Chapter 03
Vedic Mathematics in Quadratic, Cubic and Quartic Equations

Quadratic Equations:

The general quadratic equation in variable \( x \) containing \( a \), \( b \) and \( c \) as constants is
\[
a x^2 + b x + c = 0, \quad a \neq 0.
\]
The current method for solving quadratic equation is by using the quadratic formula,
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};
\]
by using the above formula, two values of \( x \) can be found out.
The quadratic equation can also solved by using Vedic method which is superior to the current method.

3.1 Special type of Quadratic Equations by using Vedic Sūtra:

Quadratics of certain forms are solved using Sūtras more effortlessly and more speedily than the above method. The solution of quadratic equations by using different Sūtras and Sub-Sūtras are explained below:

3.1.1 First Special Type under Vilokanam Sub-Sūtra:

To solve the equation of the type in which L.H.S. is in the arrangement of either addition or subtraction of the two Reciprocals.

I.e. either \( x + \frac{1}{x} \) or \( x - \frac{1}{x} \)

According to Vilokanam Sub-Sūtra, we can split R.H.S. into the same form.

Solve:
[1]

\[ z + \frac{1}{z} = \frac{10}{3} \]

By using current method,

Taking L.C.M. we get,

\[ 3z^2 + 3 = 10z \]
\[ \therefore 3z^2 - z - 9z + 3 = 0 \]
\[ \therefore z (3z - 1) - 3(3z - 1) = 0 \]
\[ \therefore (z - 3)(3z - 1) = 0 \]
\[ z = 3 \text{ OR } z = 1/3 \]
\[ \therefore \text{Roots of given Quadratic equation are } z = 3 \text{ OR } z = 1/3 \]

By using Vilokanam,

Here, the right side term is \( \frac{10}{3} \) which can be written in the form of addition of two reciprocals \( 3 + \frac{1}{3} \)
\[ \therefore z + \frac{1}{z} = 3 + \frac{1}{3} \]
\[ \therefore z = 3 \text{ OR } z = \frac{1}{3} \]
\[ \therefore \text{Roots of given Quadratic equation are } z = 3 \text{ OR } z = \frac{1}{3} \]

[2]

\[ y + \frac{1}{y} = \frac{50}{7} \]

By using Vilokanam,

We observe that the right part of given equation is \( \frac{50}{7} \), which we can write in the form of sum of two reciprocals \( 7 + \frac{1}{7} \)
\[ \therefore \frac{1}{y} = 7 + \frac{1}{7} \]
\[ \therefore y = 7 \text{ OR } y = \frac{1}{7} \]
∴ Roots of given Quadratic equation are $y = 7$ OR $y = \frac{1}{7}$

[3]
\[
\frac{y}{y+1} + \frac{y+1}{y} = \frac{26}{5}
\]
By using Vilokanam,
Here, the right term = $\frac{26}{5}$ can be written in the form of two reciprocals $5 + \frac{1}{5}$

\[
\therefore \frac{y}{y+1} + \frac{y+1}{y} = 5 + \frac{1}{5}
\]

\[
\therefore \frac{y}{y+1} = 5 \quad \text{OR} \quad \frac{y}{1+y} = \frac{1}{5}
\]

\[
\therefore y = 5 + 5y \quad \text{OR} \quad 5y = y + 1
\]

\[
\therefore y = -\frac{5}{4} \quad \text{OR} \quad y = \frac{1}{4}
\]
are roots of given quadratic equation.

[4]
\[
z + \frac{1}{z} = \frac{25}{12}
\]
By using Vilokanam,
Here, the right side term = $\frac{25}{12}$
Which is addition of two reciprocals

\[
\frac{4}{3} + \frac{3}{4}
\]

\[
\therefore z + \frac{1}{z} = \frac{4}{3} + \frac{3}{4}
\]

\[
\therefore z = \frac{3}{4} \quad \text{OR} \quad z = \frac{4}{3}
\]

\[
\therefore \text{Roots of given Quadratic equation are} \quad z = \frac{3}{4} \quad \text{OR} \quad z = \frac{4}{3}
\]

[5]
\[
\frac{w+7}{w+3} + \frac{w+3}{w+7} = \frac{53}{14}
\]
By using Vilokanam,

Here, the right side term $\frac{53}{14}$ can be written in the form addition of two reciprocals

\[
\frac{7}{2} + \frac{2}{7}
\]

\[
\therefore \frac{w + 7}{w + 3} + \frac{w + 3}{w + 7} = \frac{7}{2} + \frac{2}{7}
\]

\[
\therefore \frac{w + 7}{w + 3} = \frac{7}{2} \text{ OR } \frac{w + 7}{w + 3} = \frac{2}{7}
\]

\[
\therefore 2 (w + 7) = 7 (w + 3) \text{ OR } 7 (w + 7) = 2 (w + 3)
\]

\[
\therefore 2w + 14 = 7w + 21 \text{ OR } 7w + 49 = 2w + 3
\]

\[
\therefore 5w = -7 \text{ OR } 5w = -46
\]

\[
\therefore w = -\frac{7}{5} \text{ OR } w = -\frac{46}{5}
\]

∴ Roots of given Quadratic equation are $w = -\frac{7}{5}$ OR $w = -\frac{46}{5}$

[6]

\[
x - \frac{1}{x} = \frac{21}{10}
\]

By using Vilokanam,

Here, R.H.S. = $\frac{21}{10}$ can be split as the difference of two reciprocals $\frac{5}{2} - \frac{2}{5}$

\[
\therefore x - \frac{1}{x} = \frac{5}{2} - \frac{2}{5}
\]

\[
\therefore x = \frac{5}{2} \text{ OR } x = -\frac{2}{5}
\]

∴ Roots of given Quadratic equation are $x = \frac{5}{2}$ OR $x = -\frac{2}{5}$

[7]

\[
\frac{5x + 7}{5x - 7} + \frac{5x - 7}{5x + 7} = \frac{24}{35}
\]

By using Vilokanam,
Here, the R.H.S. \( \frac{24}{35} \) can be split as the difference of two reciprocals \( \frac{7}{5} - \frac{5}{7} \).

\[
\frac{5x + 7}{5x - 7} + \frac{5x - 7}{5x + 7} = \frac{24}{35}
\]

\[
\frac{5x + 7}{5x - 7} = \frac{7}{5} \quad \text{OR} \quad \frac{5x + 7}{5x - 7} = -\frac{5}{7}
\]

\[
\therefore 5(5x + 7) = 7(5x - 7) \quad \text{OR} \quad 7(5x + 7) = -5(5x - 7)
\]

\[
\therefore 25x + 35 = 35x - 49 \quad \text{OR} \quad 35x + 49 = -25x + 35
\]

\[
\therefore 10x = 84 \quad \text{OR} \quad 60x = -14
\]

\[
\therefore x = \frac{84}{10} = \frac{42}{5} \quad \text{OR} \quad x = -\frac{14}{60} = -\frac{7}{30}
\]

\[
\therefore \text{Roots of given Quadratic equation are } x = \frac{42}{5} \quad \text{OR} \quad x = -\frac{7}{30}
\]

### 3.1.2 Second Special Type under Śūnyamā śāmyasamuccaye Sūtra:

To solve the quadratic equation of the type

\[
\frac{kx + l}{m x + n} = \frac{px + q}{rx + s}
\]

Where \((k + p) = (m + r) \& (l + q) = (n + s);\)

We can use Śūnyamā śāmyasamuccaye Sūtra as follows:

Consider the oneness of the sum & equating it to 0, which gives one root, of the given quadratic.

i.e. \((k x + l) + (p x + q) = (m x + n) + (r x + s) = 0\)

Which gives one root of the given quadratic equation.

Consider the oneness of the difference and assuming that difference = 0; gives another root, of the given quadratic equation

i.e. \((k x + l) - (m x + n) = (p x + q) - (r x + s) = 0;\)

Which gives the second root of the given quadratic equation.

Solve the following:
\[
\frac{5z + 2}{2z + 4} = \frac{3z + 5}{6z + 3}
\]

By using Śūnyam Saṃyamuccaye Sūtra,

Consider the oneness of the sum & creating that sum = 0, which gives one root, of the given quadratic.

