4.1 Introduction:
Matrices and Determinants are important concept of Linear Mathematics.

Matrix:
A matrix is a rectangular grid of numbers, symbols or expressions that is arranged in a row or column format enclosed in square or curved brackets.
A matrix of order m × n is a matrix which contains m number of rows and n number of columns written as

\[
a_{11}a_{12}…………a_{1n}a_{21}a_{22}………….a_{2n}:………….:am_{1}am_{2}…………..am_{m}n\times n
\]

Different types of Matrices:
Rectangular Matrix:
A matrix containing m number of rows and n number of columns. A unique kind of matrix that follows the rule m ⊗ n, is said to be of Rectangular type of order m × n.

Examples of Rectangular Matrices of order 3 × 4, 4 × 3, 3 × 2, and 2 × 3:
Square Matrix:
Those kinds of numerical arrangements that strictly satisfy the constraint \( m = n \) throughout qualify to be named as Square matrices of order \( n \).
Examples of Square Matrices of 2\textsuperscript{nd}, 3\textsuperscript{rd} & 4\textsuperscript{th} order respectively, \[\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\]
\[\begin{array}{cccc}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{array}\]

Determinants:
The determinant is a unique number associated with each square matrix and it cannot be found in any other type of matrix. i.e. For finding determinant, number of rows and columns must be equal.

Examples of 2\textsuperscript{nd}, 3\textsuperscript{rd} and 4\textsuperscript{th} order determinant:

By using the method for finding 2\textsuperscript{nd} order determinant third and fourth order determinant can be obtained easily.

\[\begin{array}{ccc}
abcpqr & 2\times3; & abcpqr\text{rlnm} & 3\times3; & abcdpq\text{rsklmnefg} & 4\times4
\end{array}\]

• The determinant has definite value.
• It can be also find out by using the Vertically & Crosswise Sūtra of Vedic Mathematics.
Difference between Matrix and Determinants:
The main difference is that Matrix is an array of numbers and Determinant is a single number. The determinant has definite value while matrix has no definite value. A matrix is a function, mapping the Cartesian product (e.g. real or complex number); while determinant is a certain scalar associated with the given matrix using a particular method.

4.2 Evaluation of determinant of matrices:
The necessary condition before evaluation dictates that the matrices must be of square type whereas for rectangular ones, determinant value is not defined. The determinant of a square matrix (say Z) will be denoted by det (Z) or Z.

4.2.1 Determinant of third order matrix:
Find the determinant of 3 x 3 matrix A= 3 21-5 1 0 3-14

A=det(A)= 3 2 1-5 1 0 3-1 4=3 10-14-2-50 34+1-5 1 3-1

Φ Det (A) = 3(4 - 0) - 2 (-20 - 0) +1(5 - 3)
= 3(4) - 2(-20) + 1(2)
Determinant of 3\textsuperscript{rd} order matrix by Vertically & Crosswise Sūtra:

A third order determinant is a formula reducing $3^2 = 9$ figures to one. By using Vertically & Crosswise Sūtra determinant can be solved easily. For finding third order determinant first separate the first row (column) from the second and third row (column) by partition. Three second order determinants arise from the second and third row, out of them that elements spaced one row (column) apart contains negative sign: adjacent elements lead to a positive sign. For third order determinant out of the three second order determinants calculated after partitioning, the sign of second is negative.

Here, after partitioning first row from the second and third row calculated three second order determinants are

\[-5 \ 1 \ 3-1=2, -50 \ 34= -20 \ & \ 10-14=4\]

The result of three second order determinants are noted in a fourth row.

\[A = \det(A) = 3 \ 2 \ 1-5 \ 1 \ 0 \ 3-1 \ 4 \ + \ - \ + = 3 \ 2 \ 1-5 \ 1 \ 0 \ 3-1\]

\[4 \ ......................... \ 2-204\]

The multiplication pattern of partitioned row and fourth new row is as follows:
With the help of V & C Sutra from the figure above,

\[ \text{Det} (A) = 3(4) - 2(-20) + 1(2) = 54 \]

We can find third order determinant by separating any one row from the remaining other two rows. We can also find third order determinant by separating any one column from the remaining other two columns. The multiplication pattern of partitioned column and fourth column becomes

\[
\begin{vmatrix}
3 & 2 & 1 \\
2 & -20 & 4 \\
\end{vmatrix}
\]

Figure: 4.1

The multiplication pattern of partitioned column 3 with previous to 1st column new column is as follows:

\[
\begin{array}{ccc}
13 & 1 \\
-9 & 0 \\
2 & 4 \\
\end{array}
\]

From the above figure with the help of V & C Sutra,

\[ \text{Det} (A) = + (13)(4) - (-9)(0) + (2)(1) = 52 + 2 = 54 \]
4.2.2 **Determinant of Fourth order matrix:**

Find the determinant of $4 \times 4$ matrix $A=4557768822233494$

$A=\det(A)=4557768822233494$

$A=4688223494-5788223394+5768223344-7768222349$ \[P\]

