.16.** If the product of two spaces belongs to the class $^*\mathcal{U}_{fin}(\mathcal{O}, \mathcal{O})$, then each of them belongs to the class $^*\mathcal{U}_{fin}(\mathcal{O}, \mathcal{O})$.

**Proof.** The proof is similar to the proof of Theorem 2.1.15. \hspace{1cm} \Box

**Proposition 2.1.17.** Let $A, B$ and $C$ are any collection of subsets of $X$ and if $C$ is a collection of cover for $X$. If $^*\mathcal{U}_1(\mathcal{A}, \mathcal{B})$ and $^*\mathcal{U}_1(\mathcal{B}, \mathcal{C})$ holds, then $\{X\} \in \mathcal{C}$.

**Proof.** $\{U_n : n \in \omega\}$ be a sequence of elements of $A$. Since $^*\mathcal{U}_1(\mathcal{A}, \mathcal{B})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

Suppose $V_n = St(\bigcup_{i \in \omega} U_i, \mathcal{U}_n)$ for each $n \in \omega$ and $\mathcal{V} = \{V_n : n \in \omega\}$. Now, take the sequence $\{V_n : n \in \omega\}$ such that $\mathcal{V}_n = \mathcal{V}$, for each $n \in \omega$. Then $\{V_n : n \in \omega\}$ is sequence of elements of $B$. Since $^*\mathcal{U}_1(\mathcal{B}, \mathcal{C})$ holds, there exists a sequence $\{V'_n : n \in \omega\}$ such that for each $n$, $V'_n \in \mathcal{V}_n = \mathcal{V}$ and $\{St(\bigcup_{i \in \omega} V'_i, \mathcal{V}_n) : n \in \omega\} \in \mathcal{C}$.

We have

$$\left\{St\left(\bigcup_{i \in \omega} V'_i, \mathcal{V}\right) : n \in \omega\right\} \in \mathcal{C}$$

$$\Rightarrow \left\{St\left(\bigcup_{i \in \omega} V'_i, \mathcal{V}\right)\right\} \in \mathcal{C}, \mathcal{C} \text{ is a collection of covers for } X;$$

So, $St\left(\bigcup_{i \in \omega} V'_i, \mathcal{V}\right) = X,$

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i.e. $\{X\} \in C$.

\[\square\]

**Theorem 2.1.18.** If $X^k$ have the property $^\ast U_1(O,O)$ for any finite $k$, then $X$ has the property $^\ast U_{fin}(O,\Omega)$.

**Proof.** Let $\{U_n : n \in \omega\}$ be a sequence of open covers of $X$ and let $\omega = N_1 \cup N_2 \cup N_3 \cup \ldots$ be a countable partition of $\omega$ into countable subsets. For each $k \in \omega$ and each $m \in N_k$, let $W_m = \{U_1 \times U_2 \times \ldots \times U_k : U_1, U_2, \ldots, U_k \in U_m\}$. Then $\{W_m : m \in N_k\}$ is a sequence of open covers of $X^k$.

Since $^\ast U_1(O,O)$ holds for $X^k$, we can choose a sequence $\{H_m : m \in N_k\}$ such that for each $m$, $H_m \in W_m$ and $\{St(\bigcup_{i \in N_k} H_i, W_m) : m \in N_k\}$ is an open cover of $X^k$.

For every $m \in N_k$ and $H_m \in W_m$. Let, $H_m = U_1(H_m) \times U_2(H_m) \times U_3(H_m) \times \ldots \times U_k(H_m)$, where $U_i(H_m) \in U_m$ for $i \leq k$.

Let $F = \{x_1, x_2, x_3, \ldots, x_s\}$ be a finite subset of $X$. Then $(x_1, x_2, x_3, \ldots, x_s) \in X^s$, so there exists $n \in N_s$ such that $(x_1, x_2, x_3, \ldots, x_s) \in St(\bigcup_{i \in N_s} V_i, W_n)$, where $V_i \in W_i$ and $i \in N_s$. So, there exists a $W \in W_n$ such that $(x_1, x_2, x_3, \ldots, x_s) \in W$ and $W \bigcap(\bigcup_{i \in N_s} V_i) \neq \emptyset$. Let $W = U_1(W) \times U_2(W) \times \ldots \times U_s(W)$. $U_i(W) \in U_n$, $i \leq s$.