Here, Sum = \((5z + 2) + (3z + 5) = (2z + 4) + (6z + 3) = 8z + 7\)

\[\therefore 8z + 7 = 0 \therefore z = -\frac{7}{8}\]

Consider the oneness of difference of two sides. Now, equate that difference = 0, which gives another root, of the given quadratic equation.

i.e. \((5z + 2) - (2z + 4) = (6z + 3) - (3z + 5) = 0\)

\[\therefore 3z - 2 = 0 \therefore z = \frac{2}{3}\]

∴ Roots of given Quadratic equation are \(z = -\frac{7}{8}\) OR \(z = \frac{2}{3}\).

\[9\]

\[
\frac{11y + 2}{13y + 5} = \frac{9y + 7}{7y + 4}
\]

By using, Śūnyam Saṃyamuccaye Sūtra,

Consider the oneness of the sum, also equating that sum = 0 gives one root of the given quadratic.

i.e. \((11y + 2) + (9y + 7) = (13y + 5) + (7y + 4) = 0\)

\[20y + 9 = 0 \therefore y = -\frac{9}{20}\]

Consider the oneness of the difference of both sides, by equating that difference to 0 gives another root, of the given quadratic equation.

i.e. \((13y + 5) - (11y + 2) = (7y + 4) - (9y + 7) = 0\)

\[2y + 3 = 0 \therefore y = -\frac{3}{2}\]

∴ Roots of given Quadratic equation are \(y = -\frac{9}{20}\) OR \(y = -\frac{3}{2}\).
By using, Śūnyamāsamuccaye Śūtra,
Consider the oneness of the sum to find one of the roots of the given quadratic equation.
i.e. \((3x - 2) + (9x - 6) = (7x - 5) + (5x - 3) = 0\)
\[\therefore 12x - 8 = 0\]
\[\therefore x = \frac{8}{12} = \frac{2}{3}\]

Consider the oneness of both sides by taking the difference, which gives another root, of the given quadratic equation.
i.e. \((7x - 5) – (3x - 2) = (9x - 6) – (5x-3) = 0\)
\[\therefore 4x - 3 = 0\]
\[\therefore x = \frac{3}{4}\]

\[\therefore \text{Roots of given Quadratic equation are } x = \frac{2}{3} \text{ OR } x = \frac{3}{4}.\]
The algebraic proof of the above equation is as follows:

By using simple division,
\[
\frac{3}{w + 3} + \frac{4}{w + 4} = \frac{5}{w + 5} + \frac{2}{w + 2}
\]

\[
\therefore \left(1 - \frac{w}{w + 3}\right) + \left(1 - \frac{w}{w + 4}\right) = \left(1 - \frac{w}{w + 5}\right) + \left(1 - \frac{w}{w + 2}\right)
\]

\[
\therefore \text{Removing 1-1 and 1-1 from both sides, we get}
\]
\[
\left(\frac{w}{w + 3}\right) + \left(\frac{w}{w + 4}\right) = \left(\frac{w}{w + 5}\right) + \left(\frac{w}{w + 2}\right)
\]

\[
\therefore \text{The common factor of all terms, } w = 0 \text{ and on its removal}
\]
\[
\frac{1}{w + 3} + \frac{1}{w + 4} = \frac{1}{w + 5} + \frac{1}{w + 2}
\]

By taking L.C.M.
\[
\therefore \frac{2w + 7}{w^2 + 7w + 12} - \frac{2w + 7}{w^2 + 7w + 10} = 0
\]
\[
\therefore (2w + 7) \left[\frac{1}{w^2 + 7w + 12} - \frac{1}{w^2 + 7w + 10}\right] = 0
\]
\[
\therefore (2w + 7) \left[\frac{(w^2 + 7w + 10) - (w^2 + 7w + 12)}{(w^2 + 7w + 12)(w^2 + 7w + 10)}\right] = 0
\]
\[
\therefore (2w + 7) \left[\frac{w^2 + 7w + 10 - w^2 - 7w - 12}{(w^2 + 7w + 12)(w^2 + 7w + 10)}\right] = 0
\]
\[
\therefore (2w + 7) \left[\frac{-2}{(w^2 + 7w + 12)(w^2 + 7w + 10)}\right] = 0
\]
\[
-2 (2w + 7) = 0
\]
\[
2w + 7 = 0
\]
\[
\therefore w = -\frac{7}{2}
\]
\[
\therefore w = 0 \text{ OR } w = -\frac{7}{2}
\]

These values of w are solutions of given equation.
By using Vedic Sūtra:

\[
\frac{3}{w+3} + \frac{4}{w+4} = \frac{5}{w+5} + \frac{2}{w+2}
\]

Here, both the conditions are satisfied.

\[
\frac{3}{1} + \frac{4}{1} = \frac{5}{1} + \frac{2}{1} \quad \& \quad \frac{3}{3} + \frac{4}{4} = \frac{5}{5} + \frac{2}{2}
\]

\[\therefore\] According to the Sunyamanyat Sūtra,

One root of given equation is \(w = 0\).

Also, \((w + 3) + (w + 4) = (w + 5) + (w + 2) = 2w + 7\)

\[\therefore\] As per Śūnyam □ Sāmyasamuccaye Sūtra,

That addition = \(2w + 7 = 0\)

\[\therefore\] The another root is \(w = -\frac{7}{2}\)

\[\therefore\] The solution \(w = 0 \quad \text{OR} \quad w = -\frac{7}{2}\).

\[12\]

\[
\frac{1}{2x + 1} + \frac{1}{5x + 1} = \frac{2}{5x + 2} + \frac{3}{10x + 3}
\]

Algebraic proof is as follows:

By using simple division,

\[
\frac{1}{2x + 1} = 1 - \frac{2x}{2x + 1}
\]

\[\therefore\] \((1 - \frac{2x}{2x + 1}) + (1 - \frac{5x}{5x + 1}) = (1 - \frac{5x}{5x + 2}) + (1 - \frac{10x}{10x + 3})
\]

\[\therefore\] Removing 1-1 and 1-1 from both sides, we get

\[\therefore\] \((\frac{2x}{2x + 1}) + (\frac{5x}{5x + 1}) = (\frac{5x}{5x + 2}) + (\frac{10x}{10x + 3})
\]

\[\therefore\] \((\frac{10x}{10x + 5}) + (\frac{10x}{10x + 2}) = (\frac{10x}{10x + 4}) + (\frac{10x}{10x + 3})
\]

\[\therefore\] The common factor of all terms, \(10x = 0\)

\(x = 0\) and on its removal

We have
\[ \therefore \left( \frac{1}{10x + 5} \right) + \left( \frac{1}{10x + 2} \right) = \left( \frac{1}{10x + 4} \right) + \left( \frac{1}{10x + 3} \right) \]

\[ \therefore \left( \frac{20x + 7}{(10x + 5)(10x + 2)} \right) = \left( \frac{20x + 7}{(10x + 4)(10x + 3)} \right) \]

\[ \therefore \left( \frac{20x + 7}{(10x + 5)(10x + 2)} \right) - \left( \frac{20x + 7}{(10x + 4)(10x + 3)} \right) = 0 \]

\[ \therefore (20x + 7) \left[ \frac{(100x^2 + 70x + 12) - (100x^2 + 70x + 10)}{(100x^2 + 70x + 10)(100x^2 + 70x + 12)} \right] = 0 \]

\[ \therefore (20x + 7) \left[ \frac{2}{(100x^2 + 70x + 10)(100x^2 + 70x + 12)} \right] = 0 \]

\[ \therefore 2(20x + 7) = 0 \]

\[ \therefore (20x + 7) = 0 \]

\[ \therefore \text{Another root is } x = -\frac{7}{20} \]

\[ \therefore \text{The two roots of given quadratic are } x = 0 \text{ OR } x = -\frac{7}{20} \]

**By using Vedic Sūtra:**

\[ \frac{1}{2x + 1} + \frac{1}{5x + 1} = \frac{2}{5x + 2} + \frac{3}{10x + 3} \]

Here,

\[ \frac{1}{2} + \frac{1}{5} = \frac{2}{5} + \frac{3}{10} \]

\[ \therefore \frac{7}{10} = \frac{7}{10} \]

\[ \therefore \frac{1}{1} + \frac{1}{1} = \frac{2}{2} + \frac{3}{3} \]

both the conditions are satisfied.