Where,

$688223494=62394-82344+82249$

$= 6 (8 – 27) – 8 (8 – 12) + 8 (18 – 8)$

$= (-19) – 8 (- 4) + 8 (10)$

$= -114 + 32 + 80 = -2$
788223394 = 72394 - 82334 + 82239

\[ = 7 (8 - 27) - 8 (8 - 9) + 8 (18 - 6) \]
\[ = -133 + 8 + 96 = -29 \]

768223344 = 72344 - 62334 + 82234

\[ = 7 (8 - 12) - 6 (8 - 9) + 8 (8 - 6) \]
\[ = 7 (-4) - 6 (8 - 9) + 8 (8 - 6) \]
\[ = -28 + 6 + 16 = -6 \]

768222349 = 72249 - 62239 + 82234

\[ = 7 (18 - 8) - 6 (18 - 6) + 8 (8 - 6) \]
\[ = 7 (10) - 6 (12) + 8 (2) \]
\[ = 70 - 72 + 16 = 14 \]

Substituting the above values in [P] we get,

\[ \text{Det (A)} = 4 (-2) - 5 (-29) + 5 (-6) - 7 (14) = 8 + 145 - 30 - 98 = 9 \]
Evaluation of Determinant of Fourth order matrix by using vertically and crosswise Sūtra:

To find fourth order determinant by using vertically & crosswise Sūtra partitioning centrally horizontally, six second order determinants above the partition and the six second order determinants below the partition are obtained. The result of six second order determinants above the partition are placed above the determinant in a row, the result of six second order determinants below the partition are placed below the determinant in a row. The associated signs of both the new rows obtained from the six second order determinants are placed as below:

For the determinant of two adjacent columns ‘+’ sign; for the determinant of the column with column one space apart ‘-’ sign; for the determinant of the column with the column two space apart ‘+’ sign.

The two outermost rows of figures are multiplied together according to the pattern:

\[
\begin{array}{cccccc}
-11 & -3 & -17 & 10 & -2 & -16 \\
4 & 5 & 5 & 77 & 6 & 8 \\
2 & 12 & -1 & 10 & -4 & -19
\end{array}
\]

By using Vertically & Crosswise Sutra,

\[
\begin{array}{ccccccc}
+ & - & + & + & - & + \\
-11 & -3 & -17 & 10 & -2 & -16
\end{array}
\]

\[
\begin{array}{cccccc}
2 & 12 & -1 & 10 & -4 & -19 \\
+ & - & + & + & - & +
\end{array}
\]

Figure: 4.3

From the above figure with the help of V & C Sūtra,
\[ \text{det}(A) = +(-11)(-19) - (-3)(-4) + (-17)(10) + (10)(-1) - (-2)(12) + (-6)(2) \]

\[ = 209 - 12 - 170 - 10 + 24 - 32 = 9 \]

Similarly, to evaluate the fourth order determinant by partitioning centrally vertically, six order determinants are obtained right side of the partitions and six order determinants are obtained left side of the partitions.

The above example can be solved and compared the result by partitioning centrally vertically with the result by partitioning centrally horizontally.

Evaluation of the determinant of 4×4 matrix by centrally vertically partitioning.

\[ A=\begin{pmatrix} 4 & 5 & 7 & 6 \\ 8 & 2 & 2 & 3 \\ 8 & 2 & 2 & 3 \\ 4 & 5 & 7 & 6 \end{pmatrix} \]

\[ A=\text{det}(A)=4557768822233494 \]

\[ A=\text{det}(A)=4557768822233494 = -11 \cdot 2 + 1 + 2 + 10 + 2 + 4557768822233494-16 + 1-43 + 8-40-19 \]

By using Vertically & Crosswise Sūtra:
From the above figure,

\[
\det(A) = (-11)(-19) - (-2)(-40) + (1)(8) + (2)(-43) - (10)(1) + 2(-16)
\]

\[
= 209 - 80 + 8 - 86 - 10 - 32 = 9
\]

A pyramid layout of second-order sub-determinants:

(Centrally horizontal partition):
Figure: 4.5

Multiplication Pattern:

c = -17
b = -3
e = -2
a = -11  ................ d = 10 ........... f = -16

p = 2 .... s = 10 .... u = -19

q = 12 ........ t = -4

r = -1

Figure: 4.6

.: Det (A) = (-11)(-19) + (-16)(2) - (-3)(-4) - (-2)(12) + (-17)(10) + (10)(-1)
Also by partitioning vertically six second order determinants are obtained and find the determinant similarly.

4.3 Extraction of elements: (Expansion by individual elements):
The procedure using By Alternate Elimination and Retention Sūtra:
First eliminate the element, then everything else but that element and its expansion (the terms that go with it).

