Chapter 5

Primordial Non-Gaussianity from Inflation with a feature in the Potential

Inflation is the best motivated and predictive early universe scenario, which is invoked to specify the spectrum of initial perturbations for structure formation. The period of cosmological inflation [Starobinsky (1980); Guth (1981)] can be attained, if the energy density of the Universe is dominated by the vacuum energy density which is associated with the potential of a scalar field $\varphi$, called the inflaton field. Through its kinematic properties, namely the acceleration of the Universe, the inflationary paradigm can elegantly solve the flatness, the horizon and the monopole problems of the standard Big–Bang cosmology. In addition, inflation can explain the production of the density perturbations in the early Universe [Mukhanov and Chibisov (1981); Mukhanov and Chibisov (1982); Hawking (1982); Starobinsky (1982); Bardeen et al. (1983)] which are the seeds for the Large Scale Structure (LSS) in the distribution of galaxies and the underlying dark matter and for the Cosmic Microwave Background Radiation (CMBR) temperature anisotropies that we observe today. The inflationary scenario has several generic predictions on the properties of these density perturbations:

- They are primordial. Namely, they were frozen at super horizon scales and entering the horizon.

- They are approximately scale-invariant. This is because, during $\approx 60$ e folds, each mode experiences the same expansion when they are stretched across the horizon.

- They are approximately Gaussian. In the simplest slow-roll inflation models,
the inflaton is freely propagating in the inflationary background at the leading order. This is found to be true for other inflationary models and for different inflationary mechanisms. So, the tiny primordial fluctuations can be treated as nearly Gaussian.

These generic predictions are in good agreement with cosmological observations, most notably those arising from measurements of the temperature anisotropy and polarization of the CMBR [Komatsu et al. (2011)]. Such observations lead to strong constraints on theoretical model building. To understand the micro-physics of the early universe, it will become increasingly necessary to extend the theoretical framework beyond the leading-order effects of scale-invariant, Gaussian fluctuations. In particular, deviations away from Gaussian statistics, i.e. the presence of a three-point function \(^1\) represents a potentially powerful discriminant between competing inflationary models and have attracted considerable recent interest [Maldacena (2003); Bartolo et al. (2004); Seery and Lidsey (2005); Lyth and Rodriguez (2005); Creminelli et al. (2006); Chen et al (2008)].

The primary goal of this chapter is to study the non-Gaussianity predicted by an inflationary model, whose inflaton potential has some localized feature. In order to measure the non-Gaussianity of this model, we compute the three-point function of the curvature perturbation and study its shape and scale dependence. The chapter is organized as follows: In section 5.1, we present a brief introduction to the primordial non-Gaussianity from inflation and describe the general features of the two–point and three-point correlation functions. Section 5.2 explores a model with a feature in the potential and the non-Gaussianity predicted by this model. We conclude the chapter in section 5.3.

### 5.1 Non-Gaussianity as a Probe of the Physics of the Early Universe

Learning the physics of inflation requires a profound understanding of the evolution and interactions of quantum fields in the very early Universe. Non-Gaussianity is a sensitive probe of the aspects of inflation that are difficult to probe by other means [Komatsu et al. (2009)]. Specifically, it is a probe of the interactions of the field(s)

---

\(^1\)By measuring the reduced \(n\)-point correlation functions \((n \geq 3)\), it is possible to test whether the primordial fluctuations obey Gaussian statistics or not. That is, if a given cosmological perturbation is Gaussian distributed, then the power spectrum, which is the Fourier transform of the two-point correlation function is all what is needed to completely characterize it from a statistical point of view. In such cases, if we consider higher order correlation functions, all the odd correlation functions vanish, while the even correlation functions can be simply expressed in terms of the two-point functions [Peterson and Tegmark (2011)].
driving inflation and therefore contains vital information about the fundamental physics operative during inflation.

The assumption of Gaussianity is motivated by the following view: the probability distribution \( P(\varphi) \) of quantum fluctuations \( \varphi \) of free scalar fields in the ground state of the Bunch-Davies vacuum, is a Gaussian distribution. Thus, the probability distribution of primordial curvature perturbations (in the co-moving gauge) \( \mathcal{R} \), generated from \( \varphi \) (in the flat gauge) as \( \mathcal{R} = -[H(\phi)/\dot{\phi}_0] \varphi \) [Mukhanov and Chibisov (1981); Mukhanov and Chibisov (1982); Hawking (1982); Starobinsky (1982); Bardeen et al. (1983)] would also be a Gaussian distribution. Here, \( H(\phi) \) is the expansion rate during inflation, and \( \phi_0 \) is the mean field, i.e., \( \phi = \phi_0 + \varphi \). This argument suggests that non-Gaussianity can be generated when

(a) scalar fields are not free; but have some interactions,

(b) there are non-linear corrections to the relation between \( \mathcal{R} \) and \( \varphi \), and

(c) the initial state is not a Bunch-Davies vacuum.

