Chapter 3

Fuzzy Ideals and Real Valuation Rings

3.1. Introduction

We have introduced the notion of valuation fuzzy ideals in chapter 2 and derived the conditions required for the existence of such ideals. We also established an order preserving correspondence between such ideals and valuations of a valuation ring. In this chapter we continue this investigation. Even though the existence of a valuation fuzzy ideal guarantees that the ring is a valuation ring, it may either be a real valuation ring or a non-real one. We therefore introduce another class of fuzzy ideals called $\mathbb{R}$-valuation fuzzy ideals and prove that an integral domain is a real valuation ring if and only if it possesses a $\mathbb{R}$-valuation fuzzy ideal. We also represent the value group of real valuation rings in terms of $\mathbb{R}$-valuation fuzzy ideals and discuss a method of constructing $\mathbb{R}$-valuation fuzzy ideals in a DVR.

3.2. Fuzzy ideals and real valuation rings

3.2.1. Theorem. Let $V$ be an integral domain. Then $V$ is a real valuation ring if and only if there exists a fuzzy ideal $\mu$ satisfying the following conditions.

(i) $\mu_x = \{0\}$

(ii) $\mu(x + y) = \mu(xy) + \mu(x) \mu(y) \ \forall \ x, y \in V$ and

(iii) $\mu(y) \leq \mu(x) \Rightarrow x/y \in V$, $\forall \ x, y \in V$

Proof. Let $V$ be a real valuation ring. Then there exists a valuation $\nu$ on its quotient field $K$ with value group $G \subseteq \mathbb{R}$. Define $\mu: V \rightarrow [0, 1]$ by $\mu(x) = 1 - 2^{-\nu(x)}$. Then from the properties of valuation, it can easily be verified that

$\mu(x + y) \geq \mu(x) \wedge \mu(y)$, $\mu(xy) \geq \mu(x) \vee \mu(y)$ and $\mu(-x) = \mu(x)$.

Therefore $\mu$ is a fuzzy ideal. In order to prove condition (ii), we have
\[ \mu(x) + \mu(y) = (1 - 2^{-\upsilon(x)}) + (1 - 2^{-\upsilon(y)}) \quad \text{and} \]
\[ \mu(xy) + \mu(x) \mu(y) = 1 - 2^{-\upsilon(xy)} + (1 - 2^{-\upsilon(x)})(1 - 2^{-\upsilon(y)}) \]
\[ = 1 - 2^{-\upsilon(x)} 2^{-\upsilon(y)} + 1 - 2^{-\upsilon(x)} - 2^{-\upsilon(y)} + 2^{-\upsilon(x)} 2^{-\upsilon(y)} \]
\[ = (1 - 2^{-\upsilon(x)} + (1 - 2^{-\upsilon(y)}) \]
\[ \therefore \mu(x) + \mu(y) = \mu(xy) + \mu(x) \mu(y). \] This proves (ii)

Again,
\[ \mu(0) = 1 - 2^{-\upsilon(0)} = 1 - 2^{-\infty} = 1. \]

Also,

\[ x \in \mu_1 \Rightarrow \mu(x) = 1 \Rightarrow 1 - 2^{-\upsilon(x)} = 1 \Rightarrow \upsilon(x) = \infty \Rightarrow x = 0. \]
\[ \therefore \mu_1 = \{ 0 \}. \] This proves (i).

Finally, for \( x, y \in V \),
\[ \mu(y) \leq \mu(x) \Rightarrow \upsilon(y) \leq \upsilon(x) \Rightarrow \upsilon(x/y) = \upsilon(x) - \upsilon(y) \geq 0. \]
\[ \therefore x/y \in V. \] This proves (iii).

Conversely suppose \( \exists \) a fuzzy ideal \( \mu \) on \( V \) satisfying condition (i) and (ii) and (iii). Define \( \upsilon(x) = - \log_2 (1-\mu(x)). \) We claim that \( \upsilon(x) = \infty \Leftrightarrow x = 0. \) We have from the definition of \( \upsilon \), \( \upsilon(0) = - \log_2 (1-\mu(0)). \) But, by condition (ii), \( \mu(0) + \mu(0) = \mu(0) + \mu(0) \). Therefore \( \mu(0) (\mu(0)-1) = 0. \) Since \( \mu(0) \neq 0 \), we have \( \mu(0) = 1. \) Hence \( \upsilon(0) = \infty. \) Also,
\[ \upsilon(x) = \infty \Rightarrow - \log_2 (1 - \mu(x)) = \infty \Rightarrow 1 - \mu(x) = 0 \Rightarrow \mu(x) = 1. \]