Eg: Extract the appropriate elements from:

\[ X = 3425987214 \text{ (Solve it by extracting one element 98)} \]

It can be written after extracting 98,

\[ X = 342507214 + 983224 \quad \ldots \quad [I] \]

Where,

\[ 342507214 = 342507214 \]

For the first and last row by using Vertically & Crosswise Sūtra,
\[ 3(-7) - 4(6) + 2(5) = -21 - 24 + 10 = -35 \]

And

\[ 983224 = 9812 - 4 = 98(8) = 784 \]

By substituting both the values in [I]
The final answer = -35 + 784 = 749

[4]

\[ Y = 58234216767 \quad (\text{Solve it by extracting two elements 58 and 67}) \]

After extracting 58 & 67 given determinant can be written as,

\[ Y = 023421670 + 582170 + 670242 + 58 \times 67 \times 2 \quad \ldots \quad [II] \]
Substituting the above values in (II),

Final answer = $60 + 58 \cdot (-7) + 67 \cdot (-8) + 58 \times 67 \times 2$

$= 60 - 406 - 536 + 7772 = 7832 - 942 = 6890$
$Z = 482326992532114433120$ (Solve by extracting three elements 48, 99 and 111)

After extracting 48, 99 & 111

$$Z = 023260253204433120 + 48 \times 02520433120 + 99 \times 03230443120 + 111 \times 02260543120$$

$$+ 48 \times 99 \times 043120 + 48 \times 111 \times 053120 + 99 \times 111 \times 024120$$

$$+ 48 \times 99 \times 111 \times 120 \quad \text{[III]}$$

Where,

$$02520433120 = 02520433120 = 0 (-12) -2(228) + 5(6) = -456 + 30 = -426,$$
\[
\begin{vmatrix} 6 & 228 & -12 \\
\end{vmatrix}
\]

\[
03230443120 = 03230443120 = 0 \times (12) - 3 \times (344) + 2 \times (9) = -1032 + 18 = -1014,
\]

\[
\begin{vmatrix} 9 & 344 & -12 \\
\end{vmatrix}
\]

\[
02260543120 = 02260543120 = 0 \times (-15) - 2 \times (700) + 2 \times (18) = -1400 + 36 = -1364,
\]

\[
\begin{vmatrix} 18 & 700 & -15 \\
\end{vmatrix}
\]

\[
043120 = -12, \quad 053120 = -15, \quad 024120 = -8
\]

The fourth order determinant can be obtained by using pyramid method by centrally horizontal partition.

\[
\begin{vmatrix} -12 & \\
18 & -10 \\
-12 & 4 \\
11 & \\
\end{vmatrix}
\]

\[
023260253204433120
\]
1   6   -12
-9  -228
344

Figure: 4.7

From the above figure by using Vedic Vertically & Crosswise Sūtra,

\[
\begin{align*}
\therefore \text{Det} &= (-12) (-12) + 11(1) - (-10) (-9) - (18) (-228) + (-12) (6) + 344 (4) \\
&= 144 + 11 - 90 + 4104 - 72 + 1376 \\
&= 5635 - 162 = 5473
\end{align*}
\]

Final answer = 5473 + 48(-426) + 99(-1014) + 111(-1364) + 4752(-12)
+ 5328 (-15) + 10989(-8) + 63296640
\[
= 5473 - 20448 - 100386 - 151404 - 57024 - 79920 - 87912 + 63296640
\]
\[
= 63302113 - 497094
\]
\[
= 62805019
\]

A special case:

A special case comes under Sunyam Samyasmuccaye Sūtra

[6]

Solve: \( D = 491162 \ 16 \ 9213 \)
4.4 Applications of Determinants:
4.4.1 Use of Determinant in solving Simultaneous Linear Equations:

Solve: 3x + 2y = 5 ....... (1)
4x + y = 3........... (2)

In “Vedic Mathematics” the following method is given for solving two equations in two unknowns.

x=2.3-5.14.2-3.1=15 & y=5.4-3.34.2-3.1=115
By using Cramer’s Rule,

\[ x = \frac{-25133241}{-1-5} = 15 \quad \& \quad y = \frac{35433241}{-11-5} = 115 \]

Another Method:

By using equations (1) and (2), we get two rows of a third order determinant

\[ 325413 \]

From this three second order determinant give, on sorting out the signs, \( x \cdot D \), \( y \cdot D \) and \( D \), where

\[ D = 3241 \]

\( x \cdot D \) is given by the determinant arising on missing out the first column, \( -y \cdot D \) from missing out the second column and \( -D \) from missing out the third. This leads us to consider the third order determinant:

\[ \det A = x \cdot D - y \cdot D - 325413 \]
On evaluating the determinant the coefficient of D, x.D and y. D give the respective value of D, x. D and y. D  Thus, Det A = x. D + 11y. D + 5D

Hence, D equals five and the product of x & y with D has values of unity & eleven respectively. Whence x = 1/5 and y = 11/5.

These procedure we can also apply for solving ‘n’ equations in ‘n’ unknowns.