Therefore, according to the Śunyamanyat Sūtra one root is zero; and

We can write left side as below:

\[ \frac{1}{2x + 1} = 1 - \frac{2x}{2x + 1}; \quad \frac{1}{5x + 1} = 1 - \frac{5x}{5x + 1}; \]

And also right side can be written as below:

\[ \frac{2}{5x + 2} = 1 - \frac{5x}{5x + 2}; \quad \frac{3}{10x + 3} = 1 - \frac{10x}{10x + 3} \]

Now comparing both the sides after modification,

\[ \frac{2x}{2x + 1} = \frac{5x}{5x + 1} = \frac{5x}{5x + 2} = \frac{10x}{10x + 3} \]

Taking LCM,
\[
\frac{10x}{10x + 5} = \frac{10x}{10x + 2} = \frac{10x}{10x + 4} = \frac{10x}{10x + 3}
\]

\[(10x + 5) + (10x + 2) = (10x + 4) + (10x + 3) = 20x + 7\]

According to the Śūnyamānasāmyamuccaye Sūtra,

\[20x + 7 = 0.\]

∴ The another root of the given equation is \[x = -\frac{7}{20}\].

∴ The two roots of quadratic are \[x = 0\] OR \[x = -\frac{7}{20}\].

[13] \[
\frac{a - c}{x + a - c} + \frac{c - b}{x + c - b} = \frac{a + c}{x + a + c} + \frac{c - b}{x - c - b}
\]

Here,

\[
\frac{a - c}{1} + \frac{c - b}{1} = \frac{a + c}{1} + \frac{c - b}{1}
\]

And also,

\[
\frac{(a - c)(c - b)}{(a - c)(c - b)} = \frac{(a + c)(c - b)}{(a + c)(c - b)}
\]

Therefore, according to the Sunyamanyat Sūtra one root does not have a nonzero value;

According the Śūnyamānasāmyamuccaye Sūtra,

\[2x + a - b = 0\]

∴ Another root is \[x = -\frac{1}{2}(a - b)\].

∴ Two roots of given quadratics are,

\[x = 0\] OR \[x = -\frac{1}{2}(a - b)\].

3.1.4 Fourth Special Type under Śūnyamanyat & Parāvartya Sūtra:

The equation containing all the letters except \(x\) having fixed value as below:

\[
\frac{h}{ex + a} + \frac{i}{fx + b} = \frac{j}{gx + c}
\]

One root of the given quadratic equation is \(x = 0\); if
The quadratic equation in which two terms \( N_1 \) & \( N_2 \) on L.H.S. is compared with the right side term \( N_3 \) i.e. \( N_1 + N_2 = N_3 \)

Then Parāvartya Sūtra is applied to combine both the sides.
If \( N_1 + N_2 \neq N_3 \), then co-efficient of \( x \) in all the lower terms must be different, we can make it equal. Also check

\[
\frac{h}{e} + \frac{i}{f} = \frac{j}{g}
\]

If it is in the above form, then Parāvartya Sūtra is applied to combine both the sides.

The right side constant term is subtracted from the left side constant term & multiply those remainders by the corresponding numerators of the term on the Left counterpart.

After this process of assimilation equated with 0, which can be written in the form

\[
\frac{h(a - c)}{ex + a} + \frac{i(b - c)}{fx + b} = 0
\]

After applying merger Sūtra, if both the numerators are equal.
I.e. if the numerator \( h (a - c) = i (b - c) \) then they can be removed, and by applying the Sunyamanyat Sūtra \( D_1 + D_2 = 0 \) is another root of the given quadratic equation.

After applying the merger Sūtra, if both the numerators are not equal but different i.e. \( h (a - c) \neq i (b - c) \), then Śūnyamanyat Sūtra will not apply.

Check co-efficient of \( x \).
If \( e = f = 1 \), then one root of given equation is

\[
x = \frac{-hb(a - c) - ia(b - c)}{h(a - c) + i(b - c)}
\]

If \( e \neq f \neq 1 \), i.e. other than 1 then root of equation is

\[
x = \frac{-hb(a - c) - ia(b - c)}{e[h(a - c) + i(b - c)]}
\]

Solve the following:
\[
\frac{7}{z+7} + \frac{3}{z+3} = \frac{10}{z+10}
\]

The algebraic proof of the above equation is as follows:

\[
\therefore \frac{7(z+3) + 3(z+7)}{(z+7)(z+3)} = \frac{10}{z+10}
\]

Simplifying

\[
\therefore \frac{7z+21 + 3z+21}{z^2 + 10z + 21} = \frac{10}{z+10}
\]

\[
\therefore \frac{10z+42}{z^2 + 10z + 21} = \frac{10}{z+10}
\]

\[10z^2 + 100z + 210 = 10z^2 + 100z + 210
\]

\[42z = -210
\]

\[\therefore z = -5
\]

By Vedic Sūtra:

Here, \(\frac{7}{3} \neq \frac{10}{10}\)

Therefore, \(z = 0\) is not a root of the given quadratic equation.

\(N_1 + N_2 = 7 + 3 = 10 = N_3\)

\[\because D_1 + D_2 = 0\]

i.e. \((z + 7) + (z + 3) = 2z + 10 = 0\) is root of the given quadratic equation.

\[\therefore z = -5\] is root of given equation.

By merger method,

\[\frac{7(-3)}{z+7} + \frac{3(-7)}{z+3} = 0\]

\[\therefore \frac{-21}{z+7} + \frac{-21}{z+3} = 0\]

\[\therefore \frac{-(-21)(3) - (-21)(7)}{-21(1 + 1)} = \frac{21(3 + 7)}{-21(1 + 1)} = \frac{10}{-2} = -5\]

\[\therefore z = -5 \text{ is root of given equation.}\]

\([15]\)

\[
\frac{9}{y+3} + \frac{16}{y+4} = \frac{49}{y+7}
\]
\[
\frac{9}{3} + \frac{16}{4} = \frac{49}{7} \therefore 3 + 4 = 7
\]

By division,
\[
3 - \frac{3}{y+3} + 4 - \frac{4}{y+4} = 7 - \frac{7}{y+7}
\]
\[\therefore y = 0.\]

This can be verified by mere observation.
\[
\frac{3}{y+3} + \frac{4}{y+4} = \frac{7}{y+7}
\]

By merger method,
\[
\frac{3(-4)}{y+3} + \frac{4(-3)}{y+4} = 0
\]
\[\therefore \frac{-12}{y+3} + \frac{-12}{y+4} = 0\]
\[\therefore y = \frac{-12(4) - 12(3)}{-12 - 12} = \frac{-12(4 + 3)}{-12(1 + 1)} = \frac{7}{2} = 3 \frac{1}{2}\]
\[\therefore y = 0 \text{ OR } y = 3 \frac{1}{2}.
\]

[16]
\[
\frac{6}{3w + 2} + \frac{3}{2w + 3} = \frac{4}{6w + 1}
\]

L.C.M. = 6
\[
\frac{12}{6w + 4} + \frac{9}{6w + 9} = \frac{4}{6w + 1}
\]
\[\therefore \frac{12}{4} + \frac{9}{9} = \frac{4}{1}\]
\[\therefore \text{One solution of above quadratic is } w = 0.
\]

Applying merger method,
\[\therefore \frac{12(3)}{6w + 4} + \frac{9(8)}{6w + 9} = 0\]
\[\therefore \frac{36}{6w + 4} + \frac{72}{6w + 9} = 0\]
\[\therefore \frac{1}{6w + 4} + \frac{2}{6w + 9} = 0\]
\[ \therefore 6w = \frac{-(1)(9) - (2)(4)}{1 + 2} \]
\[ \therefore 6w = - \frac{17}{3} \]
\[ \therefore w = - \frac{17}{18} \]
\[ \therefore w = 0 \text{ OR } w = - \frac{17}{18} \] are roots of given quadratic equation.

3.2 Factorization:

Quadratic expression \( Ax^2 + Bx + C \); which uses variable \( x \) with highest degree two with always non-zero co-efficient; either \( B \) & \( C \) or both may be zero or not.

Standard formula of Quadratic equation can be obtained by equating the above expression to 0. If an example of quadratic equation is given in the form \( Ax^2 + Bx = C \) then first take all the term one side and write it in the form equal to zero and then factorize it. Factorization is one of the methods for solving quadratic equation.

3.2.1 Factorization of a Quadratic expression in one variable:

According to the current system, if we want to factorize the Quadratic expression which has prefix of \( x \) with highest degree 2 as unity. For solving quadratics in one variable \( Ax^2 + Bx + C \), divide the term \( B \) in parts such that addition of that two numbers = the coefficient of variable \( x \) and the independent quantity equals the multiplication value.

[1]

Factorize: \( z^2 + z - 12 \)

Here, co-efficient of \( z^2 \) is unity. Divide the co-efficient of \( z \) (i.e. +1) in two parts such that addition of that is +1 and the multiplication of two parts is -12.