For (a) one can think of expanding a general scalar field potential \( V(\phi) \) to the cubic order or higher, \( V(\phi) = \bar{V} + V'\varphi + (1/2)V''\varphi^2 + (1/6)V'''\varphi^3 + \ldots \). The cubic (or higher-order) interaction terms can yield non-Gaussianity in \( \varphi \) [Falk et al. (1993)]. When perturbations in gravitational fields are included, there are many more interaction terms that arise from expanding the Ricci scalar to the cubic order, with coefficients containing derivatives of \( V \) and \( \phi_0 \), such as \( \dot{\phi}_0 V'' \), \( \dot{\phi}_0^3 / H \), etc. [Maldacena (2003)]. For (b) one can think of this relation, \( \mathcal{R} = -[H(\phi)/\dot{\phi}_0] \varphi \), as the leading-order term of a Taylor series expansion of the underlying non-linear (gauge) transformation law between \( \mathcal{R} \) and \( \varphi \). Salopek and Bond (1990) show that, in the single-field models, \( \mathcal{R} = 4\pi G \int_{\phi_0}^{\phi_0 + \varphi} d\phi (\partial \ln H / \partial \phi)^{-1} \). Therefore, even if \( \varphi \) is precisely Gaussian, \( \mathcal{R} \) can be non-Gaussian due to non-linear terms such as \( \varphi^2 \) in a Taylor series expansion of this relation. One can write this relation in the following form, up to second order in \( \mathcal{R} \),

\[
\mathcal{R} = \mathcal{R}_L - \frac{1}{8\pi G} \left( \frac{\partial^2 \ln H}{\partial \phi^2} \right) \mathcal{R}_L^2,
\]

where \( \mathcal{R}_L \) is the linear part of the curvature perturbation. It is found that the second term makes \( \mathcal{R} \) non-Gaussian, even when \( \mathcal{R}_L \) is precisely Gaussian. Also, non-Gaussian fluctuations could contain a signature of any departure of the inflaton from its standard Bunch-Davies vacuum. This has been suggested as a possible signature of trans-Planckian physics, and there has been much debate of the plausibility of such modifications [some examples include Chung et al. (2003); Martin and Brandenberger (2003); Holman and Tolley (2008); Ashoorioon and Shiu (2011)].
Non-Gaussianity can be evaluated by various techniques. A standard approach is to measure non-Gaussian correlations, i.e., the correlations that vanish for a Gaussian distribution. The two lowest-order measures of non-Gaussianity are the bi-spectrum and the tri-spectrum. Just as the power spectrum $P(k)$ represents the two-point function of the co-moving curvature perturbation $\mathcal{R}$ in Fourier space, the bi-spectrum $B(k_1, k_2, k_3)$ represents the three-point function (or, its Fourier transform) and the tri-spectrum represents the four-point function. The three-point function correlates density or temperature fluctuations at three-points in space. Equivalently, the bi-spectrum, $B(k_1, k_2, k_3)$, correlates fluctuations with three wave vectors [Fig. 5.1]. These three vectors form a triangle in Fourier space, and thus there are many triangles one can form and look for. The amount of information captured by the bi-spectrum is therefore potentially far greater than that of the power spectrum, which correlates only two wave vectors with the same magnitude. For instance, a non–vanishing three-point function (or its Fourier transform, the bi-spectrum) of scalar perturbations is an indicator of non-Gaussianity in the cosmological perturbations.

A phenomenological way of parametrizing the level of non-Gaussianity in cosmological perturbations is to introduce a non–linearity parameter, $f_{\text{NL}}$ through Bardeen’s gravitational potential, $\Phi$, [Bartolo et al. (2004)]

$$\Phi = \Phi_L + f_{\text{NL}} \left( \Phi_L^2 - \langle \Phi_L^2 \rangle \right),$$

(5.2)

where $\Phi_L$ represents the gravitational potential at linear order. The non–linearity parameter, $f_{\text{NL}}$, which corresponds to the amplitude of the bi-spectrum normalized to the square of the power spectrum of primordial curvature fluctuations might have a non–trivial scale dependence. Any scale dependence of the non-linearity pa-
rameter provides a new and potentially powerful observational probe of inflationary physics. In momentum space, the three-point function (bi-spectrum), arising from the local non-Gaussianity is dominated by the so-called “squeezed” configuration, where one of the momenta is much smaller than the other two and it is parametrized by the non-linearity parameter, $f_{\text{local}}^{\text{NL}}$. Other models, such as DBI inflation [Al-\ishahiha (2004)] and ghost inflation [Arkani-Hamed (2004)] predict a different kind of primordial non-Gaussianity, called “equilateral”, because the three-point function for this kind of non-Gaussianity is peaked on equilateral configurations, in which the lengths of the three wave-vectors forming a triangle in Fourier space are equal [Babich et al. (2004)]. Non-Gaussianity for the equilateral case is parametrized by an amplitude $f_{\text{equil}}^{\text{NL}}$ [Creminelli et al. (2006)].