Solve the following simultaneous linear equations:

\[\begin{align*}
3x + 2y + z &= 10 \\
5x + 3y + 2z &= 17 \\
7x + 8y + z &= 26
\end{align*}\]

Consider the fourth order determinant

\[
\begin{vmatrix}
x.D & y. D & z. D & -D \\
-1 & 3 & 2 & 1 & 10 & -3 \\
-10 & 5 & 3 & 2 & 17 & -16 \\
19 & 7 & 8 & 1 & 26 & 35
\end{vmatrix}
\]

For finding D: Covering the fourth column, we evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure. We have

\[
\begin{vmatrix}
-1 & 1 \\
-10 & 2 \\
19 & 1
\end{vmatrix}
\]

\[\text{Figure: 4.8}\]

\[\text{i.e. } -1 \times 1 + -10 \times 2 + 19 \times 1 = -1 - 20 + 19 = -2\]
To evaluate $x$. D cover the first column and evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure.

\[
\begin{array}{cc}
2 & -3 \\
3 & -16 \\
8 & 35 \\
\end{array}
\]

Figure: 4.9

i.e. $2 \times 35 + 3 \times (-16) + 8 \times (-3) = 70 - 48 - 24 = -2$

To evaluate $-y$. D cover the second column and evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure.

\[
\begin{array}{cc}
3 & -3 \\
5 & -16 \\
7 & 35 \\
\end{array}
\]

Figure: 4.10

i.e. $3 \times 35 + 5 \times (-16) + 7 \times (-3) = 105 - 80 - 21 = 4$
To evaluate $z$. D cover the third column and evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure.

\[
\begin{array}{cc}
-1 & 10 \\
-10 & 17 \\
19 & 26 \\
\end{array}
\]

Figure: 4.11

\[
\text{i.e. } (-1) \times 26 + (-10) \times 17 + 19 \times 10 = -26 - 170 + 190 = -6
\]


∴ The final solution is $x = 1$, $y = 2$ & $z = 3$

**Procedure for eliminating one variable or two variables at a time:**

Determinants arise on eliminating unknowns from simultaneous equations: second order determinant on eliminating one unknown, and third order on eliminating two unknowns, etc.

[9]

Solve the following:

\[
4x + 8y = 20
\]

\[
2x + 5y = 12
\]
Eliminating variable $x$:

$$4825y = 420212$$

Eliminating variable $y$:

$$-4825x = 820512 \Rightarrow (20-16)y = (48-40) \Rightarrow -(20-16)x = (96-100)$$

$$16)(x = (96-100) :. 4y = 8 :. y = 2$$

$$\therefore -4x = -4 \Rightarrow x = 1$$

Ans. \(x = 1, y = 2\)

[10]

Solve the following:

1. \[4x + 2y + 2z = 22 \quad \text{(1)}\]
2. \[2x + 4y + 7z = 38 \quad \text{(2)}\]
3. \[8x + 4y + 2z = 36 \quad \text{(3)}\]

Eliminating $x$ from (1) and (2)

$$4224y + 4227z = 422238$$
\[(16-4)y + (28-4)z = (152-44)\]

\[\therefore 12y + 24z = 108 \quad \ldots (4)\]

Eliminating \(y\) from (1) and (2)

\[-424x + 2247z = 222438\]

\[\therefore -(16-4)x + (14-8)z = (76-88)\]

\[ \therefore -12x + 6z = -12 \quad \ldots (5)\]

Eliminating \(x\) from (2) and (3)

\[
\begin{array}{cccc}
2 & 48 & 4y+2 & 78 & 2z = 238836
\end{array}
\]
\[
(8-32)y + (4-56)z = (72-304) \\
\Rightarrow -24y - 52z = -232 \quad \ldots (6)
\]

Solving equations (4), (5) and (6)

We get \( x = 3, \ y = 1, \ z = 4 \).

4.4.2 Non zero solution of Linear Homogeneous Equation:

The homogeneous equation in \( x, \ y, \ z \) are

\[
a_1 x + a_2 y + a_3 z = 0 \\
b_1 x + b_2 y + b_3 z = 0 \\
c_1 x + c_2 y + c_3 z = 0
\]

A system of simultaneous equations have non-zero value or trivial solutions if all the unknowns have zero values, and is said to have non-zero solution if at least one of the unknown has the non-zero value.

The necessary and sufficient condition that the equations have non-zero solution is

\[a_1a_2a_3b_1b_2b_3c_1c_2c_3=0\]
Test whether the following equations have non-zero solution. If they have such solution, obtain the solutions.

\[\begin{align*}
  x + 2y - 3z &= 0 \\
  3x - 2y - z &= 0 \\
  2x + 2y - 4z &= 0
\end{align*}\]

The necessary and sufficient condition that the equations have non-zero solution is

\[\begin{vmatrix}
  1 & 2 & -3 & -2 & 12 & -2 & -4 \\
\end{vmatrix} = 0\]

Here, \[\begin{vmatrix}
  1 & 2 & -3 & -2 & 12 & -2 & -4 \\
\end{vmatrix} = 10 \cdot -10 - 10 \cdot 10 = 0\]

By using Vertically & Crosswise Sūtra,

\[\begin{align*}
  \therefore \text{Det} &= 1(10) - 2(-10) - 3(10) = 0 \text{ and hence the equations have non-zero solution.}
\end{align*}\]

Solving first two equations,
\[ x + 2y - z = 0 \]

\[ 3x - 2y - z = 0 \]

\[ \therefore x - 8 = -y - 8 = z - 8 \]

So that \( x = -8\lambda, y = -8\lambda, z = -8\lambda \); where \( \lambda \) is a non-zero solution.