Here, \((+4 - 3) = +1\)
\& \((+4) (-3) = -12\)
\[ z^2 + z - 12 \]
\[ = z^2 +4z - 3z - 12 \]
\[ = z (z + 4) - 3(z + 4) \]
We can calculate the product of \(z - 3\) and \(z + 4\) orally, which verifies to \(z^2 + z - 12\).

Factors of 12 is 1, 2, 3, 4, 6, and 12.

If we select + 4 and - 3 whose sum is + 1 and product is - 12.

We can verify it by vertically and crosswise Sūtra

\[
\begin{array}{c|c}
& z + 4 \\
\hline
z - 3 & z^2 + z -12 \\
\end{array}
\]

Thus, **Factors of quadratic expression** \(z^2 + z - 12 = (z + 4) (z - 3)\)

[2]

Factorize: \(2y^2 + 3y - 5\)

Here, co-efficient of \(y^2\) is 2 not unity. Evaluate two sub parts such that addition of that is + 3 and multiplication of that is - 10.

\((- 2 + 5) = + 3\) and \((- 2) (5) = - 10\)

\(2y^2 + 3y – 5 = (2y^2 – 2y + 5y – 5)\)

\= 2y(y - 1) + 5 (y - 1)

\= (y - 1) (2y + 5)

In order to find the sub parts of an equation with the highest degree prefix other than 1, mental process is not possible. But by using Vedic Sub-Sūtra Ānurūpye a (means ‘Proportionately’) first factor can be found and by using Ādyamadyenāntyamantyena (means ‘first by the first and last by last’) second factor can be found easily.

**Steps for factorization by using Vedic Sūtras:**

Divide the central term in two sub parts \(P_1\) and \(P_2\) so that

\[
\text{Ratio} = \frac{A}{P_1} = \frac{P_2}{C} \ldots \ldots \ldots \ [A]
\]

Given that \(B = P_1 + P_2\)

\(C = P_1.P_2\)

We need to find the two roots \(t\) and \(s\) for the equation. Thus,

\((x – t)(x – s) = Ax^2 + Bx + C\)
The ratio [A] is the 1st factor \((x - t)\) of given quadratic expression. 

\((x - t)\) is one factor of quadratic expression \(Ax^2 + Bx + C\) found by Vedic Sub-Sūtra Ānurūpyena. The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’. 

2nd Factor = \((x - s) = \frac{Ax^2}{x + C} / (-t)\)….. [B]

In the above expression \(2x^2 + 3x - 5\), the middle term + 3 is divided in two parts - 2 & 5 so that 

\[
\frac{2}{-2} = \frac{5}{-5} = \frac{1}{-1}
\]

The ratio is 1: -1

I.e. \(x - 1\) is one factor found by using Vedic Sub-Sūtra Ānurūpyeā. 

The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’ along with 1st factor \((x - 1)\). Applying [B]

\[
\frac{2x^2}{x} - \frac{5}{-1} = 2x + 5
\]

The second factor is \((2x + 5)\).

**Thus, Factors of quadratic expression** \(2x^2 + 3x - 5 = (x - 1) (2x + 5)\)

**An Alternative method:**

We can also split the middle co-efficient 5 in two parts 5 & - 2 such that by using [A]

\[
\frac{2}{5} \frac{2}{5}
\]

Ratio is 2: 5.

i.e. \(2x + 5\) is one factor found by using Vedic Sub-Sūtra Ānurūpyeā.

If \(2x + 5\) is one factor of quadratic expression \(2x^2 + 3x - 5\) found by Vedic sub-Sūtra used in step [1].

The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’

\[
\frac{2x^2}{2x} + \frac{5}{-5} = x - 1.
\]

Thus second factor is \((x-1)\).

**Thus, Factors of quadratic expression** \(2x^2 + 3x - 5 = (2x + 5) (x - 1)\)
3.2.2 Factorization of a Quadratic expression in two variables:

The same method of factorization of one variable is used to factorize the quadratic expression in two variables x and y whose first coefficient (coefficient of $x^2$) is not unity.

Factorize: $7x^2 - 6xy - y^2$

Divide the central term prefix in two sub parts. Now, with the help of [A], calculate the ratio which in turn gives the 1st factor of given quadratic expression.

In the above expression $7x^2 - 6xy - y^2$, divide the middle term -6 into 1 and -7

So that

\[
\frac{7}{1} = \frac{-7}{-1} = \frac{7}{1}
\]

That ratio is 7:1.

i.e. $(7x + y)$ is one factor found by using Vedic Sub-Sūtra Ānurūpya।

The second factor is obtained by using [B] as follows:

$7x + y$ is one factor of quadratic expression $7x^2 - 6xy - y^2$ founded by Vedic Sub-Sūtra as in above step. The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’. i.e. second factor is

\[
\frac{7x^2}{7x} - \frac{y^2}{y} = x - y
\]

Thus second factor is $(x - y)$.

**Thus, Factors of quadratic expression** $7x^2 - 6xy - y^2 = (7x + y) (x - y)$

An Alternative method:

We can also split the mid coefficient - 6 in two subparts - 7 & 1. By using [A], ratio is

\[
\frac{7}{-7} = \frac{1}{-1} = \frac{1}{-1}
\]

That is ratio = 1:-1

$(x - y)$ is one factor found by using Vedic Sub-Sūtra Ānurūpya।a means ‘Proportionately’.
If \((x - y)\) is one factor of quadratic expression \(7x^2 - 6xy - y^2\) found by Vedic Sub-Sūtra Ānurūpye\(\text{ī}\)a.

The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’ and the 1\(^{st}\) factor.

\[
\frac{7x^2}{x} - \frac{y^2}{-y} = 7x + y
\]

Thus, second factor is \((7x + y)\).

**Thus, Factors of quadratic expression** \(7x^2 - 6xy - y^2 = (x - y)(7x + y)\)

[4]

Factorize: \(12x^2 - 23xy + 10y^2\)

Divide the middle term - 23 into - 15 and - 8 such that ratio of the first coefficient to that first part 12: 15 i.e. 4: 5 is same as the ratio of that second part to the last co-efficient 8 : 10 = 4 : 5. That ratio 4: 5 i.e. \((4x - 5y)\) is one factor found by using Vedic Sub-Sūtra Ānurūpye\(\text{ī}\)a.

The 2\(^{nd}\) factor is obtained as follows:

\((4x - 5y)\) is one factor of quadratic expression \(12x^2 - 23xy + 10y^2\) found by Vedic Sub-Sūtra as above step. The 2\(^{nd}\) factor is obtained by Vedic Sub-Sūtra Ādyam,

\[
\frac{12x^2}{4x} + \frac{10y^2}{-5y} = 3x - 2y
\]

The 2\(^{nd}\) factor is \((3x - 2y)\).

**Thus, Factors of quadratic expression** \(12x^2 - 23xy + 10y^2 = (4x - 5y)(3x - 2y)\)

[5]

Factorize: \(15x^2 - 14xy - 8y^2\)

Divide the middle term - 14 into - 20 and 6 so that ratio is

\[
\frac{15}{20} = \frac{6}{8} = \frac{3}{4}
\]

That is ratio is 3: -4 i.e. \((3x - 4y)\) is one factor found by using Vedic Sub-Sūtra ‘Proportionately’.

The second factor is obtained as follows:

\(3x - 4y\) is one factor of quadratic expression \(15x^2 - 14xy - 8y^2\) found by Vedic Sub-Sūtra as above step. The second factor is obtained by Vedic Sub-Sūtra ‘first by the first & last by last’, 2\(^{nd}\) Factor is
\[
\frac{15x^2}{3x} - \frac{8y^2}{-4y} = 5x + 2y
\]

The second factor is \((5x + 2y)\).

Thus, Factors of quadratic expression \(15x^2 - 14xy - 8y^2 = (3x - 4y)(5x + 2y)\)

Gu\-\(\text{itasamuccaya}\)\-\(\text{Samuccayag\-\(\text{ita}\)}\) Sub-S\-\(\text{utra}\):

Meaning:
According to this, the multiplication of the variable prefix addition of the expression and the summation of the preceding constants in the final solution both are equal.
To confirm the obtained answer of the problem in multiplications, divisions and factorization of quadratic equations an additional Sub-S\-\(\text{utra}\) is Gu\-\(\text{itasamuccaya}\)\-\(\text{Samuccayag\-\(\text{ita}\)}\).