Any detection of non-Gaussianity would be a significant challenge to the currently favored models of the early universe. In the context of single-field inflation in which the scalar field is rolling down the potential slowly, the quantities, $H$, $V$, and $\phi$ are changing slowly. Therefore, one generically expects that $f_{\text{local}}^{\text{NL}}$ is small, of the order of the so-called slow-roll parameters $\epsilon$ and $\eta$, which are typically of order of $10^{-2}$ or smaller. In this sense, the single-field, slow-roll inflation models are expected to produce a tiny amount of non-Gaussianity [Maldacena (2003); Seery and Lidsey (2005); Chen et al. (2006)]. These contributions from the epoch of inflation are much smaller than those from the ubiquitous, second-order cosmological perturbations, i.e., the non-linear corrections to the relation between $\Phi$ and $R$, which produces $f_{\text{local}}^{\text{NL}}$ of the order of unity.

Various other inflationary models predict a significant amount of non-Gaussianity generated either during or immediately after inflation when the co-moving curvature perturbation becomes constant on super-horizon scales [Bartolo et al. (2004)]. Single-field [Acquaviva (2003); Maldacena (2003)] and two (multi)-field models of inflation [Bartolo et al. (2002); Lyth (2005)] generically predict a tiny level of non-Gaussianity. Isocurvature fluctuations [Linde and Mukhanov (1997); Peebles (1997)], features in a scalar-field potential [Wang and Kamionkowski (2000)], or a curvaton mechanism (by which late-time decay of a scalar field generates curvature perturbations from isocurvature fluctuations) [Enqvist and Sloth (2002); Lyth and Wands (2002); Lyth et al. (2003)] can generate stronger, potentially detectable, non-Gaussianity. Oscillatory features in the inflaton potential [Hazra et al. (2010); Aich et al. (2011)] has been studied as a way of generating large non-Gaussianity by a kind of resonance effect [Chen et al. (2008)]. Since the non-Gaussianity part of the primordial curvature perturbation is a local function of the Gaussian part generated on super-horizon scales, the level of non-Gaussianity generated by these models is local.
Although breaking of slow-roll usually results in a premature termination of inflation, it is possible to break it temporarily for a brief period, without terminating inflation, by some features (steps, dips, etc) in the shape of the potential. In such a scenario, a large non-Gaussianity may be generated at a certain limited scale at which the feature exists [Kofman et al. (1991); Wang and Kamionkowski (2000); Komatsu et al. (2003)]. The structure of non-Gaussianity from features is much more complex and model-dependent [Chen et al. (2007); Chen et al. (2008)].

The first direct comparison between inflationary non–Gaussianity and observational data was attempted for the Cosmic Background Explorer’s Differential Microwave Radiometer (COBE - DMR) data [Smoot et al. (1992)], using the angular bi-spectrum – the harmonic counterpart of the three-point correlation function [Komatsu et al. (2002)]. A very weak constraint, $|f_{NL}| < 1500$ (68% CL) was found. Although this constraint is still too weak to be useful, it explicitly demonstrates that measurements of non-Gaussianity can put quantitative constraints on inflationary models. Present limits on non-Gaussianity are summarized by $-4 < f_{NL}^{rec} < 90$ and $-151 < f_{NL}^{equil} < 253$ at 95% CL [Komatsu et al. (2011)]. An accurate calculation of the primordial bi-spectrum of cosmological perturbations has become an extremely important issue, as a number of present and future experiments, such as WMAP and Planck will allow us to constrain or detect non–Gaussianity of CMBR anisotropy with high precision.

5.1.1 Two-point correlation function and power spectrum of scalar perturbations

The adiabaticity\textsuperscript{2} and near scale-invariance of the primordial fluctuations can be determined by measuring the power spectra of fluctuations; the upper limits on the isocurvature spectrum constrains non-adiabaticity, while the slope of the scalar (curvature) power spectrum constrains the deviation from scale-invariance. The scalar power spectrum, which characterizes the properties of a perturbation field is defined as

$$P_s(k) = \frac{k^3}{2\pi^2} |u_k|^2,$$

where $u_k$ is the amplitude of the curvature perturbation and the power spectrum is evaluated at the epoch of horizon exit, $k = aH$. The Fourier modes of the curvature

\textsuperscript{2}In the context of primordial perturbations, adiabatic means that the primordial stress-energy of the universe was governed by a single, spatially uniform equation of state – in other words, on the surfaces of constant temperature, the densities of the various components (e.g. baryons, CDM, neutrinos, etc.) are uniform and these components share a common velocity field.
perturbation $\nu_k$ satisfies the equation
\[ \nu_k'' + k^2 \nu_k - \frac{z''}{z} \nu_k = 0, \tag{5.4} \]
where we have used the definitions $\nu_k \equiv z u_k$, and $z \equiv a \sqrt{2\epsilon}$. To describe the slope of the power spectrum, it is a standard practice to define a spectral index, $n_s(k)$ through
\[ n_s(k) - 1 \equiv \frac{d \ln P_s}{d \ln k}, \tag{5.5} \]
which, in terms of slow roll parameters, is given as
\[ n_s \simeq 1 - 2\epsilon - \eta, \tag{5.6} \]
where
\[ \epsilon \equiv \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2, \quad \eta \equiv \frac{1}{8\pi G} \frac{V''}{V} \tag{5.7} \]
Now, the running of the spectral index ($\alpha_s$) is defined as $\alpha_s = \left( \frac{dn_s}{d\ln k} \right)$. The shape of the two-point correlation function, characterized by the so-called primordial spectral index, $n_s$ and the running index, $\alpha_s$, and the existence or absence of primordial gravitational waves, would provide important constraints on large classes of inflationary models.