These values satisfy the third equation,

\[ 2x + 2y - 4z = 2(-8\lambda) + 2(-8\lambda) - 4(-8\lambda) \]

\[ = -16\lambda - 16\lambda + 32\lambda = 0. \]

Hence, they are non-zero solution.
2u + 3v + 4w = 0
u – 2v – 3w = 0
3u + v – 8w = 0

The necessary and sufficient condition that the equations have non-zero solution is 23 41-2-331-8 = 0

By using Vertically & Crosswise Sūtra,

\[ \text{Det} = 2(19) - 3(1) + 4(7) = 38 - 3 + 28 = 63 \neq 0 \]

Hence, the equations have no non-zero solution.

u = v = w = 0

4.5 Adjoint of a Matrix:

Minors:
Consider the determinant $A = a_{11}a_{12}a_{13}a_{21}a_{22}a_{23}a_{31}a_{32}a_{33}$

If the element $a_{ij}$ leave the row and column passing through the element $a_{ij}$. Minor of is $a_{ij}$ is written as $M_{ij}$.

The minor of the element $a_{12} = a_{21}a_{23}a_{31}a_{33} = M_{12}$

The minor of the element $a_{23} = a_{11}a_{12}a_{31}a_{32} = M_{23}$

The minor of the element $a_{33} = a_{11}a_{12}a_{21}a_{22} = M_{33}$

**Co-factors:**

The minor multiply by $(-1)^{i+j}$ is called the cofactor of the element $a_{ij}$. 
The Co-factor of the element $a_{12} = A_{12}$

$A_{12} = -1 + 2M_{12}$

$= (-1)^1 + 2a_{21}a_{23}a_{31}a_{33} = (-1)^3a_{21}a_{23}a_{31}a_{33} = -a_{21}a_{23}a_{31}a_{33}$

The Co-factor of the element $a_{23} = A_{23}$

$A_{23} = -1 + 3M_{23}$

$= (-1)^2 + 3a_{11}a_{12}a_{31}a_{32} = (-1)^5a_{11}a_{12}a_{31}a_{32} = -a_{11}a_{12}a_{31}a_{32}$
The Co-factor of the element $a_{33} = A_{33}$

$A_{33} = -13 + 3M_{33}$

$= (-1)6a_{11}a_{12}a_{21}a_{22} = a_{11}a_{12}a_{21}a_{22}$

**Adjoint of a Matrix:**

Adjoint of a Matrix is obtained by first finding a matrix formed by the co-factors of the elements of the given matrix $A$ and then by taking transpose of it.

If $A = a_{11}a_{12}a_{13}a_{21}a_{22}a_{23}a_{31}a_{32}a_{33}$ then

$\text{Adj} (A) = \text{transpose of the matrix formed by co-factor}$
\[ \text{Adj}(A) = A_{11} A_{21} A_{31} A_{12} A_{22} A_{32} A_{13} A_{23} A_{33} \]

Where, \( A_{ij} \) is the Co-factor of the element \( a_{ij} \).

If \( A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix} \) is given matrix, then find its Adjoint.

\[ A_{11} = \text{the co-factor of } a_{11} \text{ in} \]

\[ A = (-1)1+1434=(-1)2434=4334=16-9=7 \]
A12 = the co-factor of a12 in

\[ A = (-1)^{1+2} \cdot 3 = (-1)^{3} \cdot 3 = -3 = -(3+6) = -9 \]

Similarly,

\[ A_{13} = -5, \ A_{21} = -4, \ A_{22} = 1, \ A_{23} = 3, \ A_{31} = -5, \ A_{32} = 4, \ A_{33} = 1 \]

Adj (A) = transpose of the matrix formed by co-factor

\[ \text{Adj}(A) = A_{11}A_{21}A_{31}A_{12}A_{22}A_{32}A_{13}A_{23}A_{33} = 7 - 4 - 5 - 9 + 14 + 531 \]

Where, Aij is the Co-factor of the element aij.
\[ \text{Adj}(A) = 7 - 4 - 5 - 9 - 14 - 5 - 1 \]

### 4.6 Square inverse matrices:

It is mandatory for a matrix to have any determinant value other than zero so as to have its inverse in mathematical existence a possibility. Therefore, all matrices must fulfill this constraint of non singularity.

\[ A^{-1} = \text{adj}AA; A \neq 0 \]

**[14]**

Find Inverse of the matrix of order 3x3 by finding its adjoint

\[
B = \begin{bmatrix} 4 & 9 & 16 \\ 2 & 1 & 9 \\ 2 & 1 & 9 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 4 & 91621921 \\ 2 & 1 & 9 \\ 2 & 1 & 9 \end{bmatrix}
\]

\[
B = 491621921
\]

\[
\begin{bmatrix} -6 & -3 & 0 \end{bmatrix}
\]
By V & C Sūtra from the figure above,

\[ \begin{align*}
\therefore \text{Det } B &= 4(0) - 9(-3) + 1(-6) = 27 - 6 = 21 \text{ which is non-zero.} \\
\end{align*} \]

Since \(B\neq 0\), \(B\) is non-singular. Therefore, Inverse of \(B\) exits.