An example of quadratic equation for verification is given below:
\(x^2 + x - 12 = (x + 4)(x - 3)\) ................. (i)
In equation (i) add the all the co-efficients of right side
\(= 1 + 1 - 12 = -10\) and
The multiplication of the addition of the co-efficient in the left side
\(= (1 + 4)(1 - 3) = 5(-2) = -10\)

\(7x^2 - 6xy - y^2 = (7x + y)(x - y)\) ................. (ii)
In equation (ii) sum of the coefficient in the factor = 7 – 6 - 1 = 0 and
The multiplication of the summation = (7 + 1)(1-1) = 8(0) = 0

By using the above S\-\(\text{utra}\) both the sides of all the equations are equal
Thus, above quadratic equations are verified.

**3.2.3 Solution of multivariabe Quadratics:**

Equations in homogeneous form having greater than 2 variables lead to tedious & lengthy computations.
Adyamadyenāntyamantyena & Lopanasthapanaḥbhyaṃ can be applied for ease of calculation of roots of given equation.

**Procedure of factorization of long homogeneous quadratic expression with three variables x, y & z:**

[1] Initially, discarding of z is done by substituting z as zero, so we obtained the quadratic with remaining two variables which can be solved by using Adyamadyenāntyamantyena;

[2] Similarly equating y to zero eliminates it. Thus, we obtaine the quadratic with variables other than y which can be factorized by using the above sub-Sūtra used in step [1];

[3] By using the results obtained in [1] and [2], the final solution is arrived at by identifying the pattern and combining the common terms.

To factorize long homogeneous quadratic expression with three variables x, y & z, only two elimination by eliminating one letter at a time by three possible ways.

[a] Out of given 3 variables by consecutive exclusion of x and then y we get two expressions whose solutions contains two variables y & z and x & z respectively. After merging & filling the pattern, the final terms are obtained successfully.

[b] Repeat these steps of [a] after eliminating y and then z.

[c] Discard first x and then z. Reiterate the above process [a].

By taking examples procedure of factorization long quadratic expression can be explained:

[6]

Factorize: \( 2u^2 + v^2 - 3w^2 - 3uv + 2vw - wu \)

Let \( Q = 2u^2 + v^2 - 3w^2 - 3uv + 2vw - wu \)

First by considering \( w = 0 \); we obtain the expression in terms of \( u \) and \( v \).

Factorize that expression with the help of adyam Sūtra, we get

If \( w = 0 \), then \( [Q] = 2u^2 - 3uv + v^2 \)

\[= (u - v)(2u - v)\]
Similarly, eliminate v (v = 0); factorizing the obtained expression in terms of u and w.
If v = 0, then \[Q\] = \(2u^2 - wu - 3w^2\)
\[= (u + w)(2u - 3w)\]
Connecting the like terms caused by eliminating w and v, we get
\[Q = (u - v + w)(2u - v - 3w)\]

OR

If u = 0, then \[Q\] = \(v^2 + 2vw - 3w^2\)
\[= (v - w)(v + 3w)\]
Similarly, if v = 0, then as above \[Q\] = \(2u^2 - wu - 3w^2\) = \(u + w)(2u - 3w)\)

With the help of two set of factors obtained after eliminating u & v; by filling the gaps among that two factors we can get
\[Q = (u - v + w)(2u - v - 3w)\]

OR

Substituting w = 0 in \[Q\],
\[2u^2 - 3uv + v^2\]
\[= (u - v)(2u - v)\]
Similarly, if u = 0, then \[Q\] is \(v^2 + 2vw - 3w^2\)
\[= (-v - 3w)(w - v)\]
By filling the breaks of factors which are obtained after eliminating w and u
\[Q = (u - v + w)(2u - v - 3w)\]

[7]
Factorize: \(2x^2 + y^2 - 6z^2 + 2xy - 10yz + 4zx\)

Let \(R = 2x^2 + y^2 - 6z^2 + 2xy - 10yz + 4zx\)
If \(z = 0\), \([R] = 2x^2 + 2xy - 4y^2\)
\[= (x + 2y)(2x - 2y) = 2(x + 2y)(x - y)\]
Similarly,
If \( y = 0 \), then \( [R] = 2x^2 + 4zx - 6z^2 \)
\[ = (x - z) (2x + 6z) = 2(x - z) (x + 3z) \]
i.e. \( [R] = 2(x + 2y) (x - y) = 2(x - z) (x + 3z) \)

By filling the gaps among the factors after elimination of \( z \) & \( y \)
\( [R] = 2(x - y + 3z) (x + 2y - z) \) OR \( [R] = 2(x - y - z) (x + 2y + 3z) \)

Also by eliminating \( x \), we get factors in terms of \( y \) and \( z \);
If \( x = 0 \), then \( [R] = -4y^2 - 10yz - 6z^2 \)
\[ = (y + z) (-4y - 6z) = 2(y + z) (-2y - 3z) \]
\& If \( y = 0 \), then as above \( [R] = (x - z) (2x + 6z) = 2(z - x) (-x - 3z) \)
i.e. \( [R] = 2(y + z) (-2y - 3z) = 2(z - x) (-x - 3z) \) after discarding \( x \) and \( y \)

\[ \therefore \ [R] = 2(-x + y + z) (-x - 2y - 3z) \]
Thus, the final solution is \( R = 2(x - y - z) (x + 2y + 3z) \)

[8]
Factorize: \( z^2 - 2x^2 - 3y^2 - 5xy + 2yz + zx \)

Let \( S = z^2 - 2x^2 - 3y^2 - 5xy + 2yz + zx \)
If \( x = 0 \), then \( [S] = z^2 - 2yz - 3y^2 \)
\[ = (z - y) (z + 3y) \]

Similarly,
If \( y = 0 \), then given expression = \( z^2 + zx - 2x^2 \)
\[ = (z - x) (z + 2x) \]
\[ [S] = (z + 3y) (z - y) \) OR \( [S] = (z + 2x) (z - x) \)

By filling up the gaps between the factors obtained after elimination of \( x \) and \( y \).
\[ [S] = (z - x - y) (z + 2x + 3y) \) OR \( [S] = (z - x + 3y) (z - y + 2x) \)

Also, eliminating \( z \) we get factors in \( x \) and \( y \),
\[ [S] = -2x^2 - 5xy - 3y^2 \]
\[ = (x + y)(-2x - 3y) = (-x - y)(2x + 3y) \]

Also, as above \( y = 0 \), then \( [S] = (z - x)(z + 2x) \)
\[ i.e. \ [S] = (-x - y)(2x + 3y) = (z - x)(z + 2x) \]

After filling the break between two factors obtained because of the elimination of two variables \( z \) & \( y \).
Thus, the final solution is 
\[ S = (z - x - y)(z + 2x + 3y) \]

**Procedure of factorization of long homogeneous quadratic expression with three variables u, v, w & Independent term:**

Consider 3 variables (u, v, w) along with an independent term for the process of factorization. Out of the 4 terms there arise 3 possible combinations for exclusion of 2 terms satisfying the condition of retention of the pair containing an independent term & the remaining variable.
First exclude u and v by substituting the values of both u and v as zero, thereby getting a retention pair of an independent term and w. Similarly, excluding u and w, we get quadratic in terms of v & an independent term.
In this way, the third pair can be obtained by eliminating v and w. Factorizing that quadratics by Sūtra Adyam 3 different factors are obtained. By filling the gaps between the obtained three factors the final factors of the given expression in terms of 3 variables along with independent term can be found.

\[ \text{[9]} \]
Factorize: 
\[ 2u^2 - 3v^2 - 4w^2 - uv - 7vw - 2wu - 5u - 10v - 11w - 7 \]

Let \( E = 2u^2 - 3v^2 - 4w^2 - uv - 7vw - 2wu - 5u - 10v - 11w - 7 \)
Here, there are three letters u, v, w and independent term, and 3 elimination by eliminating two letters at a time so retain 1 remaining letter & independent term each time.
First eliminate $u, v$ by putting $u = v = 0$ and retain only in $w$ and independent term.

After factorizing it given expression

$$E = -4w^2 - 11w - 7 = (-4w - 7)(w + 1)$$

Then, by eliminating $v & w$, the quadratic is obtained in variable $u$ along with the independent term & then factorize it.

$$\therefore \text{ Given expression } E = 2u^2 - 5u - 7 = (2u - 7)(u + 1)$$

Then, by eliminating $u & w$, we can obtained the quadratic in variable $v$ along with the independent term & then factorize it.

$$\therefore \text{ Given expression } E = -3v^2 - 10v - 7 = (-3v - 7)(v + 1)$$

With these three sets of factors,

$$\therefore \text{ Given expression } E = (-4w - 7)(w + 1) = (2u - 7)(u + 1) = (-3v - 7)(v + 1)$$

By filling the breaks among the above 3 sets of factors, the final factors can be found with 3 variables along with independent tem.