By condition (i), \( x = 0. \) \( \therefore \upsilon(x) = \infty \Leftrightarrow x = 0. \) Again,
\[ \upsilon(xy) = - \log_2 (1 - \mu(xy)) \]
\[ = - \log_2 (1 - \mu(x) + \mu(y) - \mu(x) \mu(y)) \quad \text{(by condition (ii))} \]
\[ = - \log_2 (1 - \mu(x) (1 - \mu(y))) = \upsilon(x) + \upsilon(y) \]
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Also \( \nu(x + y) \geq \nu(x) \wedge \nu(y) \). Therefore \( \nu \) is a real valuation on the quotient field \( K \) and \( V \) is a subring of its valuation ring. In fact, the valuation ring of \( \nu \) is
\[
\{ \frac{x}{y} \in K : \mu(y) \leq \mu(x) \},
\]
But by condition (iii), this is equal to \( V \). 

3.2.2. Definition. A fuzzy ideal \( \mu \) of an integral domain \( V \) is said to be an \( \mathbb{R} \)-valuation fuzzy ideal if \( \forall \ x, y \in V, \)
(i) \( \mu^* = \{0\} \), (ii) \( \mu(x) + \mu(y) = \mu(xy) + \mu(x) \mu(y) \) and
(iii) \( \mu(y) \leq \mu(x) \Rightarrow x/y \in V. \)

In terms of the \( \mathbb{R} \)-valuation fuzzy ideal, we can restate theorem 3.2.1 as follows.

3.2.3. Theorem. An integral domain \( V \) is a real valuation ring if and only if it possesses an \( \mathbb{R} \)-valuation fuzzy ideal. 

We shall now compare \( \mathbb{R} \)-valuation fuzzy ideals with valuation fuzzy ideals defined in chapter 2.

3.2.4. Proposition. Every \( \mathbb{R} \)-valuation fuzzy ideal is a valuation fuzzy ideal.

Proof. Suppose \( \mu \) is an \( \mathbb{R} \)-valuation fuzzy ideal of an integral domain \( V \). Let \( x, y \in V \) and \( \mu(x) < \mu(y) \). Then by condition (iii), \( y/x \in V. \). \( \therefore y = r x, \) \( r \in V. \) We prove that \( r \) is a non-unit.

We have \( \mu(r x) = \mu(r) + \mu(x) - \mu(r) \). \( \mu(x) \) by condition (ii). \( \therefore \mu(y) = \mu(r) + \mu(x) - \mu(r) \). \( \mu(x) \), hence \( \mu(r) - \mu(r) \mu(x) = \mu(y) - \mu(x) > 0. \) That is \( \mu(r) (1 - \mu(x)) > 0. \) Since both the factors on the L.H.S are non-negative, we have \( \mu(r) > 0. \) We claim that \( x \in V \) is a unit \( \iff \mu(x) = 0. \) Note that
by condition (ii), we have \( \mu(1) + \mu(1) = \mu(1) + \mu(1) \). \( \mu(1) \). Hence 
\( \mu(1) (1 - \mu(1)) = 0 \). But by condition (i), \( \mu(1) \neq 1 \). \( \therefore \mu(1) = 0 \). If \( x \) is a unit, then \( xy = 1 \) for some \( y \in V \). But then \( \mu(1) = \mu(xy) \geq \mu(x) \) or \( \mu(x) \leq \mu(1) \). It follows that \( \mu(x) = \mu(1) = 0 \). Conversely, if \( \mu(x) = 0 \) for \( x \in V \), then \( \mu(x) = \mu(1) \). By condition (iii), \( 1/x \in V \). \( \therefore x \) is a unit. Since \( \mu(r) > 0 \), it follows that \( r \) is a non-unit.

Therefore \( (y) \subset (x) \). Again, \( \mu(x) = \mu(y) \Rightarrow x/y \in V \) and \( y/x \in V \) by condition (iii). \( \therefore (y) = (x) \). Thus

\[ \mu(x) < \mu(y) \Rightarrow (y) \subset (x) \text{ and } \mu(x) = \mu(y) \Rightarrow (y) = (x). \]

\( \therefore \mu \) is a valuation fuzzy ideal ■

It is not true that every valuation fuzzy ideal is an \( \mathbb{R} \)-valuation fuzzy ideal. The following example supports this statement.

