Chapter 4

A Dark Energy Model that mimics the Cosmological Constant

Three main classes of models of cosmic acceleration have so far been proposed. They include Cosmological Constant ($\Lambda$CDM) Model, dark energy models and modified gravity models. The last possibility dispenses entirely with the mysterious dark energy and instead modifies gravity at the large scales. However, this approach faces serious problems, like for e.g, inability to pass solar system constraints [Chiba (2003)]. The simplest explanation (yet the most difficult from the field theoretic point of view) for the cosmic acceleration is provided by the Cosmological Constant. Despite the fact that introduction of $\Lambda$ does not require an ad-hoc assumption and is also not ruled out by observation as an explanation for cosmic acceleration, scenarios based upon $\Lambda$ are plagued with various difficulties such as fine tuning, cosmic coincidence etc, which are detailed in section 4.1.1. In view of these problems, many alternate models have been considered.

The primary objective of this chapter is to investigate whether the spinor condensate model with a suitable interaction potential can be considered as a viable alternative to the Cosmological Constant. We analyze this model for various interaction potentials and explore the evolution of the Equation of State (EoS) parameter for each case.

The plan of the rest of the chapter is as follows: In section 4.1, we give a brief introduction to the $\Lambda$CDM model and the problems associated with it. A short review of various alternate models of dark energy constitutes section 4.2. In section 4.3, we analyze the spinor condensate model, with a power-law potential. A comparative study of the behavior of the EoS parameter obtained with two other interaction potentials is presented in section 4.4. Section 4.5 discusses the perturbation theory of the spinor condensate model with power-law interaction potential and section 4.6 concludes this chapter.
4.1 ΛCDM model

All currently available cosmological data is quite consistent with a Universe that is spatially-flat and is dominated by a Cosmological Constant $\Lambda$ with $\Omega_{\Lambda} \sim 0.74$, the rest of the energy density being mainly in non-relativistic cold dark matter with $\Omega_m \sim 0.25$. The inflationary, cold dark matter model with Cosmological Constant is considered to be the current standard model, usually called as the spatially-flat Λ-CDM model of cosmology. In the spatially-flat Λ-CDM model the background expansion of the Universe at late times is described by $H = H_0 \sqrt{\Omega_m(1+z)^3 + \Omega_{\Lambda}}$. The Cosmological Constant propelling the cosmic acceleration and eventually coming to dominate the Universe, causing it to enter a de-Sitter phase without return, seems the most straightforward explanation. This model provides an excellent fit to the wealth of high-precision observational data, on the basis of a remarkably small number of cosmological parameters. Even though the Cosmological Constant model is successful in fitting available data, it has a number of fundamental shortcomings.

4.1.1 Problems of ΛCDM

The Cosmological Constant is difficult to motivate from fundamental physics. The most plausible candidate for the Cosmological Constant is vacuum energy. In a super symmetric model every boson has a fermion of equal mass as a super symmetric partner and the vacuum energies of these partners cancel. Super symmetry (SUSY), if existent, is believed to be broken at an energy of roughly 1 TeV or so. The discrepancy between the small measured value of the Cosmological Constant and the much larger theoretically “expected” values of vacuum energy is known as the “smallness” problem [Weinberg (1989)].

As required by observations, the energy density of this constant has to be $\rho_{\Lambda} \sim (10^{-3} eV)^4$, which seems unphysically small in comparison to other physical constants in quantum gravity. At the classical level, this value does not suffer from any problems and we can measure it with progressively higher accuracy by accumulated observational data. However, questions about the origin of the Cosmological Constant—given by the energy stored in the vacuum—do not lead to a reasonable answer. In particle physics, vacuum-energy is associated with phase transitions and symmetry breaking. The vacuum of quantum electrodynamics for instance implies a $\rho_{\Lambda}$ about 120 orders of magnitude larger than what is required by observations. This is the worst fine-tuning problem of physics.