The final solution is $E = (u + v + w + 1)(2u - 3v - 4w - 7)$

3.2.4 Factorization of Quadratics expression with four variables:

Solving quadratic equations containing more than three variables is quite a challenging task especially for the equations in homogeneous form.

Let $t, u, v$ and $w$ be four variables.

Vedic Ādyamadyenāntyamantyena and Lopanasthāpanābhyaṃ Sub-Sūtra can be utilized in order to solve intricate problems in a relatively easier manner.

Procedure of factorization of long homogeneous quadratic expression with four variables $x, y, z & w$:

To factorize long homogeneous quadratic expression with four variables $x, y, z$ and $w$, three times by eliminating two letters at a time.

Either eliminate $x, y$ by putting $x = y = 0$, then we will get the expression in other two remaining variables $z & w$.

Eliminate $y, w$ by putting $y = w = 0$ retain only in $x, z$ or eliminate $x, w$ by putting
x = w = 0 and retain only in y, z.
First eliminate x, y and retain only in z, w; eliminate y, z by retain only in x, w; or by eliminating x & z to retain it in y & w.
First eliminate x, w; then eliminate z, w, and then eliminate x, z.
First eliminate y, w; then eliminate z, w and then eliminate y, z.

After eliminating two letters at a time, the quadratic with remaining two letters are obtained. By factorizing and merging it along with filling the gaps the final factors in all 4 letters can be found.

[10]
Factorize: $2x^2 + 7y^2 + 3z^2 - 5w^2 - 9xy - 10yz - 5xz - 3wx + 2wz - 2yw$

Let $T = 2x^2 + 7y^2 + 3z^2 - 5w^2 - 9xy - 10yz - 5xz - 3wx + 2wz - 2yw$

When there are four letters x, y, z and w three elimination by eliminating two letters at a time.
There are four possible ways to factorize.

[I]
First we eliminate x, y by putting x = y = 0 and retain only in z, w;
$[T] = 3z^2 + 2wz - 5w^2$
$= (z - w) (3z + 5w)$

Then eliminate y, z by putting y = z = 0 retain only in x, w;
$[T] = 2x^2 - 3wx - 5w^2$
$= (x + w) (2x - 5w) = (-x - w) (-2x + 5w)$

Also, if x = z = 0, then
$[T] = 7y^2 - 2yw - 5w^2$
$= (y - w) (7y + 5w)$

$T = (z - w) (3z + 5w) = (-x - w) (-2x + 5w) = (y - w) (7y + 5w)$
Filling the gaps in,
The solution is \((z - x + y - w) (3z - 2x + 7y + 5w)\)

[II]
First eliminate \(x\), \(w\) by putting \(x = w = 0\) and retain only in \(y\), \(z\);
\[
[T] = 7y^2 + 10yz + 3z^2
= (y + z) (7y + 3z)
\]

By eliminating \(z\) & \(w\) (i.e. \(z = w = 0\))
\[
[T] = 2x^2 - 9xy + 7y^2
= (x - y) (2x - 7y) = (y - x) (7y - 2x)
\]

If \(x = z = 0\), then
\[
[T] = 7y^2 - 2yw - 5w^2
= (y - w) (7y + 5w)
\]

With these three sets of factors,
Thus, \(T = (y + z) (7y + 3z) = (y - x) (7y - 2x) = (y - w) (7y + 5w)\)
Filling in the gaps,
The solution is \(T = (y + z - x - w) (7y + 3z - 2x + 5w)\)

[III]
By eliminating \(z\) & \(w\) (\(z = w = 0\)) as above
\(T = (y - x) (7y - 2x)\)

By eliminating \(y\) & \(z\) (i.e. substituting \(y = z = 0\)),
\(T = (- x - w) (-2x + 5w)\)

Then, eliminate \(y\), \(w\) by putting \(y = w = 0\) and retain only in \(x\), \(z\);
If \(y = w = 0\), then given expression \(T = 2x^2 - 5xz + 3z^2\)
\[
= (x - z) (2x - 3z)
\]
= (z - x) (3z - 2x)

With these three sets of factors,
\[T = (y - x) (7y - 2x) = (-x - w) (-2x + 5w) = (z - x) (3z - 2x)\]

Filling in the gaps,
The solution is \[T = (y - w + z - x) (7y + 5w + 3z - 2x)\]

[IV]
First eliminate \(x, w\) by putting \(x = w = 0\) and retain only in \(y, z\);
\[T = 7y^2 + 10yz + 3z^2\]
\[= (y + z) (7y + 3z)\]

Also, eliminate \(y, w\) by putting \(y = w = 0\) and retain only in \(x, z\);
\[T = 2x^2 - 5xz + 3z^2\]
\[= (x - z) (2x - 3z) = (z - x) (3z - 2x)\]

Then eliminate \(x, y\) by putting \(x = y = 0\) and retain only in \(z, w\);
\[T = 3z^2 + 2wz - 5w^2\]
\[= (z - w) (3z + 5w)\]

With these three sets of factors,
Given expression \(T = (y + z) (7y + 3z) = (z - w) (3z + 5w) = (z - x) (3z - 2x)\)

By filling the breaking variables, we found the final factor,
The solution is \(T = (y - w + z - x) (7y + 5w + 3z - 2x)\)

All four possible ways we get same answer,
\[T = (-x + y + z - w) (-2x + 7y + 3z + 5w) = (x - y - z + w) (2x - 7y - 3z - 5w)\]

**Factorization of Cubic Expressions:**

[11]
Factorize: \(x^3 + 2x^2 - 23x - 60\)
Let \(E = x^3 + 2x^2 - 23x - 60\)

Sum of coefficient of all the term = 1 + 2 - 23 - 60 = -80

Factors of 80 = 1, 2, 4, 5, 8, 10, 16, 20, 40, 80
And Factors for 60 = 1, 60, 2, 30, 3, 20, 4, 15, 5, 12, 6, 10
We want to find the factors such that product of any three is 60 & total of them must be 2.
∴ Possible factors are 3, 4,-5
Also co-efficient of \(x^2\), 12 - 20 - 15 = - 23
∴ \(E = x^3 + 2x^2 - 23x - 60 = (x + 3) (x + 4) (x - 5)\)

3.3 Factorization of Cubic Equations:
We can also solve cubic equation by using Parāvartya Śūtra, Lopanasthāpabhyām Sub-Śūtra, and Pūranapūrnabhyām Śūtra which means by completion as well as incompletion from second to fourth powers.

[12]
Solve: \(z^3 + 5z^2 - z - 5 = 0\)
∴ \(z^3 + 5z^2 = z + 5\)
But \((z + 2)^3 = z^3 + 6z^2 + 12z + 8\)
By Puraśarpabhyām,
∴ Substituting the value of \(z^3 + 6z^2\) from above, we have:
\((z + 2)^3 = z^2 + z + 5 + 12z + 8\)
\= z^2 + 13z + 13
\= z^2 + 13z + 22 - 9
\= (z + 2) (z + 11) - 9
\= (z + 2) (z + 2 + 9) - 9 (z + 2)^3
By substituting \(z + 2 = y\), \(y^3 = y (y + 9)\)
∴ \(y^3 = y^2 + 9y - 9\)
∴ \(y^3 - y^2 - 9y + 9 = 0\)
Sum of co-efficient of all the terms = 1 - 1 - 9 + 9 = 0
∴ (y-1) is one factor of this cubic equation.
By synthetic division we can find other two factors \((y + 3)\) and \((y - 3)\).
Thus, \(y = 1\) and \(y = \pm 3\) but \(z + 2 = y\)
∴ \( z + 2 = 1 \) & \( z + 2 = \pm 3 \)

Thus, final solution is \( z = -1, z = 1 \) & \( z = -5 \)

[13]

Solve: \( z^3 + 12z^2 + 47z + 60 = 0 \)

Let \( z^3 + 12z^2 = -47z - 60 \)

But \( (z + 4)^3 = z^3 + 12z^2 + 48z + 64 \)

By Pūrāṇapūrābhyām,

Substituting the value of \( z^3 + 12z^2 \) from above, we have:
\[ (z + 4)^3 = -47z - 60 + 48z + 64 = z + 4 \]

Consider \( z + 4 = y \) \∴ \( y^3 = y \) \∴ \( y^3 - y = 0 \)

\[ \therefore y (y^2 - 1) = 0 \quad \therefore y (y -1) (y + 1) = 0 \quad \therefore y = 0 \text{ and } y = \pm 1 \]

I.e. \( z + 4 = 0 \) and \( z + 4 = \pm 1 \text{ since } z + 4 = y \)

\[ \therefore z = 0, z = 3 \text{ and } z = -5 \]

[14]

Solve: \( 2z^3 - 12z^2 + 22z -12= 0 \)

\[ 2z^3 - 12z^2 + 22z -12 = 0 \]

\[ \therefore 2(z^3 - 6z^2 + 11z - 6) \]

Here, sum of all the coefficient = 1- 6 + 11 - 6 = 0

\[ \therefore z - 1 \text{ is one factor of the given expression.} \]

Here, sum = 0 and the last term is 12 whose factors are 1, 2, 3, 4, 6, 12

We find three numbers co-efficient of \( z^2 = a + b + c = - 6 \)

\[ \therefore \text{We select the group } -1,-2 \text{ and } -3 \text{ [Since } -1- 2 - 3 = - 6 \]}

Also testing for the co-efficient of \( z \).