3.2.5. Example. Let \( Z_{(2)} \) represents the ring consisting of all rational numbers of the form \( r/s \) where \( r, s \in Z \) and \( s \) is odd. Then \( Z_{(2)} \) is called the localization of the ring of integers \( Z \). Its ideal structure is

\[ Z_{(2)} \supset (2/1) \supset (2^2/1) \supset (2^3/1) \supset \ldots \ldots \supset (0) \]

where \( S \) is the set of all odd integers. Define \( \mu : Z_{(2)} \rightarrow [0,1] \) by

\[ \mu(x) = 0, \text{ if } x \in Z_{(2)} - (2/1) \]

\[ = n/(n+1), \text{ if } x \in (2^n/1) - (2^{n+1}/1), n = 1, 2, \ldots \]

\[ = 1, \text{ if } x = 0 \]

Then \( \mu \) is a valuation fuzzy ideal.

Let \( x = 4/3 \) and \( y = 12/5 \). Then \( x, y \in (2^2/1) - (2^3/1). \therefore \mu(x) = \mu(y) = 2/3 \).
\( xy = 48/15 = 16/5 \in (2^1/1) - (2^2/1). \) Now \( \mu(x) + \mu(y) = 4/3 = 60/45 \) and \( \mu(xy) + \mu(x) \mu(y) = 4/5 + 4/9 = 56/45 \). \( \therefore \mu(x) + \mu(y) \neq \mu(xy) + \mu(x) \mu(y) \). Hence \( \mu \) is not an \( \mathbb{R} \)-valuation fuzzy ideal ■
We have seen from theorem 3.2.3 that $\mathbb{R}$-valuation fuzzy ideals and real valuations are equivalent for an integral domain. We therefore denote the valuation equivalent to an $\mathbb{R}$-valuation fuzzy ideal $\mu$ by $\upsilon_\mu$ and the $\mathbb{R}$-valuation fuzzy ideal equivalent to a valuation $\upsilon$ by $\mu_\upsilon$. We shall extend this idea to the context of quotient rings. For this, we need the following results.

3.2.6. Proposition [36]. If $\upsilon$ is a valuation on an integral domain $V$ and $P$ is a prime ideal of $V$ then the quotient ring $V/P$ is an integral domain and the map $\upsilon/P$ on $V/P$ defined by

$$\upsilon/P (x+P) = \vee \{ \upsilon(x+z) : z \in P \}$$

is a valuation on $V/P$.

3.2.7. Proposition [28]. If $\mu$ is a fuzzy ideal on $R$ and $P$ is a prime ideal of $R$ then the map $\mu/P$ defined on the quotient ring $R/P$ by

$$(\mu/P)(x+P) = \vee \{ \mu(x+z) : z \in P \}$$

is a fuzzy ideal on $R/P$.

Let $\mu$ be an $\mathbb{R}$-valuation fuzzy ideal on an integral domain $V$ and $\upsilon_\mu$ be the equivalent valuation. If $P$ is a prime ideal of $V$, then there are two quotient fuzzy ideals on $V/P$ (i) $\mu/P$, the fuzzy ideal of $\mu$ relative to $P$ and (ii) $\mu_{(\upsilon_\mu)/P}$, the $\mathbb{R}$-valuation fuzzy ideal equivalent to the valuation $\upsilon_\mu/P$. In the following proposition, we prove that they are identical.

3.2.8. Proposition. Let $\mu$ be an $\mathbb{R}$-valuation fuzzy ideal on an integral domain $V$ and $\upsilon_\mu$ be the equivalent valuation. Let $P$ be a prime ideal of $V$. Then $\mu_{(\upsilon_\mu)/P} = \mu/P$.

**Proof.** We have $\upsilon_\mu(x) = - \log_2(1-\mu(x))$. Also by definition

$$(\upsilon_\mu/P)(x+P) = \vee \{ \upsilon_\mu(x+z) : z \in P \}$$
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\[ \mu_{(\nu/P)}(x + P) = 1 - 2^{-\nu(\nu/P)(x + P)} \]

\[ = 1 - 2^{-\nu(\nu(x+z); z \in P)} \]

\[ = 1 - 2^{-\nu(-\log_2 (1-\mu(x+z)); z \in P)} \]

\[ = 1 - 2^\nu(\log_2 (1-\mu(x+z)); z \in P) \]

Since the function \( f(x) = \log_2 x \) is strictly increasing,

\[ \nu(\log_2 (1-\mu(x+z): z \in P) = \log_2 (\nu(1-\mu(x+z): z \in P)) \].