For finding co-factor of \(B\), find all the minors of matrix \(B\),

\[ \begin{align*}
\text{Minor of 4} &= (-1)^{1+1} = 0 \\
\text{Minor of 9} &= (-1)^{1+2} = 3 \\
\text{Minor of 1} &= (-1)^{1+3} = -6
\end{align*} \]
Minor of $6 = (-1)^{2/1}$

Minor of $2 = (-1)^{2/2}$

Minor of $1 = (-1)^{2/3}$

Minor of $9 = (-1)^{3/1}$

Minor of $2 = (-1)^{3/2}$
Minor of 1 = (-1)^3 = -1

Co-factor of B = 03-6-7-57372-46

Transforming rows into columns of above matrix Adjoint matrix can be obtained as follows:

Adjoint of B = 0-773-52-673-46

∴ B^-1 = adjBB; B ≠ 0

∴ B^-1 = 1210-773-52-673-46
Find Inverse of the matrix of order 4x4 by finding its adjoint

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 1 & 4 \end{bmatrix} \]

\[ A = \det(A) = 1001021221012014 \]

By using Pyramid method by centrally horizontal partition very easily.

\[ \begin{bmatrix} 2 \\ 1 & -2 \\ 2 & 0 & -1 \end{bmatrix} \]

\[ 1001021221012014 \]

\[ \begin{bmatrix} -2 & 1 & -1 \\ 2 & 4 \\ 6 \end{bmatrix} \]

Figure: 4.12

From the figure with the help of Vertically & Crosswise Sūtra,
\[ \det (A) = 2 (-1) + (-1) (-2) + 1(4) + (-2) (2) + 2 (1) + 0 (6) \]

\[ = -2 + 2 + 4 – 4 + 2 = 2 \]

To find co-factor of A, taking minors of all the elements of matrix A

\[ \begin{vmatrix} 1 & 4 & -1 \\ 2 & 6 & -1 \end{vmatrix} \]

\[ \text{Minor of } a_{11} = 1 = (-1)^{1+1} = (-1)^2 [2 (-1) -1(4) + 2(1)] = -2 - 4 + 2 = -4 \]

\[ \begin{vmatrix} 1 & 4 & -1 \\ 2 & 6 & -1 \end{vmatrix} \]

\[ \text{Minor of } a_{12} = 0 = (-1)^{1+2} = (-1)^3 [0(-1)-1(6) + 2(2)] = -[0 - 6 + 4] = 2 \]

\[ \begin{vmatrix} 1 & 4 & -1 \\ 2 & 6 & -1 \end{vmatrix} \]

\[ \text{Minor of } a_{13} = 0 = (-1)^{1+3} = (-1)^4 [0 - 2(6) + 0] = 12 \]

\[ \begin{vmatrix} 1 & 4 & -1 \\ 2 & 6 & -1 \end{vmatrix} \]
Minor of $a_{14} = 1 = (-1)^{1+4}$

$= (-1)^5[0 - 2(2) + 1(-2)] = -[-4 - 2] = 6$

\[-2 \quad 2 \quad 2\]

Minor of $a_{21} = 0 = (-1)^{2+1}$

$= (-1)^3[0 + 0 + 1(1)] = -1$

\[1 \quad 4 \quad -1\]

Minor of $a_{22} = 2 = (-1)^{2+2}$

$= (-1)^4[1(-1) - 0(6) + 1(2)] = -1 - 0 + 2 = 1$

\[2 \quad 6 \quad -1\]

Minor of $a_{23} = 1 = (-1)^{2+3}$

$= (-1)^5[1(4) - 0 + 1(-2)] = -[4 - 0 - 2] = -2$

\[-2 \quad 6 \quad 4\]

Minor of $a_{24} = 2 = (-1)^{2+4}$

$= (-1)^6[1(1) - 0 + 0] = 1$
Minor of $a_{31} = 2 = (-1)^{3 \cdot 1} = (-1)^4 [0 - 0 + 1(2)] = 2$

Minor of $a_{32} = 1 = (-1)^{3 \cdot 2} = (-1)^5 [1(2) - 0 + 1(-2)] = -[2 - 2] = 0$

Minor of $a_{33} = 0 = (-1)^{3 \cdot 3} = (-1)^6 [1(8) - 0 + 1(-4)] = 8 - 4 = 4$

Minor of $a_{34} = 1 = (-1)^{3 \cdot 4} = (-1)^7 [1(2) - 0 + 0] = -2$
\[
\begin{align*}
\text{Minor of } a_{41} &= 2 = (-1)^{4+1} = (-1)^5 [0 - 0 + 1(-1)] = - (-1) = 1 \\
&= 
\begin{pmatrix}
-1 & 0 & 1 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{Minor of } a_{42} &= 0 = (-1)^{4+2} = (-1)^6 [1(-2) - 0 + 1(-2)] = -2 - 2 = -4 \\
&= 
\begin{pmatrix}
-2 & -4 & 1 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{Minor of } a_{43} &= 1 = (-1)^{4+3} = (-1)^7 [1(0) - 0 + 1(-4)] = -(-4) = 4 \\
&= 
\begin{pmatrix}
-4 & -4 & 0 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{Minor of } a_{44} &= 4 = (-1)^{4+4} = (-1)^8 [1(-1) - 0 + 0] = -1 \\
&= 
\begin{pmatrix}
-4 & -2 & -1 \\
\end{pmatrix}
\end{align*}
\]
Co-factors of A = -42 - 126 - 11 - 21 - 204 - 21 - 44 - 1