Here, \( (a.b + b.c + c.a) =2 + 6 + 3 = 11 = \text{co-efficient of } z. \)

\[ \therefore z^3 - 6z^2 + 11z - 6 = (z - 1) (z - 2) (z - 3) = 0 \]

Thus, the final solution is \( z = 1, 2 \text{ and } 3 \)
Solve: $z^3 - 6z^2 + 5z + 12 = 0$

Here sum of co-efficient of all the term = $1 - 6 + 5 + 12 = 12 \neq 0$

∴ (z - 1) is not a factor of the given expression.

Sum of co-efficient of even power of $z$ = Sum of co-efficient of odd power of $z$.

i.e. $1 + 5 = -6 + 12 = 6$

∴ (z + 1) is a factor of the given expression.

∴ To find another two factors, dividing $E$ by the factor (z + 1), by using Sub - Sūtra Ādyamadyenāntyamantyena 1 as a 1st co-efficient and 12 as the last.

The co-efficient of the middle term = $6 - (1 + 12)$

= $6 - 13 = -7$

∴ $Q = z^2 - 7z + 12$

By using Ānurūpya and Ādyam Sub-Sūtra $Q = (z - 3)(z - 4)$

∴ $E = z^3 - 6z^2 + 5z + 12 = (z + 1)(z - 3)(z - 4) = 0$

Thus, the final solution is $z = -1, 3$ and $4$

OR

Here, sum of co-efficient of all the term = $1 - 6 + 5 + 12 = 12$ and the last term is 12

Factors of 12 = 1, 2, 3, 4, 6, 12

But we want three numbers such that their total should be -6

∴ We select the group 1, -3, -4.  [Since $1 - 3 - 4 = -6$]

Also testing for the co-efficient of $z$.

Here, $(a.b + b.c + c.a) = -3 + 12 - 4 = 5 = $ co-efficient of $z$.

∴ Given expression = $z^3 - 6z^2 + 5z + 12 = (z + 1)(z - 3)(z - 4) = 0$

∴ $z = -1, 3$ and $4$

[16]

Solve: $x^3 + x^2 - 19x - 34 = 0$

Here sum of co-efficient of all the term = $1 + 1 - 19 - 34 = -51 \neq 0$

∴ (x-1) is not a factor of the given expression.

Sum of co-efficient of even power of $x$ = Sum of co-efficient of odd power of $x$
i.e. \( 1 - 19 \neq 1 - 34 \)

\[
(x + 1) \text{ is not a factor of the given expression.}
\]

By trial and error method \((x + 2)\) is one factor of the given expression.

To find another two factors, dividing \(E\) by the factor \((x + 2)\), by using

By Ādyam Sūtra, \(1\) is 1\(^{st}\) coefficient & -17 is last.

\[
\text{\therefore Co-efficient of the middle term} = -17 - (1 - 17) = -17 - (-16) = -1
\]

\[
Q = x^2 - x - 17
\]

\[
\therefore \text{Given expression} = x^3 + x^2 - 19x - 34 = (x + 2) (x^2 - x - 17) = 0
\]

Factors of \(x^2 - x - 17\)

\[
2x - 1 = \pm \sqrt{1 - 4(1)(-17)}
\]

\[
\therefore 2x - 1 = \pm \sqrt{69}
\]

\[
\therefore 2x = 1 \pm \sqrt{69}
\]

\[
\therefore x = \frac{1}{2} \pm \frac{\sqrt{69}}{2}
\]

i.e. The value of \(x\) above are two roots of \(x^2 - x - 1\)

\[
\therefore \text{The final solution: } x = -2, x = \frac{1}{2} \pm \frac{\sqrt{69}}{2}
\]

\[
\text{OR}
\]

Here, sum of co-efficient of all the term \(= 1 + 1 - 19 - 34 = -51\) and the last term is 34

Factors of -34 = \(\pm 1, \pm 2, \pm 17\)

But we want three numbers such that their total should be \(+1\)

\[
\therefore \text{It is not possible to select the group of three numbers such that sum of three numbers is 1.}
\]

But if we substitute \(x = -2\) then we get remainder \(R = 0\).

\[
\therefore (x + 2) \text{ is one factor of the given expression.}
\]

\[
\therefore \text{To find another two factors, dividing } E \text{ by the factor } (x + 2), \text{ by using } \text{sub- sūtra}
\]

Ādyamadyenāntyamantyena orders substitution of 1 and -17 as the start & end prefixes.

\[
\therefore \text{The middle coefficient is } -17 - (1-17) = -17 - (-16) = -1
\]

\[
Q = x^2 - x - 17
\]

\[
\therefore \text{Given expression} = x^3 + x^2 - 19x - 34 = (x + 2) (x^2 - x - 17) = 0
\]
Factors of $x^2 - x - 17$ and final answer can be obtained as above.

$. \therefore$ The final solution is $x = -2, x = \frac{1}{2} \pm \frac{\sqrt{69}}{2}$

**Special equations (resembling Cubics):**

Equations which are of this special kind bring to mind cubic ones. Post simplifying we come to know that it is the simple equation of the first degree by using the formula Śūnyam Samuccaya.

[17]

Solve: $(y - 2)^3 + (y - 8)^3 = 2(y - 5)^3$

The given equation is solved by using the current system,

Post rigorous simplifications and quantitative manipulations, finally after so many steps we get the answer $y = 5$. But Vedic formula gives answer at first sight.

By Śūnyam Śāmyasamuccaye Śūtra,

$(y - 2) + (y - 8) = 2(y - 5)$

Avoiding the numeric term, right side $y - 5$ emerges.

i.e. $(y - 5) = 0$

$. \therefore$ The solution is $y = 5$.

[18]

Solve: $(z - 147)^3 + (z + 145)^3 = 2(z - 1)^3$

By Śūnyam Śāmyasamuccaye Śūtra,

$(z - 147) + (z + 145) = 2(z - 1)$

Avoiding the numeric term, right side $z - 1$ emerges.

i. e. $z - 1 = 0$

Therefore, $z = 1$ is a factor of the above given equation.

3.4 Factorization of biquadratic equations:

[19]

Solve: $z^4 + 3z^3 - 21z^2 - 83z - 60 = 0$
Here, the sum of co-eff. of odd power of $z = \text{the sum of co-eff. of even power of } z$

$\therefore \ 3 - 83 = 1 - 21 - 60 \ \therefore -80 = -80$

$\therefore (z + 1)$ is one factor of given equation.

By using Synthetic division,

\[
z^4 + 3z^3 - 21z^2 - 83z - 60 = (z + 1) (z^3 + 2z^2 - 23z - 60)
\]

Now we factorize, \((z^3 + 2z^2 - 23z - 60)\)

\[
\pm 1, \pm 60, \pm 2, \pm 30, \pm 3, \pm 20, \pm 4, \pm 15, \pm 5, \pm 12, \pm 6, \pm 10 \text{ are factors of } 60.
\]

But we want three numbers such that their total should be 2.

We select the group 3, 4, -5 \[\text{[since, } 3 + 4 - 5 = 2]\]

Also testing for the co-efficient of $z$.

Here, $a \cdot b + b \cdot c + c \cdot a = 12 - 20 -15 = - 23 = \text{co-efficient of } z$.

$\therefore$ Possible factors of -60 = (3) (4) (-5)

\[
z^3 + 2z^2 - 23z - 60 = (z + 3) (z + 4) (z - 5)
\]

$\therefore$ Given expression \(= z^4 + 3z^3 - 21z^2 - 83z - 60 = (z + 1) (z + 3) (z + 4) (z + 5) = 0\)

Thus, the final solution is $z = -1, -3, -4$ and -5.