## 5.1.2 Three-point correlation function

Measurements of power spectrum alone have limited potential in revealing the physics of inflation. The power spectrum is determined by the inflationary expansion rate and its time dependence which in turn relates to the evolution of inflationary energy density. However, it does not strongly constrain the interactions of the field(s) associated with this energy density. The power spectrum is therefore degenerate in terms of the inflationary action that can lead to it.

Non-Gaussianity typically manifests itself through a non-vanishing three-point correlation function of the curvature perturbation, and can potentially discriminate between models of inflation with degenerate power spectra. More optimistically, if the amplitude of the three-point function is large enough for us to map its dependence on both the scale and shape of the momenta triangle, this will be an enormous boon to early universe cosmology, since the three-point function encodes information on both these properties. However, to realize this possibility, we need to compute the three-point correlation function for a given inflationary model, and compare the predictions with the data.
The three-point function is written in terms of integrals over conformal time $\tau$, the integrand being a finite expansion in slow roll parameters. Since it is convenient to work in the Arnowitt - Deser - Misner (ADM) formalism [Arnowitt et al. (1960)], we write the metric as

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt)$$  \hspace{1cm} (5.8)

In the ADM formulation, spatial coordinate re-parametrizations are an explicit symmetry while time re-parametrizations are not so obviously a symmetry. The ADM formalism is designed so that one can think of $h_{ij}$ and $\phi$ as the dynamical variables and $N$ and $N^i$ as Lagrange multipliers [Mukhanov et al. (1992); Maldacena (2003)]. We work in a co-moving gauge where the three dimensional metric $h_{ij}$ takes the form

$$h_{ij} = a^2 e^{2\zeta} \delta_{ij}$$  \hspace{1cm} (5.9)

where we have neglected the tensor perturbations. $a$ is the scale factor of the universe and $\zeta$ parameterizes the scalar perturbations which remains constant outside the horizon in this gauge. The inflaton fluctuation $\delta \phi$ vanishes in this gauge, which makes the computations simpler.

Using the ADM metric ansatz, the action [for details, see Embacher (1995)] is given by

$$S = \frac{1}{2} \int \sqrt{h} \left[ NR^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^i \partial_i \phi)^2 - Nh^{ij} \partial_i \phi \partial_j \phi \right]$$  \hspace{1cm} (5.10)

where the reduced Planck mass $M_{pl}$ is set to unity for convenience. The three-dimensional Ricci curvature $R^{(3)}$ is computed from the metric $h_{ij}$. The symmetric tensor $E_{ij}$ is defined as

$$E_{ij} = \frac{1}{2} (h_{ij} - \nabla_i N_j - \nabla_j N_i), \hspace{1cm} E = E^i_i$$

As the non-Gaussian effects come from the cubic interaction terms in the full action, which arise from the non-linearities of the Einstein action as well as from non-linearities in the potential for the scalar field, we must compute the action Eq. (5.10) to cubic order$^3$ in the perturbation [Maldacena (2003); Seery and Lidsey (2005);

$^3$Details of this computation are provided in the appendix.
Chen et al. (2007): \[
S_3 = \int dtd^3x \left\{ a^3 \epsilon^2 \dot{\zeta}^2 + ae^2\zeta(\partial \zeta)^2 - 3ae\dot{\zeta}(\partial \zeta)(\partial \chi) + \frac{a^3 \epsilon d\eta}{2 dt} \zeta^2 \dot{\zeta} + \frac{\epsilon}{2a}(\partial \zeta)(\partial \chi) \partial^2 \chi + \frac{\epsilon}{4a}(\partial^2 \zeta)(\partial \chi)^2 + 2f(\zeta) \frac{\delta L}{\delta \zeta} \right\} \] (5.11)

where \[
\chi = a^2 \epsilon \dot{\zeta} \] (5.12)
\[
\frac{\delta L}{\delta \zeta} |_{1} = a \left( \frac{d}{dt} \partial^2 \chi \right) + H \partial^2 \chi - \epsilon \partial^2 \zeta \] (5.13)
\[
f(\zeta) = \frac{\eta}{4} \zeta^2 + \text{terms with derivatives of } \zeta \] (5.14)

Here \(\partial^{-2}\) is the inverse Laplacian and \(\frac{\delta L}{\delta \zeta} |_{1}\) is the variation of the quadratic action with respect to the perturbation \(\zeta\). The cubic action Eq. (5.11) is exact for arbitrary \(\epsilon\) and \(\eta\). It is to be noted that the highest power of slow-roll parameters, which appears is of \(O(\epsilon^3)\), when we recall that \(\chi\) contains a multiplicative factor of \(\epsilon\). The last term in Eq. (5.11) can be absorbed by a field redefinition of \(\zeta\),

\[
\zeta \rightarrow \zeta_n + f(\zeta_n) \] (5.15)