It follows that

\[ \mu_{(\nu/P)}(x + P) = 1 - \nu(1-\mu(x+z): z \in P) \]

\[ = \nu(\mu(x+z): z \in P) \]

\[ = (\mu/P)(x+P) \]

\[ \therefore \mu_{(\nu/P)} = \mu/P \quad \square \]

3.3. Value group from \( \mathbb{R} \)-valuation fuzzy ideals

3.3.1. Proposition. Let \( V \) be an integral domain having an \( \mathbb{R} \)-valuation fuzzy ideal \( \mu \). Then \( V \) is a real valuation ring with value group

\[ P = \left\{ \log_2 \left[ \frac{1-\mu(y)}{1-\mu(x)} \right] : x, y \in V; \ x \neq 0, y \neq 0 \right\}. \]

Proof. In theorem 3.2.1 we have proved that an integral domain \( V \) having an \( \mathbb{R} \)-valuation fuzzy ideal is a real valuation ring and we know that for a valuation ring \( V \), the value group is \( G = \mathbb{K}/U \) where \( \mathbb{K} \) is the multiplicative group of all non-zero elements of its quotient field \( K \) and \( U \) is the subgroup consisting of all non units of \( V \). Therefore we need only to prove that \( P \) is a subgroup of \( \mathbb{R} \) and that \( P \) is isomorphic to \( G \).
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Since \( \mu \) is an \( \mathbb{R} \)-valuation fuzzy ideal, we have \( \mu^* = \{0\} \), \( \mu(x) + \mu(y) = \mu(xy) + \mu(x) \mu(y) \) and \( \mu(y) \leq \mu(x) \Rightarrow x/y \in V \) for all \( x, y \in V \). In order to prove that \( P \) is a subgroup of \( \mathbb{R} \), let \( X, Y \in P \) where

\[
X = \log_2 \left[ \frac{1 - \mu(y)}{1 - \mu(x)} \right] \quad \text{and} \quad Y = \log_2 \left[ \frac{1 - \mu(y')}{1 - \mu(x')} \right].
\]

Then

\[
X + Y = \log_2 \left[ \frac{1 - \mu(y)}{1 - \mu(x)} \right] + \log_2 \left[ \frac{1 - \mu(y')}{1 - \mu(x')} \right] = \log_2 \left[ \frac{(1 - \mu(y))(1 - \mu(y'))}{(1 - \mu(x))(1 - \mu(x'))} \right]
\]

\[
= \log_2 \left[ \frac{1 - (\mu(y) + \mu(y') - \mu(y)\mu(y'))}{1 - (\mu(x) + \mu(x') - \mu(x)\mu(x'))} \right] = \log_2 \left[ \frac{1 - \mu(yy')}{1 - \mu(xx')} \right]
\]

\[\therefore X + Y \in P.\]

Again,

\[
- X = -\log_2 \left[ \frac{1 - \mu(y)}{1 - \mu(x)} \right] = \log_2 \left[ \frac{1 - \mu(x)}{1 - \mu(y)} \right] \in P.
\]

\[\therefore P \text{ is a subgroup of } \mathbb{R}.\]

In order to prove that \( G \) is isomorphic to \( P \), note that any element of \( G \) is of the form \( (x/y)U \). Consider the map from \( G \) into \( P \) defined by

\[
(x/y)U \rightarrow \log_2 \left[ \frac{1 - \mu(y)}{1 - \mu(x)} \right]
\]

The map is well defined. For, if \( (x/y)U = (x'/y')U \), then \( xy'/yx' \in U \).

\[\therefore xy'/yx' = u \quad \text{where } u \text{ is a unit in } V. \therefore xy' = uyx', \quad \text{hence } \mu(xy') = \mu(yx').\]

\[\therefore \mu(x) + \mu(y') - \mu(x)\mu(y') = \mu(y) + \mu(x') - \mu(y)\mu(x').\]