Thus $x_1 \in U_1(W)$, $x_2 \in U_2(W)$, $\ldots$, $x_s \in U_s(W)$ and $(U_1(W) \times U_2(W) \times \ldots \times U_s(W)) \bigcap(\bigcup_{i \in N_s} V_i) \neq \emptyset$. i.e. $(U_1(W) \times U_2(W) \times \ldots \times U_s(W)) \bigcap(\bigcup_{i \in N_s} U_1(H_i) \times U_2(H_i) \times \ldots \times U_s(H_i)) \neq \emptyset$ which implies $(U_1(W) \times U_2(W) \times \ldots \times U_s(W)) \bigcap(\bigcup_{i \in N_s} U_1(H_i) \times \bigcup_{i \in N_s} U_2(H_i) \times \bigcup_{i \in N_s} U_3(H_i) \times \ldots \times \bigcup_{i \in N_s} U_s(H_i)) \neq \emptyset$.

Thus, for each $j \leq s$, $U_j(W) \bigcap(\bigcup_{i \in N_s} U_j(H_i)) \neq \emptyset$. Hence, for each $j \leq s$, $U_j(W) \bigcap(\bigcup_{i \in N_s} \bigcup_{j=1}^s U_j(H_i)) \neq \emptyset$.

The set $\{U_1(H), U_2(H), \ldots, U_s(H)\} = \mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $n \in \omega$ and for each $j \leq s$, $x_j \in U_j(W) \subseteq St(\bigcup_{i \in N_s} (\bigcup_{j=1}^s U_j(H_i)), U_n)$, $n \in \omega$i.e. for each $j \leq s$, $x_j \in U_j(W) \subseteq St(\bigcup_{i \in \omega} (\bigcup \mathcal{V}_i)), U_n)$. Thus $F \subseteq St(\bigcup_{i \in \omega} (\bigcup \mathcal{V}_i)), U_n)$, i.e. $F \subseteq St(\bigcup_{i \in \omega} (\bigcup \mathcal{V}_i)), U_n)$.

For each $n$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ satisfying: for each finite set $F \subset X$ there is an $n$ such that $F \subset St(\bigcup_{i \in \omega} (\bigcup \mathcal{V}_i)), U_n)$ and $\{St(\bigcup_{i \in \omega} (\bigcup \mathcal{V}_i)), U_n) : n \in \omega\} \in \Omega$. 26
This implies that $X$ satisfies $^*\mathcal{U}_{fin}(\mathcal{O},\Omega)$.

**Note 2.1.19.** For a finite collection of open covers $\{\mathcal{U}_i : i = 1, 2, 3, ..., n\}$ we define $\cap\{\mathcal{U}_i : i = 1, 2, 3, ..., n\} = \{U_1 \cap U_2 \cap U_3 \cap ... \cap U_n : U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, U_3 \in \mathcal{U}_3, ..., U_n \in \mathcal{U}_n\}$.

**Theorem 2.1.20.** If a space has the property $^*\mathcal{U}_1(\mathcal{O},\Gamma)$, then it has the property $^*\mathcal{U}_1(\mathcal{O},\mathcal{O}^{sp})$.

**Proof.** Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of $X$. We construct new open covers follows.

$$V_n = \cap \left\{ \mathcal{U}_i : \frac{n(n+1)}{2} \leq i < \frac{(n+1)(n+2)}{2} \right\}, \text{ for each } n \in \omega.$$

So, $\{V_n : n \in \omega\}$ is also a sequence of open covers of $X$. Since $X$ has the property $^*\mathcal{U}_1(\mathcal{O},\Gamma)$, we can find a sequence $\{W_n : n \in \omega\}$ such that $W_n \in V_n$ for each $n \in \omega$ and every $x \in X$ belongs to all but finitely many members of $\{\text{St}(\bigcup_{i \in \omega} W_i, V_n) : n \in \omega\}$.