After this redefinition, the interaction Hamiltonian in conformal time is

\[
H_{int}(\tau) = - \int d^3x \left\{ a \epsilon^2 \dot{\zeta}^2 + ae^2\zeta(\partial \zeta)^2 - 2\epsilon \zeta'(\partial \zeta)(\partial \chi) \right.
+ \left. \frac{a}{2} \eta' \zeta^2 \right. + \left. \frac{\epsilon}{2a}(\partial \zeta)(\partial \chi) \right. + \left. \frac{\epsilon}{4a}(\partial^2 \zeta)(\partial \chi)^2 \right\},
\]

where ‘ represents derivative with respect to conformal time \(\tau\). Like the power-spectrum, higher order correlation functions are also computed in Fourier-space. While the power spectrum depends only on one wave number, the three-point function depends on three, which form a triangle in \(k\)-space. That is,

\[
\langle \zeta(x)\zeta(x)\zeta(x) \rangle = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} \langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle e^{i(k_1+k_2+k_3) \cdot x} \] (5.16)

The three-point function after field redefinition becomes

\[
\langle \zeta(x_1)\zeta(x_2)\zeta(x_3) \rangle = \langle \zeta_n(x_1)\zeta_n(x_2)\zeta_n(x_3) \rangle
+ \frac{\eta}{2} (\langle \zeta_n(x_1)\zeta_n(x_2) \rangle \langle \zeta_n(x_1)\zeta_n(x_3) \rangle + \text{sym})
+ O(\eta^2(P^C_k)^3), \] (5.17)
where the slow roll parameters are evaluated at the end of inflation.

We consider only the first term in Eq. (5.14), since all other terms involve at least one derivative of $\zeta$, and vanish when evaluated outside the horizon. The three-point correlation function at some time $\tau$ after horizon exit is then the vacuum expectation value of the three-point function in the interaction vacuum

$$\langle \zeta(\tau, k_1)\zeta(\tau, k_2)\zeta(\tau, k_3) \rangle = -i \int_{\tau_0}^{\tau} d\tau' a \langle [\zeta(\tau, k_1)\zeta(\tau, k_2)\zeta(\tau, k_3), H_{int}(\tau')] \rangle \quad (5.18)$$

The three-point on the left hand side is evaluated with the interaction vacuum while the the right hand side is evaluated at the true vacuum. The interaction Hamiltonian, $H_{int}$ evolves the true vacuum to the interaction vacuum at the time we evaluate the three-point function. Quantizing $\zeta$, by decomposing into its Fourier modes and writing

$$\zeta(\tau, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \zeta(\tau, p)e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (5.19)$$

with associated operators and mode functions,

$$\zeta(\tau, k) = u(\tau, k)a(k) + u^*(\tau, -k)a^\dagger(-k) \quad (5.20)$$

where $a$ and $a^\dagger$ satisfy the commutation relation

$$[a(k), a^\dagger(k')] = (2\pi)^3 \delta^3(k - k').$$

The “true” vacuum is annihilated by the lowering operator $a(k)$.

From equations (5.16) and (5.18) and recalling that $\chi_k \propto \epsilon \dot{\zeta}_k$, it is clear that the three-point correlation function consists of a sum of integrals of the form

$$I_{\epsilon^2} \propto \Re \left[ \prod_i u_i(\tau_{end}) \int_{\tau_0}^{\tau_{end}} d\tau e^{2a^2\zeta_1(\tau)\zeta_2(\tau)\zeta_3(\tau)} + \mathcal{O}(\epsilon^3) \right] \quad (5.21)$$

and

$$I_{\epsilon\eta'} \propto \Re \left[ \prod_i u_i(\tau_{end}) \int_{\tau_0}^{\tau_{end}} d\tau e^{\eta'a^2\zeta_1(\tau)\zeta_2(\tau)\zeta_3(\tau)} \right] \quad (5.22)$$

$\xi_n$ is either $u_{k_n}^*$ or $du_{k_n}^*/d\tau$. In a single field model $u_k(\tau) \to \text{constant}$ after Hubble crossing as it freezes out, while $u_k(\tau) \to e^{-ik\tau}$ oscillates rapidly at early times, so its contribution to the integral tends to cancel. Thus the integral is dominated by the range of $\tau$ during which the modes leave the horizon.
5.2 Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

It is evident from the foregoing discussion that detection of a significant amount of non-Gaussianity and its shape, either from the CMBR or from the LSS offers the possibility of opening a window into the dynamics of the universe during the very first stages of its evolution and to the mechanism which gave rise to the cosmological perturbations. There are many possible sources of non-Gaussianities, such as features in the potential, non-canonical kinetic terms, preheating, the curvaton scenario, particle production during inflation, etc. It is expected that large non-Gaussianity, produced when the inflaton potential has some localized feature becomes shape and scale dependent, and modes that exit the Hubble scale around the time the field crosses the feature can pick up large non-Gaussianities.

In this section, we study the non-Gaussianity predicted by a model with a Higgs-like potential, which is also used in the hybrid inflationary scenario. In order to measure the non-Gaussianity of this model, we compute the three-point function of the curvature perturbation and show that there will be three more terms which contribute at order two of slow roll parameters, to the three-point function, as compared to standard slow roll inflation. These terms affect scales that exit Hubble horizon around the time the field crosses the feature in the potential. We compute these extra terms and study the shape and scale dependence of the three-point function.