A possible explanation for the “smallness” problem is based on anthropic arguments, since it is difficult to find a solution from fundamental physics. In string theory, multiple vacuum states with all possible values of vacuum energy are possi-
ble. Different causally disconnected patches of the Universe, spontaneously choose vacuum states that are independent of each other. If the Universe is infinite, there will always be parts of it that have a given value of the vacuum energy, no matter how unlikely, and we just happen to be living in one of those regions which has a very small value of the vacuum energy density [Susskind (2003); Bousso and Polchinski (2004)].

Another interesting fact that is difficult to explain in the ΛCDM model is that both non-relativistic matter and dark energy have comparable energy densities today. This is surprising since the matter and dark energy components scale with redshift differently. For radiation, it is $\Omega_r \sim (1 + z)^4$; for CDM and baryons $\Omega_m \sim (1 + z)^3$ and for the Cosmological Constant $\Omega_\Lambda \sim \text{const}$. At the beginning of cosmic evolution the Universe was radiation dominated, today radiation contributes less than 1% of the total energy density. The contribution of DE was negligible in the past. It has become a dominant component only very recently, and in the future will be the only component driving cosmic expansion. There is only a short period of time when the energy densities of matter and Cosmological Constant are comparable. It is unclear why we happen to live in this narrow window of time. This is called the “coincidence” problem.

Besides these two problems, there are other observational facts that appear to conflict with the expectations of the ΛCDM model, at possibly more than $2\sigma$ confidence level. These are:

- High redshift SNIa data are consistent with spatially-flat ΛCDM. However, it favors models with $w_{\text{DE}} < -1$. Initially, this was thought to be a statistical fluke that would go away as more data accumulated. This discrepancy however still persists even as larger data sets have become available. The discrepancy is caused by high ($z > 1$) redshift SNIa which are systematically brighter than what we would expect in the ΛCDM model [Kowalski et al. (2008)]. This could be due to unknown systematic effects, possibly associated with high-redshift SNIa evolution, or a statistical effect that will go away with even more data. If the discrepancy persists, it would mean that the Universe in the past was decelerating faster than the ΛCDM model predicts.

- Large-scale velocity flows have amplitudes of 400 km/s, larger than what is expected in a ΛCDM model [Watkins et al. (2009); Kashlinsky et al. (2009a); Kashlinsky et al. (2009b); Lavaux et al. (2010)]. Velocity flows extend to $z = 0.2$ and could be as large as 1600 km/s. In ΛCDM the probability of having velocity flows with such large amplitudes is less than 1%. The explanation for this could be that it is just a big statistical fluctuation or there
could be some physical reason, such as time-dependent Newton’s constant, presence of noninflationary perturbations, or a giant void at the distance of a few Gpc.

- Cosmological simulations based on the ΛCDM model predict that large voids should be filled with many dwarf dark matter halos. This turns out to be true for very large voids (larger than 10 Mpc). Smaller voids however are observed to be surprisingly empty of dark matter halos [Peebles (2007)]. For example, based on ΛCDM we would expect to observe on average, ten dwarf galaxies in our local void, but there are none. Possible resolutions of this problem could be related to the incompleteness of the observational sample, or incorrect bias model that fails to account for specific environmental properties.

As the ΛCDM model suffers from these various difficulties, it will be interesting to look for alternative models.

4.2 Alternate Models

Dark energy models postulate the existence of a dark energy fluid, with equation of state \( P \approx -\rho \) (where, \( \rho \) and \( P \) are respectively the energy density and pressure of the fluid), which comes to dominate towards the end of the matter era. Even though various dark energy models have been proposed [for a detailed review, see Copeland et al. (2006); Rakhi and Indulekha (2009)], none of these is totally convincing or is free from fine-tuning problems, or can be demonstrated to be the “correct” one.