Hence

\[1 - \mu(x) - \mu(y') + \mu(x)\mu(y') = 1 - \mu(y) - \mu(x') + \mu(y)\mu(x').\]

\[\therefore (1 - \mu(x))(1 - \mu(y')) = (1 - \mu(y))(1 - \mu(x')).\]
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Now
\[
\frac{1-\mu(y)}{1-\mu(x)} = \frac{1-\mu(y')}{1-\mu(x')}. \therefore \log_2 \left( \frac{1-\mu(y)}{1-\mu(x)} \right) = \log_2 \left( \frac{1-\mu(y')}{1-\mu(x')} \right)
\]

Hence the map is well defined. Again,
\[
(x/y)U \cdot (x'/y')U = (xx'/yy')U
\]

\[
\Rightarrow \log_2 \left[ \frac{1-\mu(yy')}{1-\mu(xx')} \right] = \log_2 \left[ \frac{1-\mu(y) - \mu(y') + \mu(y)\mu(y')}{1-\mu(x) - \mu(x') + \mu(x)\mu(x')} \right]
\]

\[
= \log_2 \left( \frac{1-\mu(y)}{1-\mu(x)} \right) \left( \frac{1-\mu(y')}{1-\mu(x')} \right)
\]

\[
= \log_2 \left( \frac{1-\mu(y)}{1-\mu(x)} \right) + \log_2 \left( \frac{1-\mu(y')}{1-\mu(x')} \right)
\]

Therefore the map is a homomorphism. The map is 1-1, for,
\[
\log_2 \left( \frac{1-\mu(y)}{1-\mu(x)} \right) = \log_2 \left( \frac{1-\mu(y')}{1-\mu(x')} \right)
\]

\[
\Rightarrow 1-\mu(x') - \mu(y) + \mu(y)\mu(x') = 1-\mu(y') - \mu(x) + \mu(x)\mu(y')
\]

\[
\Rightarrow \mu(xy') = \mu(xy) \Rightarrow (xy') = (xy) \Rightarrow xy' = uxy', u \text{ is a unit in } V
\]

\[
\Rightarrow x'/y' = u x/y \Rightarrow (x'/y')U = (x/y)U.
\]

Also the map is on to. Hence G is isomorphic to P \(\square\)

3.3.2. Definition. Let V be an integral domain and \(\mu\) be an \(\mathbb{R}\)-valuation fuzzy ideal of V. Then the group
\[
P = \left\{ \log_2 \left( \frac{1-\mu(y)}{1-\mu(x)} \right) : x, y \in V; x \neq 0, y \neq 0 \right\}
\]

is called the **value group** of \(\mu\).
3.3.3. **Theorem** [36]. Let $R$ be an integral domain and $G$ an ordered abelian group. If there exists a map $\nu_0$ from $R$ into $G$ satisfying the conditions of a valuation, then there exists a valuation $\nu$ on its quotient field $K$ with value group $G$ and valuation ring $V$ such that $\nu(a) = \nu_0(a)$ for all $a \in R$ and $R$ is a subring of $V$.

3.3.4. **Proposition.** Given an ordered subgroup $G \subseteq \mathbb{R}$, there exist an integral domain $R$ and an $\mathbb{R}$-valuation fuzzy ideal $\mu$ on $R$ having value group $G$.

**Proof.** Let $G$ be a subgroup of $\mathbb{R}$. Let $k$ be a field say $k = \mathbb{Z}_2$, the binary field. Let $R$ be the vector space over $k$ with basis $\{ x^a : a \in G, a \geq 0 \}$. Define multiplication of basis elements by $x^a \cdot x^b = x^{a+b}$ and extend this to a product operation on $R$ by linearity. Then $R$ is an integral domain, the unit element being $x^0 = 1$.

Define the map $\nu_0 : R \to G$ by
\[
\nu_0(0) = \infty \quad \text{and} \quad \nu_0(x^{a_1} + \ldots + x^{a_k}) = \min\{a_1, \ldots, a_k\}
\]

If $X = x^{a_1} + \ldots + x^{a_k}$ and $Y = x^{b_1} + \ldots + x^{b_k}$ are elements of $R$, then
\[
\nu_0(X + Y) \geq \nu_0(X) \land \nu_0(Y) \quad \text{and} \quad \nu_0(XY) = \nu_0(X) + \nu_0(Y).
\]