Adjoint of A = -4 - 121 - 20 - 4 - 12 - 24 - 46 - 1 - 2

A⁻¹ = adj AA

Also determinant of A is 2.

\[ A⁻¹ = \begin{bmatrix} 12 & 4 & -12 & 1 & 2 & 10 & -4 & 12 & -24 & 4 & 6 & 1 & -2 & -1 \end{bmatrix} \]

4.7 Rank of Algebraic Equation:

Sub Matrix:
Sub matrix of any matrix is obtained by omitting specific vertical & horizontal number arrangements from original matrix of order $m \times n$.

E.g. \[
\begin{array}{ccccccc}
6 & 04 & 4-2 & 148 & 18 & 14-140-10 \\
\end{array}
\]

Contains four 3$^{rd}$ order sub matrices,

\[
\begin{array}{ccccccc}
604-2 & 148 & 14-140, & 04 & 4 & 148 & 18-140-10, & 64 & 4-28 & 18 & 140-10, & 604-21418 \\
14-14-10 \\
\end{array}
\]

Contains fifteen 2$^{nd}$ order sub matrices,

\[
\begin{array}{ccccccc}
60-214,6014-14,-21414-14,04148,04-140, & & & & & & & & & & \\
148-140,44818, 440-10,8180-10,64-28, & & & & & & & & & & \\
\end{array}
\]
Contains three row matrix 6044,-214818,14-140-10

Contains column matrix 6-214,014-14,480,418-10 and so on.

**Rank of a Matrix:**

We can say that the rank of a matrix is $r$ if,

When starting from smallest sub matrices, all the determinants result in trivial solution up until the sub matrix of order $r$, beyond which determinant of higher sub matrices are of non trivial significance.

[16]

Evaluate rank:

6044 4-2148 1814-140-10
The matrix is of order 3 x 4.

Hence, rank of the given matrix i.e. $\rho(A) \leq 3$.

The matrix contains four 3x3 matrices

\[
\begin{pmatrix} 6 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \end{pmatrix} ; \quad \begin{pmatrix} 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 8 & 4 \end{pmatrix} ; \quad \begin{pmatrix} 6 & 4 & 4 \end{pmatrix} \begin{pmatrix} -2 & 8 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 1 \end{pmatrix} ; \quad \begin{pmatrix} 6 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 10 & -1 \end{pmatrix} ;
\]

The determinant of all these are zero.

Hence, rank of the given matrix is less than or equal to 2 i.e. $\rho(A) \leq 2$

\[
\begin{array}{ccc}
60 & -2 & 14 = 84 \\&\neq 0 : \cdot \rho(A) = 2
\end{array}
\]
∴ Rank of the given matrix is 2.

4.8 Consistency of Linear Equations:

Consider a set of equations
\begin{align*}
a_1x + a_2y + a_3z &= d_1 \\
b_1x + b_2y + b_3z &= d_2 \\
c_1x + c_2y + c_3z &= d_3.
\end{align*}

The matrix arrangement of these three relationships are displayed:

\[
a_{1a} a_{2a} a_{3a} b_{1b} b_{2b} b_{3b} c_{1c} c_{2c} c_{3c} x y z = d_{1d} d_{2d} d_{3d}
\]

If we join the matrices i.e. \([A:D] = a_{1a} a_{2a} a_{3a} b_{1b} b_{2b} b_{3b} c_{1c} c_{2c} c_{3c} : :: d_{1d} d_{2d} d_{3d}\).

It is called as Augmented Matrix. Reduce \([A: D]\) to Echelon form, and find rank of \(A\) and rank of \([A: D]\).
If \( \rho(A) \neq \rho(AD) \), then result cannot be obtained.

If \( \rho(A) = \rho(AD) \), then the system is consistent and

If \( \rho_A = \rho_{AD} = \text{No. of unknowns} \) we can say that all equations are consistent & its final result must be distinct.

If \( \rho_A = \rho_{AD} < \text{Number of unknowns} \) then the system is consistent and has infinitely many solution.

[17]
Discuss the consistency of
\[
x + y + z = 1,
2x + 4y - 3z = 9,
3x + 5y - 2z = 11
\]
In the matrix form
\[
A = 11 \begin{bmatrix} 1 & 2 & -3 & 5 & -2 \end{bmatrix}, \quad [A:D] = 11 \begin{bmatrix} 12 & 3 & 5 & 2 \end{bmatrix} \quad \text{:::} \quad 1911
\]
First we will find Rank of A,

Since \[
\begin{bmatrix}
1 & 1 & 2 \\
-3 & 3 & 5 \\
-2 & -2 & 2
\end{bmatrix}
= 1 \cdot (7) - 1 \cdot (5) + 1 \cdot (-2) = 7 - 5 - 2 = 0
\]

\[-2 \quad 5 \quad 7\]

i.e. \[
\begin{bmatrix}
1 & 1 & 2 \\
-3 & 3 & 5 \\
-2 & -2 & 2
\end{bmatrix}
= 0.
\]

Therefore, Rank (A) must be less than or equal to two.