5.2.1 The model with a feature in the potential

Our basic model is one which is proposed to explain the small wiggles or local spikes super imposed on an approximately scale invariant spectrum. Here the inflaton potential experiences a sudden small change in its second derivative (the effective mass of the inflaton) \[ M. Joy \text{ et al. (2008); M. Joy \text{ et al. (2009)} \]. In this model, the resulting density perturbation has a quasi flat power spectrum with a break in its slope (a step in spectral index \( n_s \)). The step in the spectral index is modulated by characteristic oscillations and results in large running of the spectral index, localized over a few e-folds of scales. A field theoretical model giving rise to such behavior of the inflationary potential is based on a fast phase transition experienced by a second scalar field weakly coupled to the inflaton. Such a transition is similar to that which terminates inflation in the hybrid inflationary scenario. This scenario suggests that the observed local running of spectral index in the WMAP data may be caused by a fast second order phase transition which occurred during
5.2: Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

Inflation.

The model is described by the following potential:

\[
V(\psi, \phi) = \frac{1}{4\lambda} (M^2 - \lambda \psi^2)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g^2}{2} \phi^2 \psi^2. \tag{5.23}
\]

At \( \phi_c = M/g \), the curvature of \( V(\psi, \phi) \) along the \( \psi \) direction vanishes so that \( m_\psi^2 = \frac{\partial^2 V}{\partial \psi^2} = g^2 \phi^2 - M^2 > 0 \) for \( \phi > \phi_c \), while \( m_\psi^2 < 0 \) for \( \phi < \phi_c \). This implies that for large values of the inflaton \( \phi \), the auxiliary field \( \psi \) rolls towards \( \psi = 0 \). However, once the value of \( \phi \) falls below \( \phi_c \), the \( \psi = 0 \) configuration is destabilized, resulting in a rapid cascade (mini-waterfall) which takes \( \psi \) from \( \psi = 0 \) to its minimum value. That is, just before the (weakly second order) phase transition, \( \phi > \phi_c, \psi = 0 \) so

\[
V(\phi) = \frac{M^4}{4\lambda} + \frac{m^2 \phi^2}{2}, \tag{5.24}
\]

and \( \partial^2 V / \partial \phi^2 = m^2 \). Soon after the transition, \( \phi < \phi_c, \psi^2 = (M^2 - g^2 \phi^2) / \lambda \) and

\[
V(\phi) = \frac{1}{2} (m^2 + \frac{g^2 M^2}{\lambda}) \phi^2 - \frac{g^4 \phi^4}{4\lambda}. \tag{5.25}
\]

Thus, the second derivative of the potential has a discontinuity at \( \phi = \phi_c \) [Fig. 5.2].

The potential has four parameters, namely, \( M, m, g \) and \( \lambda \) whose typical values are given in Table 5.1 below, where \( N \) represents the number of e-folds after the phase transition.

For the potential given by Eq.(5.23), one can immediately notice that \( n_s \) will have discontinuity due to its dependence on \( \eta \) (Recall Eqs. (5.6) and (5.7)). This
5.2: Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values for N=60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M/M_{pl}$</td>
<td>$8 \times 10^{-4}$</td>
</tr>
<tr>
<td>$m/M_{pl}$</td>
<td>$5.3 \times 10^{-7}$</td>
</tr>
<tr>
<td>$g$</td>
<td>$3 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 5.1: Typical values for the potential parameters

will result in the power spectrum having a jump in its slope at a scale set by $\phi_c$. Thus the power spectrum ($P(k) \propto k^{n_s-1}$) has small oscillations superimposed around the scale of change of slope, as shown in M. Joy et al. (2008). Fig. 5.3 shows the quasi flat $P(k)$ for the mini waterfall hybrid model.

Figure 5.3: Quasi-flat power spectrum for the model with a step in $V''$

5.2.2 Background evolution

By initiating evolution with initial field value $\phi_i > \phi_c$ with the potential given by Eq. (5.23), the coupled system of equations of the background inflaton and the scale factor of expansion of space time exhibits inflation with the inflaton rolling down the potential, till inflation ends, and $\phi$ finally oscillates about the potential minimum. The slow roll conditions $\epsilon, \eta < 1$, where $\epsilon, \eta$ are given by

$$\epsilon \equiv 3 \frac{\dot{\phi}^2/2}{\phi^2/2 + V}; \quad \eta = \frac{\dot{\epsilon}}{\epsilon H},$$

(5.26)
5.2: Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

are satisfied throughout the inflationary period. Therefore, $\eta$ has a discontinuity at $\phi_i = \phi_c$ since it is proportional to $V''$. Figs. 5.4 and 5.5 shows the behavior of $\eta$ and $\eta'$ with respect to the conformal time $\tau$. We explore the consequences of this discontinuity in the behavior of the three-point functions of the perturbations of the dynamical variables. For our model, $\eta'$ can be written as

$$\eta' = 6aH \left( 2\epsilon - \frac{\eta}{2} - \frac{5}{6}\epsilon \eta + 2\epsilon^2 - \frac{\eta^2}{12} - \frac{V_{\phi\phi}}{3H^2} \right)$$

(5.27)

Figure 5.4: Slow-roll parameter $\eta$ vs. conformal time $\tau$

Therefore,

$$\epsilon \eta' = 6aH \left( 2\epsilon^2 - \frac{\epsilon \eta}{2} - \frac{5}{6}\epsilon^2 \eta + 2\epsilon^3 - \frac{\epsilon \eta^2}{12} - \frac{\epsilon V_{\phi\phi}}{3H^2} \right)$$

(5.28)

Since the three-point function is going to be dependent on $\eta'$, we want to calculate exactly how large it will be.