Since the fundamental theory of nature that would explain the microscopic physics of dark energy (DE) is unknown at present, a phenomenological approach can be taken and various models based on its macroscopic behavior constructed. A basic way to explore a dark energy model in the light of observational data is to parametrize dark energy by an EoS parameter \( w \), relating the dark energy pressure \( p \) to its density \( \rho \) via \( w = p/\rho \). From a theoretical point of view [see, Sahni and Starobinsky (2006); Arefeva et al. (2007) and references therein], the domain of \( w \) covers three essentially different regions. Thus, nearly all DE models can be classified by the behavior of their EoS parameters as follows:

1. Cosmological Constant: its EoS is exactly equal to \( w_\Lambda = -1 \).

2. Quintessence: its EoS remains above the Cosmological Constant boundary, that is \( 1 > w_Q > -1 \) [Ratra and Peebles (1988); Wetterich (1988)].
3. Phantom: its EoS lies below the Cosmological Constant boundary, that is $w_P \leq -1$ [\textit{Caldwell} (2002); \textit{Caldwell} et al. (2003); \textit{Vikman} (2005); \textit{Saridakis} et al. (2009)].

4. Quintom: its EoS is able to evolve across the Cosmological Constant boundary [\textit{Feng} et al. (2005); \textit{Cai} et al. (2008)].

Current observational data mildly favor $w_{DE}$ crossing the phantom divide during the evolution of universe [\textit{Padmanabhan and Choudhury} (2003); \textit{Alam} et al. (2004); \textit{Hannestad and Mortsell} (2004); \textit{Xia} et al. (2005); \textit{Wang} and \textit{Mukherjee} (2007); \textit{Wright} (2007) etc.]. However, the $\Lambda$CDM model is still in good agreement with the data.

All consistent theories in physics should satisfy the so-called Null Energy Condition (NEC). This implies a restriction on the dark energy equation of state parameter, $w_{DE}$; otherwise the theory might be unstable and unbounded. The condition $w_{DE} < -1$ implies violation of the NEC. The parameter space of many different models can lead to similar observational consequences and hence it is difficult to single out the best model of dark energy on the basis of the existing and planned cosmological observations. However, each particular model can be studied for checking consistency with observations.

### 4.3 An Alternative for the Cosmological Constant

We have demonstrated in the previous chapters that a spinor condensate field can act as an inflaton field for the early universe, and later on, as a dark energy field. Here, we explore whether a spinor condensate model can have an EoS parameter close to $-1$, for some interaction potential, and hence mimic a Cosmological Constant. Also, we analyze the behavior of perturbations, of the model with a power-law interaction potential for which we find the EoS parameter $\simeq -1$. From the phenomenological point of view, for such an analysis, it is allowed to treat the background classically, while dealing with the perturbations at the quantum level, like what it is usual in inflation theory.

In field dark energy models, one can find potential independent solutions [\textit{Scherrer} and \textit{Sen} (2008)], but in general cosmological evolution depends significantly on the choice of potential. In the quintessence case, a well studied potential is the inverse-power law one [\textit{Peebles} and \textit{Ratra} (1988)]. In such models, the quintessence energy density remains small at early and intermediate times, thus the known cosmological epochs are unaffected. By extracting the early-time, tracker solutions under the assumption of matter domination, it can be found that energy posi-
tivity for phantom models requires normal power-law potentials instead of inverse power-law ones, with the potential exponent being bounded by a quadratic form [Saridakis (2009)]. A theoretical justification for the power-law potentials can arise from super-symmetry considerations also [Binetruy (1999)]. In a dynamical DE model with a power-law potential, spinor matter (Spinor Quintom Model), which behaves like a linear-barotropic perfect fluid, is able to realize the acceleration of the universe, under certain conditions. Also, this class of quintom models by virtue of the spinor field can avoid quantum instabilities, while a scalar-type quintom model leads to such problems (due to the ghost field with a negative kinetic term) [Cai et al (2008)]. These results motivate us to explore an alternative model for the Cosmological Constant, with spinor condensate field having an interaction potential of the form of a power law.