By theorem 3.3.3, the extension $\nu$ of $\nu_0$ to the quotient field $K$ of $R$ defined by $\nu(X/Y) = \nu_0(X) - \nu_0(Y)$ is a valuation on $K$. The valuation ring being $V = \{ X/Y : X, Y \in R; \nu(X/Y) \geq 0 \}$. The fuzzy ideal $\mu$ of $V$ defined by $\mu(X/Y) = 1 - 2^{-\nu(X/Y)}$ is an $\mathbb{R}$-valuation fuzzy ideal of $V$ with value group $G$. In fact,
\[
\left\{ \log_2 \frac{1-\mu(X_i/Y_i)}{1-\mu(X_j/Y_j)} : 0 \neq X_i/Y_i \in V, \ 0 \neq X_j/Y_j \in V \right\}
\]
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\[ \{v(X_i/Y_i) - v(X_i/Y_i) : 0 \neq X_i/Y_i \in V, \ 0 \neq X_j/Y_j \in V\} = G \]

3.4. Construction of an \( \mathbb{R} \)-valuation fuzzy ideal in a discrete valuation ring

Since discrete valuation rings are real valuation rings, they always possesses \( \mathbb{R} \)-valuation fuzzy ideals. Instead of generating \( \mathbb{R} \)-valuation fuzzy ideals from valuations, we here develop a method for finding \( \mathbb{R} \)-valuation fuzzy ideals by assigning a value to its single irreducible element.

An \( \mathbb{R} \)-valuation fuzzy ideal \( \mu \) essentially satisfy \( \mu = \{0\} \) and \( \mu(x) + \mu(y) = \mu(xy) + \mu(x) \mu(y) \) for all \( x, y \). It follows from the proof of proposition 3.2.4 that \( \mu(x) = 0 \iff x \) is a unit. Again, if \( y \in V \),

\[ \mu(0) + \mu(y) = \mu(0) + \mu(0) \cdot \mu(y). \therefore \mu(y) [\mu(0) - 1] = 0. \]

Since \( y \) is arbitrary, \( \mu(0) - 1 = 0 \), hence \( \mu(0) = 1 \).

The ideal structure of a DVR \( V \) is \( V \supset M = (a) \supset (a^2) \supset (a^3) \supset \ldots \supset (0) \) where ‘\( a \)’ is the single irreducible element of \( V \). Now,

\[ \mu(a) + \mu(a) = \mu(a^2) + \mu(a). \mu(a). \therefore \mu(a^2) = \mu(a) (1-\mu(a)) + \mu(a). \]

Again,

\[ \mu(a^2) + \mu(a) = \mu(a^3) + \mu(a^2). \mu(a). \therefore \mu(a^3) = \mu(a^2) (1-\mu(a)) + \mu(a). \]

Similarly,

\[ \mu(a^4) = \mu(a^3) (1-\mu(a)) + \mu(a) \] and \[ \mu(a^5) = \mu(a^4) (1-\mu(a)) + \mu(a). \]

In general,

\[ \mu(a^k) = \mu(a^{k-1}) (1-\mu(a)) + \mu(a), k = 1,2,3,\ldots \]

Thus to get an \( \mathbb{R} \)-valuation fuzzy ideal, fix a membership value for ‘\( a \)’. It must be less than 1, for, otherwise \( \mu(x) = 1 \ \forall x \in M \) so that \( \mu_* \neq 0 \) which
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is not possible. Then use the above recurrence formula (1) to find the values of \( \mu(a^2), \mu(a^3), \mu(a^4) \) etc. Now define \( \mu \) by

\[
\mu(x) = \begin{cases} 
0, & \text{if } x \in V - (a) \\
\mu(a^k), & \text{if } x \in (a^k) - (a^{k+1}), \ k = 1, 2, \ldots \\
1, & \text{if } x = 0.
\end{cases}
\]

To prove that \( \mu \) is an \( \mathbb{R} \)-valuation fuzzy Ideal, note that from the definition of \( \mu \) itself, \( \mu_* = \{0\} \). In order to prove that \( \mu(x) + \mu(y) = \mu(xy) + \mu(x)\mu(y) \), we first prove the following.