For each $i \in \omega$, $W_i \subset U_j$, for some $U_j \in \mathcal{U}_j$ with $\frac{i(i+1)}{2} \leq j < \frac{(i+1)(i+2)}{2}$. We consider the set of non-negative integers $n_0 < n_1 < ... < n_p < ...$ defined by $n_p = \frac{p(p+1)}{2}$.

If $x \in X$ belongs to $\text{St}(\bigcup_{i \in \omega} W_i, V_k)$ for some $k \in \omega$, then $x$ belongs to $\text{St}(\bigcup_{i \in \omega} W_i, \mathcal{U}_l)$, for each $l$ such that $n_k \leq l < n_{k+1}$. i.e. $x \in \bigcup_{n_k \leq l < n_{k+1}} \text{St}(\bigcup_{i \in \omega} W_i, \mathcal{U}_l)$.

So, for each $x \in X$, we have $x \in \bigcup_{n_k \leq l < n_{k+1}} \text{St}(\bigcup_{i \in \omega} W_i, \mathcal{U}_l)$ for all but infinitely many $k \in \omega$. $\bigcup_{i \in \omega} W_i \subset \bigcup_{i \in \omega} U_i$. So, for each $x \in X$, we have $x \in \bigcup_{n_k \leq l < n_{k+1}} \text{St}(\bigcup_{i \in \omega} U_i, \mathcal{U}_l)$ for all but infinitely many $k \in \omega$. Thus the cover $\{\text{St}(\bigcup_{i \in \omega} U_i, \mathcal{U}_n) : n \in \omega\}$ is groupable. \qed
2.2 Selection Principles $SS^*_c(A, B)$ and $SS^*_{c, fin}(A, B)$

Generally selection principles deal with two collections of families of subsets. In this section we introduce some new star-selection principles termed as $SS^*_c,1(A, B)$ and $SS^*_{c, fin}(A, B)$ which deal with three collections of subsets or families of subsets of a set. As a initiation of a systemic study, some properties of the new selection principles are investigated which show them to be distinct from existing selection principles.

Let $A$ be a collection of subsets of an infinite set $X$ and $B$ and $C$ be collections of families of subsets of $X$. Then

**Definition 2.2.1.** $SS^*_c,1(A, B)$ denotes the following selection hypothesis:

For every sequence $\{K_n : n \in \omega\}$ of elements of $A$ and for every $U \in C$, there exists a sequence $\{x_n : n \in \omega\}$ of points of $X$ such that $x_n \in K_n$ for each $n \in \omega$ and $\{St(x_n, U) : n \in \omega\} \in B$.

**Definition 2.2.2.** $SS^*_{c, fin}(A, B)$ denotes the following selection hypothesis:

For every sequence $\{K_n : n \in \omega\}$ of elements of $A$ and for every $U \in C$, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of $X$ such that $F_n \subseteq K_n$ for each $n \in \omega$ and $\{St(F_n, U) : n \in \omega\} \in B$.

**Proposition 2.2.3.** (a) If $A$ and $A'$ are two collections of subsets of an infinite set $X$ such that $A \subseteq A'$ then $SS^*_c,1(A', B) \Rightarrow SS^*_c,1(A, B)$.

(b) If $B$ and $B'$ are two collections of families of subsets of an infinite set $X$ such that $B \subseteq B'$ then $SS^*_c,1(A, B) \Rightarrow SS^*_c,1(A, B')$.

(c) If $C$ and $C'$ are two collections of families of subsets of an infinite set $X$ such that $C \subseteq C'$ then $SS^*_{c,1}(A, B) \Rightarrow SS^*_{c,1}(A, B)$.

**Proof.**

(a) Let $\{K_n : n \in \omega\}$ be an arbitrary sequence of elements of $A$. Since $A \subseteq A'$, so $\{K_n : n \in \omega\}$ is a sequence of elements of $A'$. But selection principle $SS^*_c,1(A', B)$
holds. So, for every \( U \in \mathcal{C} \), there exists a sequence \( \{x_n : n \in \omega\} \) of points of \( X \) such that \( x_n \in K_n \) for each \( n \in \omega \) and \( \{\text{St}(x_n, U) : n \in \omega\} \in \mathcal{B} \). Therefore, selection principle \( SS_{\mathcal{C},1}^*(A, B) \) holds.