4.3.1 Evolution of the EoS parameter

The basic model with spinor field, inspired by string/M- theory demonstrates that, fermionic terms to all orders in the Lagrangian lead to inflation at early times and acceleration at late times for the universe [for details, see Chapter 2]. The model had investigated whether the introduction of a non-Dirac fermionic field—characterized by an interaction term—affects the cosmological evolution. It has been seen that the model behaves like an inflation field for the early Universe and later on, as a dark energy field. In addition, the results showed that the higher order non-linear terms do not significantly change the overall behavior, obtained with $L_f$ alone; except that the late time acceleration is even more prominent.

Though this model produces early inflation and late time acceleration with interaction potential as a combination of scalar and pseudo-scalar invariants, this cannot be considered as an alternative to the Cosmological Constant, since it does not produce an EoS parameter, which is close to unity (as required by observations). Cosmological evolution, in general, depends significantly on the choice of the potential. Therefore, in search of an alternative to the Cosmological Constant, we take the fermionic interaction potential in the form of a power-law and study its EoS.

The potential takes the form [Armendariz-Picon and Greene (2003); Cai and Wang (2004)]

$$V = V_0 \left( \frac{\bar{\psi} \psi}{N} \right)^n,$$

with $n$ a non-zero constant and $N$ a positive time-dependent constant which we
define as today's value of $\bar{\psi}\psi$. The action for this model reads

$$S = \int \sqrt{-g} \, d^4 x \left( L_g + L_m + a_1 L_f + a_2 L_f^2 \right), \quad (4.2)$$

where $L_g = R/2$, with $R$ denoting the curvature scalar, is the gravitational Einstein Lagrangian density, $L_m$ is the Lagrangian density of the matter field and $a_1$ and $a_2$ are coupling constants. Also we have set $M_{pl} = (8\pi G)^{-1/2} = 1$. The Lagrangian density $L_f$ of the fermionic field, for a fermionic mass $m$ is given by

$$L_f = -\frac{i}{2} \left[ \bar{\psi} \Gamma^\mu D_\mu \psi - (D_\mu \bar{\psi}) \Gamma^\mu \psi \right] - m \langle \bar{\psi}\psi \rangle - V. \quad (4.3)$$

In Eq. (4.3), $V = V(\bar{\psi}\psi)$ describes the potential density of self-interaction between the fermions, which is given by Eq. (4.1). Similar to the previous cases, the connection between general relativity and the Dirac equation is done via the tetrad formalism and the components of the tetrad play the role of gravitational degrees of freedom.

Through Euler-Lagrange equations, from Eqs. (4.2) and (4.3), we can obtain the equations of motion for the spinor field [Rakhi et al., 2010a] as

$$i\Gamma^\mu D_\mu \psi - m\psi - \frac{\partial V}{\partial \bar{\psi}} = 0. \quad (4.4)$$

The variation of the action Eq.(4.2) with respect to the tetrad gives the Einstein’s field equations as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -T_{\mu\nu}, \quad (4.5)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g} g^{\mu\nu}} \delta S = T^{\text{f}}_{\mu\nu} + T^{\text{m}}_{\mu\nu}$, $T^{\text{f}}_{\mu\nu}$ being the energy-momentum tensor of the fermionic field and $T^{\text{m}}_{\mu\nu}$, that of the matter field. The symmetric form of the energy-momentum tensor of the fermionic field [Birrel and Davies (1982); Green et al. (1987)] which follows from Eq. (4.2) is

$$T^{\text{f}}_{\mu\nu} = \frac{i}{4} \left\{ \bar{\psi} \Gamma^\mu D^\nu \psi + \bar{\psi} \Gamma^\nu D^\mu \psi - D^\nu \bar{\psi} \Gamma^\mu \psi - D^\mu \bar{\psi} \Gamma^\nu \psi \right\} +$$

$$\frac{i}{2} L_f \left\{ \bar{\psi} \Gamma^\mu D^\nu \psi + \bar{\psi} \Gamma^\nu D^\mu \psi - D^\nu \bar{\psi} \Gamma^\mu \psi - D^\mu \bar{\psi} \Gamma^\nu \psi \right\} -$$

$$g^{\mu\nu} a_1 L_f - g^{\mu\nu} a_2 L_f^2. \quad (4.6)$$

From Eqs.(4.1),(4.3),(4.6), we get the energy density and pressure of the spinor
field:

\[ \rho_f = a_1 (m \langle \bar{\psi} \psi \rangle + V) - a_2 (m \langle \bar{\psi} \psi \rangle + V)^2 + a_2 (m \langle \bar{\psi} \psi \rangle + nV)^2 \]  