(i) \( \mu(a^h) - \mu(a^k) = \mu(a^{h-k})(1 - \mu(a))^{k} \); \( h,k = 1, 2, 3, \ldots \), \( h \geq k \)

(ii) \( \mu(a^k) = \mu(a)[k - (\mu(a) + \mu(a^2) + \ldots + \mu(a^{k-1}))], \)

\[ \quad k = 2, 3, 4, \ldots \]

(iii) \( \mu(a^k) = \mu(a)[1 + t + t^2 + \ldots + t^{k-1}], \) where \( t = 1 - \mu(a), \)

\[ \quad k = 2, 3, 4, \ldots \]

From the recurrence formula (1),

\[
\mu(a^h) = \mu(a^{h-1}) (1 - \mu(a)) + \mu(a) \text{ and } \\
\mu(a^k) = \mu(a^{k-1}) (1 - \mu(a)) + \mu(a).
\]

Hence,

\[
\mu(a^h) - \mu(a^k) = (1 - \mu(a)) (\mu(a^{h-1}) - \mu(a^{k-1})) \\
= (1 - \mu(a))^2 (\mu(a^{h-2}) - \mu(a^{k-2})).
\]

Proceeding like this, we get

\[
\mu(a^h) - \mu(a^k) = (1 - \mu(a))^{k-1} (\mu(a^{h-k+1}) - \mu(a)) \\
= (1 - \mu(a))^{k-1} \mu(a^{h-k})(1 - \mu(a)), \text{ using (1)} \\
= (1 - \mu(a))^k \mu(a^{h-k}). \text{ This proves (i).}
\]

Again, from the recurrence relation (1),

\[
\mu(a) = 0 \ (1 - \mu(a)) + \mu(a) \text{ and } \\
\mu(a^{i+1}) = \mu(a) (1 - \mu(a)) + \mu(a), \ i = 1, 2, \ldots, k-1.
\]

Adding these \( k \) equations, we get
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\[
\mu(a^k) = -\mu(a)[\mu(a) + \mu(a^2) + \ldots + \mu(a^{k-1})] + k\mu(a)
\]
\[
= \mu(a)[k - (\mu(a) + \mu(a^2) + \ldots + \mu(a^{k-1}))].
\]
This proves (ii).

Finally,
\[
\mu(a^2) = \mu(a)(1 - \mu(a)) + \mu(a)
\]
\[
= \mu(a)[1 + (1 - \mu(a))]
\]
\[
= \mu(a)[1 + t], \text{ where } t = 1 - \mu(a).
\]
Hence,
\[
\mu(a^3) = \mu(a^2)(1 - \mu(a)) + \mu(a)
\]
\[
= \mu(a)[1 + t]t + \mu(a)
\]
\[
= \mu(a)[1 + t + t^2].
\]
Therefore by induction,
\[
\mu(a^k) = \mu(a)[1 + t + t^2 + \ldots + t^{k-1}].
\]
This proves (iii).

Now if \(i \geq j\), we have
\[
\mu(a^i) + \mu(a^j) - \mu(a^{i+j}) = \mu(a)[i - \mu(a) + \ldots + \mu(a^{i+j-1})]
\]
\[
\mu(a)[j - \mu(a) + \ldots + \mu(a^{i+j-1})] - \mu(a)[(i + j) - \mu(a) + \ldots + \mu(a^{i+j-1})]
\]
(\text{using (iii)})
\[
= \mu(a)[\mu(a^i) + \ldots + \mu(a^{i+j-1})] - \mu(a) + \ldots + \mu(a^{i+j-1})]
\]
\[
= \mu(a)[\mu(a^i) + \mu(a^j) - \mu(a^i) - \mu(a^j) + \ldots + \mu(a^{i+j-1})]
\]
\[
= \mu(a)[\mu(a^i) + \mu(a^j)(1 - \mu(a)) + \ldots + \mu(a^j)(1 - \mu(a))]
\]
(\text{using (i)})
\[
= \mu(a)(\mu(a^i) + \mu(a^j) + \ldots + \mu(a^j) t^{i-1})
\]
\[
= \mu(a^i) \mu(a^j) + \mu(a^j) t^{i-1}
\]
\[
\mu(a^i) \mu(a^j) \text{ using (ii)}.
\]

Hence,
\[
\mu(a^i) + \mu(a^j) = \mu(a^{i+j}) + \mu(a^i) \mu(a^j)
\]
(2)

We now prove that \(\mu(x) + \mu(y) = \mu(xy) + \mu(x) \mu(y)\), for all \(x, y \in V\).
Chapter 3. Fuzzy Ideals and Real Valuation Rings

(i) If \( x, y \in V - (a) \), then \( x \) and \( y \) are units. \( \therefore xy \) is a unit and hence belongs to \( V - (a) \). By definition of \( \mu \), \( \mu(x) = \mu(y) = \mu(xy) = 0 \).
\[
\therefore \mu(x) + \mu(y) = \mu(xy) + \mu(x) \mu(y).
\]