(b) Let \( \{K_n : n \in \omega\} \) be an arbitrary sequence of elements of \( A \). Since selection principle \( SS_{\mathcal{C},1}^*(A, B) \) holds, for every \( U \in \mathcal{C} \), there exists a sequence \( \{x_n : n \in \omega\} \) of points of \( X \) such that \( x_n \in K_n \) for each \( n \in \omega \) and \( \{\text{St}(x_n, U) : n \in \omega\} \in \mathcal{B} \). Since \( \mathcal{B} \subseteq \mathcal{B}' \), \( \{\text{St}(x_n, U) : n \in \omega\} \in \mathcal{B}' \). Thus selection principle \( SS_{\mathcal{C},1}^*(A, B') \) holds.

(c) Let \( \{K_n : n \in \omega\} \) be an arbitrary sequence of elements of \( A \). Let \( U \in \mathcal{C} \) be arbitrary. Since \( \mathcal{C} \subseteq \mathcal{C}' \), \( U \in \mathcal{C}' \). But selection principle \( SS_{\mathcal{C}',1}^*(A, B) \) holds. So, there exists a sequence \( \{x_n : n \in \omega\} \) of points of \( X \) such that \( x_n \in K_n \) for each \( n \in \omega \) and \( \{\text{St}(x_n, U) : n \in \omega\} \in \mathcal{B} \). Thus selection principle \( SS_{\mathcal{C}',1}^*(A, B') \) holds. \( \square \)

**Proposition 2.2.4.** (a) If \( A \) and \( A' \) are two collections of subsets of an infinite set \( X \) such that \( A \subseteq A' \) then \( SS_{\mathcal{C},f.in}^*(A', B) \Rightarrow SS_{\mathcal{C},f.in}^*(A, B) \).

(b) If \( B \) and \( B' \) are two collections of families of subsets of an infinite set \( X \) such that \( B \subseteq B' \) then \( SS_{\mathcal{C},f.in}^*(A, B) \Rightarrow SS_{\mathcal{C},f.in}^*(A, B') \).

(c) If \( C \) and \( C' \) are two collections of families of subsets of an infinite set \( X \) such that \( C \subseteq C' \) then \( SS_{\mathcal{C}',f.in}^*(A, B) \Rightarrow SS_{\mathcal{C},f.in}^*(A, B) \).

*Proof.* The proof is similar to the proof of proposition 2.2.3, so omitted. \( \square \)

So, we conclude that \( SS_{\mathcal{C},1}^*(A, B) \) is monotonic in the second collection and anti-monotonic in the first collection and subscripted collection. Similarly \( SS_{\mathcal{C},f.in}^*(A, B) \) is monotonic in the second collection and anti-monotonic in the first collection and subscripted collection. Thus we have,
The notion of star-Lindelöfness was introduced in [20], and has been studied extensively by many mathematicians. In this section we are going to show that star-Lindelöfness comes under the selection principle $SS_{\mathcal{O},1}^*(\{X\}, \mathcal{O})$.

**Proposition 2.2.5.** A space $X$ is star-Lindelöf if and only if it has the property $SS_{\mathcal{O},1}^*(\{X\}, \mathcal{O})$.

**Proof.** Let, the space $X$ be star-Lindelöf. Therefore for every open cover $\mathcal{U}$ of $X$, there exists $F \in [X]^\omega$ such that $St(F, \mathcal{U}) = X$. Without loss of generality suppose $F = \{x_1, x_2, \ldots, x_n, \ldots\}$. Therefore, $\bigcup\{St(x_n, \mathcal{U}) : n \in \omega\} = X$. But, for each $n \in \omega$, $\ldots$
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\( St(x_n, \mathcal{U}) \) is open. Which in turn implies that \( \{ St(x_n, \mathcal{U}) : n \in \omega \} \in \mathcal{O} \). Therefore \( X \) has the property \( SS^*_{\mathcal{O},1}(\{X\}, \mathcal{O}) \).