5.2.3 Computation of the three-point correlation function and the non-Gaussianity

Our approach is based on the numerical evaluation of both the perturbation equations and the integrals which contribute to the three-point function. To compute the three-point correlation function, the mode solutions, $u_k$ are simply substituted into Eq. (5.18), and integrated from $\tau_0$ through to the end of inflation. This integral can be done semi–analytically for simple models, provided the slow roll parameters
are small and relatively constant. For standard single field slow roll inflation, the three terms of order $\epsilon^2$ in (Eq. 5.16) are the dominant contributors to the three-point function and the other terms of order $\epsilon \eta'$ and $\epsilon^3$ were neglected in Maldacena (2003); Seery and Lidsey (2005) and Chen et al. (2008). In the presence of a step in the potential, the $\epsilon \eta'$ term becomes large and dominant [Chen et al. (2007); Chen et al. (2008)]. As shown by Fig. 5.6, a step in the second order derivative of the potential will make the $\epsilon \eta'$ term much larger compared to $\epsilon^2$ term, thereby leading to a modification to the standard slow-roll answer. Hence we now focus on the $I_{\epsilon \eta'}$ term, and discuss the sub-leading terms in an appendix.

\[ I_{\epsilon \eta'} \propto \epsilon \left( \prod_i u_i(\tau_{\text{end}}) \right) \int_{-\infty}^{\tau_{\text{end}}} d\tau a^2 \epsilon \eta' \left( u_1^*(\tau) u_2^*(\tau) \frac{d}{d\tau} u_3^*(\tau) \right) + \text{two perm} \times (2\pi)^3 \delta^3 \left( \sum_i k_i \right) + c.c \]  

(5.29)

where the two perm stands for two other terms that are symmetric under permutations of the indices 1, 2 and 3, where 1, 2, 3 are shorthand for $k_1$, $k_2$ and $k_3$.

In order to integrate Eq. (5.29) numerically, we split the integral into two parts:

\[ I = \int_{-\infty}^{\tau_0} + \int_{\tau_0}^{\tau_{\text{end}}} = I_1 + I_2 \]  

(5.30)

Here, $\tau_0$ is an arbitrary time when all three modes are well inside the horizon and $\tau_{\text{end}}$ corresponds to a moment long after horizon exit. The integral $I_2$ can be numer-
5.2: Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

Figure 5.6: Contribution of $I_{\epsilon\eta'}$ [red] and $I_{\epsilon^2}$ terms [black], at $k = k_{\text{feature}}$, indicating that the contribution of $|\epsilon^2|$ term is almost zero whereas that of $|\epsilon\eta'|$ term becomes larger, soon after the conformal time $\tau$ corresponding to the phase transition

Physically evaluated in a straightforward manner. However, $I_1$ suffers from the cut-off dependence. That is, these integrals are formally convergent in the limit $\tau \to -\infty$, but cutting them off at a finite value of $\tau$ exposes the oscillatory nature of the integrand whose amplitude blows up rapidly as $\tau$ grows large and negative. Physically, when the modes are well within the horizon, they oscillate rapidly compared to the rate of change of the interaction terms; thus the contribution to the integrals almost cancels. When all three modes are well inside the horizon, their phase and amplitude is well described by the WKB approximation:

$$\nu_k \approx \frac{1}{\sqrt{2\alpha(k)}} \exp \left[ i \int \sqrt{\alpha(k)} d\tau \right] + \text{c.c.} \quad (5.31)$$

where $\alpha(k) = k^2 + z''/z$. Deep within the horizon, $k^2 \gg z''/z$, the modes would not see the curvature term and hence it will propagate as a plane wave $\nu_k \propto \exp[ik\tau]/\sqrt{k}$. Therefore, $I_1$ can be obtained as:

$$I_1 = \int_{-\infty}^{\tau_0} d\tau \theta(\tau) \frac{1}{\sqrt{8k_1k_2k_3}} e^{iK(\tau-\tau_0)} \quad (5.32)$$
5.2: Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

where \( K = k_1 + k_2 + k_3 \) and for our model,

\[
\nu_k(\tau_0) = \frac{\sqrt{\pi} \tau_0}{2} H_{\mu_1}^{(2)}(k\tau_0),
\]

where \( H_{\mu_1}^{(2)}(k\tau_0) \) is the Hankel function and \( \mu_1 = \frac{3}{2} - \frac{V''}{3H_0^2} + 3\epsilon_0 \), where \( V'' \equiv \left( \frac{d^2 V}{d\phi^2} \right) \) before phase transition.