(4.7)

\[ p_f = a_1 (n - 1) V + a_2 [(n - 1) V]^2. \]  

(4.8)

The EoS parameter of the DE field will then become

\[ w_f = \frac{p_f}{\rho_f} = \frac{a_1 (n - 1) V + a_2 [(n - 1) V]^2}{a_1 (m \langle \bar{\psi} \psi \rangle + V) - a_2 (m \langle \bar{\psi} \psi \rangle + V)^2 + a_2 (m \langle \bar{\psi} \psi \rangle + nV)^2}, \]  

(4.9)

with

\[ V = V_0 \left( \frac{\bar{\psi} \psi}{N} \right)^n. \]

We have studied the dependence of the evolution of the EoS parameter on the potential parameters \( V_0 \) and \( n \), for \( N = 0.051, a_1 = 1, a_2 = 0.5 \). Fig. 4.1 shows the dependence of EoS parameter on the power, \( n \), while that on \( V_0 \) is depicted in Fig. 4.2. The initial conditions, we have chosen at \( t=0 \) are: \( a(0) = 1, \psi_1(0) = 0.1i, \psi_2(0) = 1, \psi_3(0) = 0.3, \psi_4(0) = i, \rho_m(0) = 0.254 \)

![Figure 4.1: Time evolution of EoS parameter for the power-law potential model. In this figure, we have kept \( V_0 = 0.00005 \).](image)
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Figure 4.2: Dependence of EoS parameter on $V_0$, for the power-law potential

From Figs. 4.1 and 4.2, we find that the EoS parameter of the dark energy field at late times, is practically constant with a value nearly equal to $-1$ as expected from observations. Fig. 4.1 also shows that the choice of the parameter $n$ does not significantly alter the time when accelerating phase begins. However, we see that the time taken for $w_f$ to change from 0 to $-1$ becomes shorter and shorter, for smaller and smaller values of $n$. From Fig. 4.2, one can notice that $V_0$ determines whether $w_f$ crosses the Cosmological Constant boundary. For high values of $V_0$ ($\gtrsim \mathcal{O}(10^{-2})$), $w_f$ is less than $-1$ and increases monotonically to $-1$. However, for $10^{-2} > V_0 \gtrsim 10^{-4}$, $w_f$ crosses the phantom divide line from above and goes to $-1$ asymptotically from below. It is remarkable to note that the spinor condensate field is able to achieve the phantom divide line crossing without the introduction of any ghost fields, just like the spinor field [Cai et al. (2008)]. Finally, for $V_0 \lesssim 10^{-4}$, the EoS parameter of the spinor condensate field drops from 0 to $-1$ monotonically.

4.3.2 Field equations in terms of the redshift

In this section, we study the behavior of the density parameters for the proposed model, with the power-law interaction potential using the redshift as a variable, instead of time. Time and redshift are related by the following expressions:

$$z = \frac{1}{a} - 1, \quad \frac{d}{dt} = -H(1 + z) \frac{d}{dz} \tag{4.10}$$
It is possible to obtain the evolution equations for $\psi$ and $\bar{\psi}$, from Eq. (4.4), as

$$\dot{\psi} + \frac{3}{2} H \psi + i \gamma^0 \psi + i \gamma^0 \frac{\partial V}{\partial \bar{\psi}} = 0$$

(4.11)

$$\dot{\bar{\psi}} + \frac{3}{2} H \bar{\psi} - i \gamma^0 \bar{\psi} - i \gamma^0 \frac{\partial V}{\partial \psi} = 0$$