Conversely, suppose \( X \) has the property \( SS^*_{\mathcal{O},1}(\{X\}, \mathcal{O}) \). The only sequence of elements of \( \{X\} \) is \( \{K_n = X : n \in \omega \} \). So for any open cover \( \mathcal{U} \) of \( X \), we can choose \( x_n \in K_n \), i.e. \( x_n \in X \) for each \( n \in \omega \) such that \( \{ St(x_n, \mathcal{U}) : n \in \omega \} \in \mathcal{O} \) i.e. \( St(F, \mathcal{U}) = X \), where \( F = \{x_1, x_2, x_3, \ldots\} \). Therefore, \( X \) is star-Lindelöf.  

\[ \square \]

**Corollary 2.2.6.** Every Lindelöf space \( X \) has the property \( SS^*_{\mathcal{O},1}(\{X\}, \mathcal{O}) \).

**Proof.** The proof follows from the proposition 2.2.5 and the fact that every Lindelöf space is star-Lindelöf.  

\[ \square \]

**Example 2.2.7.** (\( AC_\omega \)) The converse of the above corollary is not true. For example Consider \( X = [0, \omega_1) \) and \( \tau(X) = \{[0, \alpha) : \alpha \in \omega_1 \} \cup \{X, \emptyset\} \). Consider the open cover \( \mathcal{U} = \{[0, \alpha) : \alpha \in \omega_1 \} \). Suppose \( \mathcal{U}' = \{[0, \alpha_n) : n \in \omega \} \subseteq \mathcal{U} \) covers \( X \). Then \( \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \) is a countable set of countable ordinals. If, \( \alpha_{\text{max}} = \text{Sup}\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \), \( \alpha_{\text{max}} < \omega_1 \). Therefore, \( [\alpha_{\text{max}} + 1, \omega_1) \) remains uncovered. Therefore, \( \mathcal{U} \) cannot have a countable subcover. Hence \( X \) is not Lindelöf.

Now, \( \{K_n = X : n \in \omega \} \) is the only sequence of elements of \( \{X\} \). Also, 0 belongs to every open subset of \( X \) other than \( \emptyset \). Now, for any \( \mathcal{U} \in \mathcal{O} \), if we select \( x_0 = 0 \in K_0 \) and select \( x_n \in K_n(n \in \omega \setminus \{0\}) \) in any way, then \( \{ St(x_n, \mathcal{U}) : n \in \omega \} \in \mathcal{O} \). Therefore \( X \) has the property \( SS^*_{\mathcal{O},1}(\{X\}, \mathcal{O}) \).

**Proposition 2.2.8.** For a space \( X \),

\[ SS^*_{1}(\mathcal{A}, \mathcal{B}) \Rightarrow SS^*_{\mathcal{A},1}(\{X\}, \mathcal{B}) \]

**Proof.** Let \( SS^*_{1}(\mathcal{A}, \mathcal{B}) \) holds for the space \( X \). Applying \( SS^*_{1}(\mathcal{A}, \mathcal{B}) \) to the sequence \( \{\mathcal{U}, \mathcal{U}, \mathcal{U}, \ldots\} \), there is a sequence \( \{x_1, x_2, x_3, \ldots\} \) such that \( \{ St(x_n, \mathcal{U}) : n \in \omega \} \). Hence, \( X \) has the property \( SS^*_{\mathcal{A},1}(\{X\}, \mathcal{B}) \).  

\[ \square \]
Example 2.2.9. The converse of the above proposition may not be true. There exists a space $X$ which have the property $SS^*_{\mathcal{O},1}(\{X\}, \mathcal{O})$ but does not have the property $SS^*_1(\mathcal{O}, \mathcal{O})$.

Let, $X = (0, 3]$.

We consider the topology $\tau(X) = \{(x, y) : x, y \in [0, 3) \text{ and } x < y \} \cup \emptyset, X \}$ i.e. $X$ having the upper limit topology induced from the upper limit topology of $\mathbb{R}$.