[20]

Solve: $z^4 - 8z^3 + 8z^2 + 32z - 48 = 0$

$z^4 - 8z^3 = -8z^2 - 32z + 48$

But, \((z - 2)^4 = z^4 - 8z^3 + 24z^2 - 32z + 16\)

\[
= -8z^2 - 32z + 48 + 24z^2 - 32z + 16 \text{ [Since } z^4 - 8z^3 = - 8z^2 - 32z + 48] \\
= 16z^2 - 64z + 64
\]

$\therefore (z - 2)^4 = 16 (z - 2)^2$

By Pūrāṇapūrāṇabhyām,

Let $z - 2 = y$

$\therefore y^4 - 16y^2 = 0 \therefore y^2 (y^2 - 16) = 0$

$\therefore y^2 = 0 \text{ OR } (y^2 - 16) = 0$

Since $z - 2 = y$, $(z - 2)^2 = 0 \text{ OR } (z - 2)^2 - (4)^2 = 0$

The solution of $z^4 - 8z^3 + 8z^2 + 32z - 48 = 0$ is $z = 2, 2 \text{ OR } z = 6, -2$

Special type “biquadratic”:
One special type of biquadratic equation is the type in which the L.H.S. consists of the sum of the fourth power of two binomials whose exact middle binomial is possible and the R.H.S. is given.
The common format of biquadratic is \[(x + m) + n]^4 + [(x + m) - n]^4 = p\]

This type of problem can be solved by Vya\-\-sam\-\-i Sūtra or the Lopanasthāpanābhyām Sub-Sūtra which teaches us by taking mean of two quantities OR by dividing the central term the quartic can be converted in basic equation without odd powers, i.e. \(x^3\) and \(x\).

By using Vya\-\-sam\-\-i Sūtra, we can also solve the equation involving more complex numbers, fractions, surds, imaginary quantity and literal co-efficient.

**[21]**

Solve: \((x + 8)^4 + (x + 6)^4 = 3026\)

By applying Vya\-\-sam\-\-i Sutra, \((x + 7) = a\) = the average of the two binomials.
\[
\therefore (a + 1)^4 + (a - 1)^4 = 3026 \\
\therefore (a^4 + 4a^3 + 6a^2 + 4a + 1) + (a^4 - 4a^3 + 6a^2 - 4a + 1) = 3026 \\
\therefore 2a^4 + 12a^2 + 2 = 3026 \therefore 2a^4 + 12a^2 = 3024 \therefore a^4 + 6a^2 - 1512 = 0 \\
\therefore a^4 - 36a^2 + 42a^2 - 1512 = 0 \therefore (a^2 - 36) (a^2 + 42) = 0 \\
\therefore ([x + 7]^2 - 36] = 0 OR [(x + 7)^2 + 42] = 0 \\
\therefore (x + 7)^2 = 36 = (6)^2 OR (x + 7)^2 = - 42 \\
\therefore (x + 7) = \pm 6 \\
\text{Since} (x + 7)^2 = - 42 \text{ is not possible.}
\]

There will always be a non-negative value for self multiplication.

\[
\therefore (x + 1) (x + 13) = 0 \text{ is the only possible factors.}
\]

Thus, the final solution is \(x = -1\) and \(-13\)
Solve: \((x + 5)^4 + (x + 3)^4 = 1146\)

According to *Vyaśāsimā Sūtra*, \((x + 4) = a = \text{the average of the two binomials.}\)

\[\therefore (a +1)^4 + (a - 1)^4 = 1146\]
\[\therefore 2a^4 +12a^2 + 2 = 1146 \quad \therefore 2a^4 + 12a^2 = 1144\]
\[\therefore a^4 +6a^2 - 572 = 0 \quad \therefore a^4 +31a^2 - 25a^2 - 572 = 0\]
\[\therefore (a^2 - 25) (a^2 + 31) = 0 \quad \therefore a^2 = 25 \text{ OR } a^2 = -31\]

\(a^2 = -31\) is not possible.

Value by exponentiation to an even power will never be negative.

\[\therefore a^2 = 25\] is the only possible factor.

\[\therefore (x + 4)^2 - (5)^2 = 0; \text{ since } (x + 4) = a\]
\[\therefore (x + 9)(x -1) = 0\]

Thus, the final solution is \(x = 1, -9\)

**Special type of equations which look like “biquadratic”:**

- Some special equations whose appearance is similar to the “biquadratic” we come to know that it is simple equation of the first degree after writing it in a simplified form and can be solved by using *Śūnyamā Śāmyasamuccaye Sūtra.*

The following are the particular characteristics of this type of equations:

[a] Out of given the four binomial the summation of the 1\(^{st}\) & 2\(^{nd}\) should be equal to the summation of the 3\(^{rd}\) & 4\(^{th}\) i.e. \(N_1 + D_1 = N_2 + D_2\)

[b] Absolute term of the binomials \(N_2, N_1, D_1,\) and \(D_2\) must be in A.P.

[c] The subtraction of two right side binomials and three times the result achieved after subtracting two left side binomials must be equal i.e. \(D_2 - N_2 = 3(D_1 - N_1)\)

[d] Thus, if in each direction, the addition of numerator and denominator matches, then that sum \(= 0\) is the solution.
Solve the following:

\[
\frac{(x + 3)^3}{(x + 4)^3} = \frac{x + 2}{x + 5}
\]

By neglecting cube; \( N_1 + D_1 = N_2 + D_2 \)

Here, \((x + 2), (x + 3), (x + 4)\) and \((x + 5)\) are in A.P and \((5 - 2) = 3 (4 - 3)\)

Also, \( N_1 + D_1 = N_2 + D_2 \)

Therefore, by using Śūnyam Śāmyasamuccaye,

\[2x + 10 = 0\]

Thus the final solution is \(x = -5\).

Solve the following:

\[
\frac{(x - 4)^3}{(x - 6)^3} = \frac{x - 2}{x - 8}
\]

By neglecting cube; \( N_1 + D_1 = N_2 + D_2 \)

\[N_1 + D_1 = N_2 + D_2 = 2x - 10\]

Using Śūnyam Śāmyasamuccaye,

\[\therefore 2x - 10 = 0\]

\[\therefore x = 5\]

**Corollary:**

Because of the A.P. relationship between the binomial factors, if cross-addition of these factors as the addition of both the sides equal to zero gives us the solution.

By cross-multiplication, we have

\[(x - 8)(x - 4)^3 = (x - 2)(x - 6)^3\]

And by cross-addition of these factors as the addition of both the terms = \(4x - 20\).

\[\therefore 4x - 20 = 0 \implies x = 5\]
• The second special kind of special equations similar to “biquadratic” but afterward simplifying it we come to know that it is simple equation of the first degree and solved by using “Śūnyam Sāmuccaye” Sūtra.

If it is in the form of
\[(x + k) (x + l)(x + m) (x + n) = (x + p) (x +q) (x + r) (x + s)\]

It must contains the following characteristics:

The cross-addition must give same total on both sides
\[\text{i.e. } k + l + m + n = p + q + r + s\]

The summation of any two binomials on right side = the summation of any two binomials on left side.

After arranging all the terms in ascending order on both the sides,
\[k.l + m.n = p.q + r.s\]

By using “Śūnyam Sāmyasamuccaya” Sūtra, solving the total on both sides equal to zero is the required solution.

[25]

Solve:
\[
\frac{(x + 2)(x - 2)}{(x + 5)(x + 9)} = \frac{(x + 1)(x - 1)}{(x + 6)(x + 8)}
\]

\[(x + 5) (x + 9) (x + 1) (x - 1) = (x + 2) (x - 2) (x + 8) (x + 6)\]

The cross-addition must give same total on both sides.
\[\text{i.e. } 2 - 2 + 6 + 8 = 1 - 1 + 5 + 9\]

The sum of any two binomial on the right side
\[(x + 6) + (x + 1) = (x + 8) + (x - 1) = 2x + 7\]

The sum of any two binomial on the left side
\[(x + 5) + (x + 2) = (x + 9) + (x - 2) = 2x + 7\]
After arranging all the terms in ascending order on both the sides,

\((-2) .2 + 6 .8\) on the left = \((-1) .1 + 5.9\) on the right = 14

By using “Śūnyam śāmyṣamuccaye” Śūtra

The total on both sides = 4x + 14 = 0

\(\therefore 4x = -14\)

Therefore, the required solution is \(x = -\frac{14}{4} = -\frac{7}{2}\)