Also, \( \theta \) is some function of \( \tau \) given by

\[
\theta(\tau) = \frac{a^2}{z^3} \epsilon \eta'
\]

Integrating Eq. (5.32) by parts

\[
I_1 = \frac{1}{\sqrt{8k_1k_2k_3}} \left[ \theta(\tau) - \frac{i}{K} e^{iK(\tau-\tau_0)} \right]_{-\infty}^{\tau_0} - \int_{-\infty}^{\tau_0} d\tau \frac{d\theta(\tau)}{d\tau} \frac{1}{\sqrt{8k_1k_2k_3}} \left( -\frac{i}{K} \right) e^{iK(\tau-\tau_0)}.
\]

Applying the limits and integrating by parts a second time, we get

\[
I_1 = \frac{1}{\sqrt{8k_1k_2k_3}} \left[ \left( -\frac{i}{K} \right) \theta(\tau_0) - \left( -\frac{i}{K} \right)^2 \frac{d\theta(\tau_0)}{d\tau} \right] + \int_{-\infty}^{\tau_0} d\tau \frac{d^2\theta(\tau)}{d\tau^2} \frac{1}{\sqrt{8k_1k_2k_3}} \left( -\frac{i}{K} \right)^2 e^{iK(\tau-\tau_0)}.
\]

The resulting equation is convergent and can be evaluated efficiently. Every three-point correlation function has two main attributes: shape and scale. Chen et al. (2007) introduced a new parameter \( \mathcal{G} \) to describe non-Gaussianities with both shape and scale dependence:

\[
\mathcal{G}(k_1, k_2, k_3) \equiv \frac{1}{k_1k_2k_3} \frac{(k_1k_2k_3)^2}{\delta^3(k_1 + k_2 + k_3) P_k^2(2\pi)^7} \langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle,
\]

where \( \langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle \) can be can be obtained from Eqs. (5.18) and (5.29). In the absence of a sharp feature [Chen et al. (2008)], Eq. (5.37) reduces to the local form with

\[
\mathcal{G} = (3/10)f_{NL}^{local} \sum k_i^3.
\]

Using Eqs. (5.37) and (5.38), we can calculate the non-Gaussianity parameter, \( f_{NL}^{local} \) for our model.

\[
\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle_{local} = (2\pi)^7 \delta^3(k_1 + k_2 + k_3) \left( -\frac{3}{10}f_{NL}ight) \frac{\Sigma_i k_i^3}{\Pi_i k_i^3}.
\]
5.2: Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

The local non-Gaussianity, $f_{NL}^{equil}$ for the equilateral configuration ($k_1 = k_2 = k_3$) in momentum space is shown in Fig. (5.8) whereas for the $f_{NL}^{local}$ squeezed configuration ($k_3 \ll k_1 = k_2$) is depicted in Fig.(5.7). It is found that the non-Gaussianity parameter, $f_{NL}$ is $\lesssim 1$, which is much bigger than the slow-roll parameters. As expected, the mini waterfall model is giving a slightly higher $f_{NL}$ value compared to the standard single field inflationary model. The distinctive
5.2: Non-Gaussianity from Inflation with a Step in the Second derivative of the Potential

Feature of this non-Gaussianity is its characteristic ringing behavior. We see that the oscillations in $f_{NL}$ in this model last for a much longer range of $k$ values, as compared to the previously studied models [Chen et al (2007); Chen et al. (2008)]. In this sense, the model is potentially distinguishable from models with other features in the potential.

Figure 5.9: The shape of non-Gaussianity $G/k^3$ for the equilateral configuration, x axis is $k_1/k_{feature}$ and the y axis is $k_2/k_{feature}$.
5.3 Conclusions

A large number of inflationary models have been predicted which can produce a power spectrum which fits the CMB data. But the single field, slow-roll models of inflation generically yield a negligible primordial non-Gaussianity, which is not even observable. Thus, bi-spectrum analysis of CMB data has become the most promising candidate to discriminate between degenerate inflationary models.

In this chapter, we considered a variant of hybrid inflation where the potential has a discontinuity in its second derivative with respect to field. This describes a fast second order phase transition during inflation that occurs in some other scalar field weakly coupled to the inflaton. During inflation, the effective mass of the inflaton changes rapidly, and this change results in a universal local feature being imprinted onto the primordial spectrum of density perturbations. Also, a fast second order phase transition which occurred during inflation may cause the observed local running of the spectral index in the WMAP data.

The three-point correlation function is numerically integrated for this anomalous inflationary model in which slow roll is violated for a brief moment. With a step in the second derivative of the potential, the resulting transient violation of the slow-
roll leads to an oscillating and scale dependent three-point function. For typical values of the parameters in the potential, non-Gaussianity associated with the model with a feature in the potential is found to be ten times larger than those in standard single field, slow-roll inflation. These large non-Gaussianities are generated when the non-linear coupling between the modes, $\epsilon \eta'$ becomes temporarily large during the period of phase transition. The distinctive feature of this non-Gaussianity is its characteristic ringing behavior. Since the oscillations in $f_{NL}$ last for a much longer range of $k$ values when compared to the previously studied models, the model is potentially distinguishable from models with other features in the potential. It is also remarkable that the non-Gaussianity $f_{NL}$ associated with the model is larger than that in standard slow-roll inflation and may even be within the range of next generation CMB experiments such as Planck.