Now, we construct a sequence $\{U_n : n \in \omega\}$ of open covers of $X$ where,

$$U_n = \left\{\left(0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \left(\frac{2}{2^n}, \frac{3}{2^n}\right], \ldots, \left(\frac{3^{n} - 1}{2^n}, \frac{3^n}{2^n}\right]\right\}.$$

Now, for each $n \in \omega$, the length of each interval (open set) contained in $U_n$ is $\frac{1}{2^n}$. The maximum length covered by $\{St(x_n, U_n) : n \in \omega\} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^n + \ldots = (1 - \frac{1}{2})^{-1} = 2$. But the total length of the space is 3. So, $\{St(x_n, U_n) : n \in \omega\} \notin B$. So $X$ does not have the property $SS^*_1(\mathcal{O}, \mathcal{O})$.

But the space is Lindelöf. Hence by corollary 2.2.6 it has the property $SS^*_{\mathcal{O},1}(\{X\}, \mathcal{O})$.

Theorem 2.2.10. If a topological space has the property $SS^*_{\mathcal{O},1}(\mathcal{D}, \mathcal{O})$, then every clopen subspace of the space has the property $SS^*_{\mathcal{O},1}(\mathcal{D}, \mathcal{O})$.

Proof. Let $X$ be a topological space which has the property $SS^*_{\mathcal{O},1}(\mathcal{D}, \mathcal{O})$ and $Y$ be a clopen subspace of $X$. Let $\{K_n : n \in \omega\}$ be a sequence of dense subsets of $Y$ and $U$ be an open cover of $Y$ in the subspace $Y$. Now $\{K_n \cup (X \setminus Y) : n \in \omega\}$ is a sequence of dense subsets of $X$ and $U \cup \{X \setminus Y\}$ is an open cover of $X$. So by the $SS^*_{\mathcal{O},1}(\mathcal{D}, \mathcal{O})$ property of $X$ there are points $x_n \in K_n \cup \{X \setminus Y\}$ for each $n \in \omega$ such that $\{St(x_n, \{U \cup \{X \setminus Y\}\}) : n \in \omega\} \in \mathcal{O}$.

Without loss of generality, we can suppose $x_0 \in X \setminus Y$ and $x_n \in K_n$ for each $n \in \omega \setminus \{0\}$. Then $\{St(x_n, U) : n \in \omega\}$ is an open cover of $Y$, i.e. $Y$ has the property $SS^*_{\mathcal{O},1}(\mathcal{D}, \mathcal{O})$. \hfill \Box

Theorem 2.2.11. If a topological space has the property $SS^*_{\mathcal{O},fin}(\mathcal{D}, \mathcal{O})$, then every clopen subspace of the space has the property $SS^*_{\mathcal{O},fin}(\mathcal{D}, \mathcal{O})$. 

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Proof. The proof of the theorem is similar to the proof of the theorem 2.2.10, so omitted.

\textbf{Theorem 2.2.12.} \((AC_{\omega})\) Every Lindelöf space has the property \(SS_{O,1}(D, O)\).

\textbf{Proof.} Let, \(X\) be a Lindelöf space. \(\{K_n : n \in \omega\}\) be a sequence of elements of \(D\) and \(U \in O\). So, there exists \(U' \in [U]^{\leq \omega}\) such that \(U' \in O\). Suppose \(U' = \{U_n : n \in \omega\}\).

For each \(n \in \omega\), since \(K_n \cap U_n \neq \emptyset\) we can choose \(x_n \in K_n \cap U_n\).

\[
\therefore St(x_n, U') \supseteq U_n \text{ for each } n \in \omega.
\]

Thus \(\{St(x_n, U') : n \in \omega\} \in O\). Hence \(X\) has the property \(SS_{O,1}^*(D, O)\).

\textbf{Example 2.2.13.} Converse of the above theorem is not true. Consider the topological space constructed in example 2.2.7. The space is not Lindelöf.

Let, \(\{K_n : n \in \omega\}\) be any sequence of elements of \(D\). Also, 0 is an isolated point. \(0 \in K_n\) for all \(n \in \omega\). For any \(U \in O\), if we select \(x_0 = 0 \in K_0\) and select \(x_n \in K_n(n \in \omega \setminus \{0\})\) in any way, then \(X \in \{St(x_n, U) : n \in \omega\} \in O\). Therefore \(X\) has the property \(SS_{O,1}^*(D, O)\).