Since \[
1124=4-2=2\neq0\therefore \rho(A)=2
\]

Now find rank of augmented Matrix [A: D]

\[
\begin{bmatrix}
1 & 1 & 24-3 \\
35-2 \\
\vdots & \vdots & \vdots
\end{bmatrix}
\]

Since \[
1114-395-211 = 1 \cdot (-15) - 1 \cdot (-1) + 1 \cdot (7) = 7 \neq 0
\]
Thus, the system is inconsistent i.e. it has no solution.

[18]
Discuss the consistency of
\begin{align*}
5u + 3v + 7w &= 4 \\
3u + 26v + 2w &= 9 \\
7u + 2v + 10w &= 5
\end{align*}

The above equations can be written in the matrix form
First we will find Rank of matrix $A = 53732627210$

$53732627210 = 5(256) - 3(16) + 7(-176) = 0$

\[-176 \quad 16 \quad 256\]

i.e. $53732627210 = 0$.

Therefore, Rank (A) must be less than or equal to two.

$262210 = 260 \cdot 4 = 256 \neq 0 \therefore \rho(A) = 2$
Now find rank of augmented Matrix [A: D]

\[ A:D=\begin{bmatrix} 53732627210 & 4 & 9 & 5 \end{bmatrix} \]

Contains four 3rd order sub-matrices,

\[ 53732627210, 5743297105, 5343269725, 37426292105 \]

All 4 sub-matrices have determinant zero. \( \therefore \rho[A:D] \leq 2 \)

But, \( 262210 = 260 - 4 = 256 \neq 0 \). \( \therefore \rho[A:D] = 2 \)

\( \therefore \rho[A:D] = \rho(A) = 2 \)

\( \therefore \rho(A) = \rho(A:D) = 2 < 3 = \text{Number of unknown} \)
Thus the system is consistent.
I.e. it has infinitely many solutions.

[19]
Discuss the consistency of

\[ \begin{align*}
3u + v + 2w &= 3 \\
2u - 3v - w &= -3 \\
u + 2v + w &= 5
\end{align*} \]

In the matrix form

\[ \begin{bmatrix}
3 & 1 & 2 & -1 & 1 & 2 & 1
\end{bmatrix} \]

And
\[ \begin{bmatrix}
3 & 1 & 2 & -1 & 1 & 2 & 1 & 3 & -3 & 5
\end{bmatrix} \]

First for finding Rank of A.

Since

\[ \begin{bmatrix}
3 & 1 & 2 & -1 & 1 & 2 & 1
\end{bmatrix} = \begin{bmatrix}
3 & 1 & 2 & -1 & 1 & 2 & 1
\end{bmatrix} \]

\[ \begin{bmatrix}
7 & 3 & -1
\end{bmatrix} \]

By using V & C Sutra for 1st & last row,

\[ \det (A) = 3 (-1) - 1 (3) + 2 (7) = 8 \neq 0 \]

Therefore, Rank (A) must be 3.
Now find rank of augmented Matrix \([A; D]\)

\[
[A; D] = \begin{bmatrix} 3 & 1 & 2 & 2 & -3 & 11 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \end{bmatrix}
\]

Since the above augmented matrix contain four 3rd order Sub-matrices in order to find out rank, the determinants of all sub-matrices are obtained as follows:

\[
\begin{vmatrix} 3 & 1 & 2 & 2 & -3 & 11 & 2 & 1 \end{vmatrix} = 3
\]

\[
\begin{vmatrix} 7 & 3 & -1 \end{vmatrix} = 3
\]

\[
\begin{vmatrix} 3 & 1 & 3 & 2 & -3 & 31 & 2 & 4 \end{vmatrix} = 4
\]

\[
\begin{vmatrix} 7 & 11 & -6 \end{vmatrix} = 11
\]

\[
\begin{vmatrix} 3 & 2 & 3 & 2 & -1 & 31 & 1 & 4 \end{vmatrix} = 16
\]
\[
\begin{array}{ccc}
1 & 2 & 3 \cdot 3 \cdot 1 \cdot 32 \\
1 \neq 1 & 2 & 3 \cdot 3 \cdot 1 \cdot 32 \\
4 \cdot 1 \cdot 2 \cdot 6 + 3 \cdot 1 = 8 & 4 \cdot 1 \cdot 2 \cdot 6 + 3 \cdot 1 = 8 & \neq 0
\end{array}
\]

\[
\begin{array}{ccc}
-1 & -6 & -1
\end{array}
\]

\[
\therefore \rho_A : \rho_D = 3 = \rho_A : \rho_D = 3 = \rho_A : \rho_D = 3 = \rho_A : \rho_D = 3 = \text{Number of unknowns}
\]

Thus, the system is consistent. i.e. it has unique solution.