(4.12)

From Eqs. (4.11) and (4.12), one can easily write

$$\frac{d(\bar{\psi} \psi)}{dt} + 3H \bar{\psi} \psi = 0$$

We define the bilinear $\Psi(z)$ as $\Psi(z) = \bar{\psi} \psi$. Using Eq. (4.10), the evolution equation for $\Psi(z)$ can be written as

$$-(1 + z) H \Psi' + 3H \Psi = 0$$

where $'$ represents differentiation with respect to $z$. The evolution equation for the matter density reads

$$\dot{\rho}_m + 3H \rho_m = 0.$$  

(4.13)

Here we take into account the fact that matter is considered as a pressure-less fluid, i.e., $p_m = 0$. The evolution equation for the energy density of the fermionic field is

$$\dot{\rho}_f + 3H (\rho_f + p_f) = 0.$$  

(4.14)

We transform Eq. (4.13) with the help of Eq. (4.10), and integrate it to get the dependence of the energy density of the matter field on the redshift:

$$\rho_m(z) = \rho_m(0)(1 + z)^3,$$  

(4.15)

where, $\rho_m(0)$ is the value of the energy density at the present time $z = 0$. The energy density and pressure of the fermionic field [Eqs. (4.7) & (4.8)] are expressed in terms of the redshift as

$$\rho_f = a_1 (m\Psi(z) + V) - a_2 (m\Psi(z) + V)^2 + a_2 (m\Psi(z) + nV)^2$$

(4.16)

$$p_f = a_1 (n - 1) V + a_2 [(n - 1) V]^2$$

(4.17)

where the interaction potential has the form $V = V_0 \left(\frac{\bar{\psi} \psi}{N}\right)^n$. The evolution equation for $H(z)$ can be obtained from the modified forms of the Friedmann and acceleration equations: $H^2 = \frac{\rho}{3}, \frac{\dot{a}}{a} = H^2 + \frac{dH}{dt} = -\frac{1}{6}(\rho + 3p)$. Thus, the system of coupled differential equations for the bilinear $\Psi(z)$ and for the Hubble Parameter $H(z)$
reads

\[-(1 + z)\Psi'(z) + 3\Psi(z) = 0\] (4.18)

\[2H(1 + z)H' = \rho_m + \rho_f + p_f + p_m\] (4.19)

To obtain numerical solutions of the system of differential equations (4.18) and (4.19), we have to specify the initial conditions for \(\Psi(z)\) and \(H(z)\). We take the present day values of the density parameters, \(\Omega_i(z) = \frac{\rho_i(z)}{\rho(z)} = \rho_i \left( \frac{1+z_0}{1+z} \right)^{3+\Omega_f(z)}\) from WMAP–7 data [Komatsu et al. (2011)] as \(\Omega_m(0) = 0.254 \pm 0.022\) and \(\Omega_f(0) = 0.746 \pm 0.022\). The expression for the initial value for the dimensionless Hubble parameter is given by

\[H(z_0) = \sqrt{\frac{\Omega_m(0)(1+z_0)^3 + \Omega_f(z_0)}{3}}\] (4.20)

in the matter-dominated era. This gives the present value of the Hubble parameter \(H(0)\) as \(H(0) = 70.4 \pm 2.5\, \text{km s}^{-1}\, \text{Mpc}^{-1}\) [Komatsu et al. (2011)]. The initial condition for \(\Psi(z)\) can be obtained by combining Eq. (4.16) with the present value of \(\Omega_f\).

Fig. 4.3 shows the time-dependent density parameters as a function of redshift for our model. We notice that dark energy dominates only at very low redshifts \((z < 0.5)\) and the evolution drives \(\Omega_m\) close to unity and \(\Omega_f\) towards zero by redshifts, \(z \sim 10\) itself. This behavior matches well with the Λ-CDM model and we conclude that the spinor condensate model, with power-law interaction potential can be considered as an alternative to the Cosmological Constant.
4.4 The EoS Parameter for Different Interaction Potentials

In this section, we make a comparative study between the behavior of the EoS parameter obtained with the power-law potential and two other interaction potentials. We choose the potentials phenomenologically, without any constraints from quantum field theory.