(ii) If \( x, y \in (a^i) - (a^{i+1}) \) and \( i \geq 1 \), then \( x = ra^i \), \( y = sa^i \) where \( r \) and \( s \) are not multiples of \( a \).
\( \therefore xy = rs a^{2i} \in (a^{2i}) - (a^{2i+1}) \). Now
\[
\mu(x) = \mu(y) = \mu(a^i) \quad \text{and} \quad \mu(xy) = \mu(a^{2i}).
\]
By (2),
\[
\mu(x) + \mu(y) = \mu(a^i) + \mu(a^i) = \mu(a^{2i}) + \mu(a^i) \mu(a^i)
\]
\[
= \mu(xy) + \mu(x) \mu(y).
\]

(iii) If \( x \in V - (a) \) and \( y \in (a^i) - (a^{i+1}) \), \( j \geq 1 \), then \( x \) is a unit, hence \( xy \in (a^i) - (a^{i+1}) \). \( \therefore \mu(x) = 0 \) and \( \mu(y) = \mu(xy) = \mu(a^i) \).
\( \therefore \mu(x) + \mu(y) = \mu(xy) + \mu(x) \mu(y) \).

(iv) If \( x \in (a^i) - (a^{i+1}) \) and \( y \in (a^i) - (a^{i+1}) \) and \( i > j \), then \( x = ra^i \) and \( y = sa^j \) where \( r \) and \( s \) are not multiples of \( a \).
\( \therefore xy = rs a^{i+j} \in (a^{i+j}) - (a^{i+j-1}) \).

Now
\[
\mu(x) = \mu(a^i), \mu(y) = \mu(a^j) \quad \text{and} \quad \mu(xy) = \mu(a^{i+j}).
\]
By (2),
\[
\mu(x) + \mu(y) = \mu(a^i) + \mu(a^j) = \mu(a^{i+j}) + \mu(a^i) \mu(a^j)
\]
\[
= \mu(xy) + \mu(x) \mu(y).
\]

Finally, to prove that \( \mu(y) \leq \mu(x) \) \( \Rightarrow x/y \in V \), let \( x, y \in V \). If \( \mu(y) < \mu(x) \), then there exist \( i \) and \( j \) with \( i < j \) such that \( y \in (a^i) - (a^{i+1}) \) and \( x \in (a^j) - (a^{j+1}) \). Now \( y = ra^i \) and \( x = sa^j \) where \( r \) and \( s \) are not multiples of \( a \), hence units. \( \therefore x/y = (s/r) a^{i-j} \in V \). If \( \mu(y) = \mu(x) \), then \( x, y \in (a^i) - (a^{i+1}) \) for some \( i \). \( \therefore y = ra^i \) and \( x = sa^j \) where \( r \) and \( s \) are units, hence \( x/y = s/r \in V \). Thus \( \mu \) is an \( \mathbb{R} \)-valuation fuzzy ideal \( \blacksquare \).
3.4.1. Example. Consider the localization $\mathbb{Z}_{(2)}$ of the ring of integers $\mathbb{Z}$. Then $V = \mathbb{Z}_{(2)} \supset (2/1) \supset (2^2/1) \supset (2^3/1) \supset \ldots \supset (0)$ is a discrete valuation ring.

Take
\[
\mu(2/1) = 1/2, \quad \mu(2^2/1) = (1/2)(1 - 1/2) + 1/2 = 3/4,
\]
\[
\mu(2^3/1) = 3/4(1 - 1/2) + 1/2 = 7/8. \text{ etc.}
\]

Define $\mu$ by
\[
\mu(x) = 0, \text{ if } x \in \mathbb{Z}_{(2)} - (2/1)
\]
\[
= (2^i - 1)/2^i, \text{ if } x \in (2^i/1) - (2^{i+1}/1), i = 1, 2, \ldots \text{ and}
\]
\[
\mu(0) = 1. \text{ Then } \mu \text{ is an } \mathbb{R}\text{-valuation fuzzy ideal on } V.
\]

3.4.2. Remark. Arbitrarily fixing a value for $\mu(a)$ as in example 3.4.1, we can define infinitely many $\mathbb{R}$-valuation fuzzy ideals in a DVR. Equivalently, we can define infinitely many valuations.