\textbf{Corollary 2.2.14.} \((AC_{\omega})\) Every compact, \(\sigma\)-compact space has the property \(SS_{O,1}^*(D, O)\).

\textbf{Proof.} The corollary follows from theorem 2.2.12 and the fact that every compact, \(\sigma\)-compact space is a Lindelöf space.

\textbf{Note 2.2.15.} A space having \(S_{fin}(D, D)\) is separable. Also a space having \(S_1(D, D)\) is separable.

\textbf{Theorem 2.2.16.}

\(a)\ S_1(D, D) \Rightarrow SS_{O,1}^*(D, O)\)
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(b) \( S_{\text{fin}}(D, D) \Rightarrow SS^*_{\text{fin}}(D, D) \)

Proof. (a) Let a topological space \( X \) have the property \( S_1(D, D) \), \( \{K_n : n \in \omega\} \) be a sequence of elements of \( D \) and \( U \in O \). Therefore, we can select points \( x_n \in K_n \) for each \( n \in \omega \) such that \( \bigcup_{n \in \omega} \{x_n\} \in D \). Clearly \( \{St(x_n, U) : n \in \omega\} \) is a collection of open sets of \( X \), and the set \( \{x_n : n \in \omega\} \) intersects every element of \( U \). So, \( \{St(x_n, U) : n \in \omega\} \in O \). Hence the theorem.

(b) Let a topological space \( X \) have the property \( S_{\text{fin}}(D, D) \), \( \{K_n : n \in \omega\} \) be a sequence of elements of \( D \) and \( U \in O \). So, there exists a sequence \( \{F_n : n \in \omega\} \) of finite subsets of \( X \) such that \( F_n \subseteq K_n \) for each \( n \in \omega \) and \( \bigcup_{n \in \omega} \{F_n\} \in D \). Clearly \( \{St(F_n, U) : n \in \omega\} \) is a collection of open sets of \( X \), and the set \( \bigcup \{F_n : n \in \omega\} \) intersects every element of \( U \). So, \( \{St(F_n, U) : n \in \omega\} \in O \). Hence the theorem.

Example 2.2.17. \( (AC_\omega) \) The converse of theorem 2.2.16 may not be true.

Let \( X \) be an uncountable set equipped with co-countable topology. Then \( X \) has no countable dense subset. Hence \( X \) does not have the properties \( S_1(D, D) \) and \( S_{\text{fin}}(D, D) \).

Let, \( U \in O \). \( \{K_n : n \in \omega\} \) be a sequence of elements of \( D \). We take \( U_0 \in U \). Clearly \( U_0 \) is of the form \( U_0 = X \setminus C \), where \( C \) is countable. Suppose, \( C = \{y_i : i \in \omega \setminus \{0\}\} \). Now, \( U_0 \cap K_0 \neq \emptyset \). We select \( x_0 \in U_0 \cap K_0 \).

Since, \( U \) is an open cover of \( X \), there exists \( V_n \in U \) such that \( y_n \in V_n \in U \) for each \( n \in \omega \setminus \{0\} \). Because each \( K_n (n \in \omega \setminus \{0\}) \) is dense in \( X \), it follows that \( V_n \cap K_n \neq \emptyset \) for each \( n \in \omega \setminus \{0\} \). We select \( x_n \in V_n \cap K_n \) for each \( n \in \omega \setminus \{0\} \). Then \( U_0 \subseteq St(x_0, U) \) and \( y_n \in V_n \subseteq St(x_n, U) \).

Hence \( X = U_0 \cup \{y_1, y_2, y_3, \ldots\} \subseteq U_0 \cup V_0 \cup V_1 \cup V_2 \cup \ldots \subseteq \bigcup_{n \in \omega} St(x_n, U) \).

Further, \( \{St(x_n, U) : n \in \omega\} \in O \)

\( \therefore X \) have the property \( SS^*_{\text{fin}}(D, O) \). But \( SS^*_{\text{fin}}(D, O) \Rightarrow SS^*_{\text{fin}}(D, O) \) (by definition). Thus, \( X \) also have the property \( SS^*_{\text{fin}}(D, O) \).