4.4.1 Perturbatively re-normalizable potential of the form:

\[ V(\bar{\psi}\psi) = V_0 + \alpha \bar{\psi}\psi + \beta (\bar{\psi}\psi)^2 \]

It has been demonstrated that a non-free, Lorentz invariant, non-standard spinor will have a well-defined quantum theory, provided the interactions are perturbatively re-normalizable [Böhmer et al. (2010)]. Hence, we consider a massless spinor field model with a perturbatively re-normalizable potential of the form given by Böhmer et al. (2010):

\[ V(\bar{\psi}\psi) = V_0 + \alpha \bar{\psi}\psi + \beta (\bar{\psi}\psi)^2 \]

(4.21)

and study the evolution of the EoS parameter. The equation of motion for the spinor field can be written as

\[ \dot{\psi} + \frac{3}{2}H\psi + i\gamma^0V'\psi = 0, \]

(4.22)

where ' represents differentiation with respect to \( \bar{\psi}\psi \). The energy density and pressure of the spinor field under consideration reads

\[ \rho_f = a_1 V - a_2 V'^2 (\langle \bar{\psi}\psi \rangle)^2 - 2a_2VV' \langle \bar{\psi}\psi \rangle \]

(4.23)

and

\[ p_f = a_1 \left( \frac{V'}{V} \langle \bar{\psi}\psi \rangle - 1 \right) V + a_2 V^2 \left( \frac{V'}{V} \langle \bar{\psi}\psi \rangle - 1 \right)^2 \]

(4.24)

respectively.

From Eqs. (4.23) and (4.24), we get the expression for the EoS parameter as:

\[ w_{f1} = \frac{a_1 \left( V' \langle \bar{\psi}\psi \rangle - V \right) + a_2 \left( V' \langle \bar{\psi}\psi \rangle - V \right)^2}{a_1 V - a_2 \left( V' \langle \bar{\psi}\psi \rangle \right)^2 - 2a_2VV' \langle \bar{\psi}\psi \rangle} \]

(4.25)

where the interaction potential takes the form given by Eq. (4.21). Eq. (4.25) is numerically evaluated and the evolution of EoS parameter is depicted in Figs. (4.4). Here, we have taken \( \alpha = \beta = V_0 = a_1 = 1 \) and \( a_2 = 0.5 \), for simplicity. We
Figure 4.4: EoS parameter $w_{f_1}$ vs. time, for a perturbatively re-normalizable potential

see that the EoS parameter satisfies the NEC; but it is not close to $-1$.

4.4.2 Potential as a combination of the scalar and the pseudo scalar invariants

We now consider a potential $V$ that satisfies the Pauli-Fierz theorem and can be written as a function of the scalar invariant $(\bar{\psi}\psi)^2$ and the pseudo-scalar invariant $(i\bar{\psi}\gamma^5\psi)^2$: i.e., $V = V\left[(\bar{\psi}\psi)^2, (i\bar{\psi}\gamma^5\psi)^2\right] \quad [Nambu \text{ and } Jona-Lasinio \text{ (1961); Ribas et al. \text{ (2005)}}]$. Therefore, we now analyze self-interaction potentials of the form:

$$V = \lambda \left[ \beta_1 \left( \langle \bar{\psi}\psi \rangle \right)^2 + \beta_2 \left( i \langle \bar{\psi}\gamma^5\psi \rangle \right)^2 \right]^n$$ \quad (4.26)

where $\lambda$ is a coupling constant and $n$ is a constant exponent. Both the scalar and pseudo-scalar densities play equally important roles and we assume the coefficients $\beta_1$ and $\beta_2$ to be both $O(1)$. For simplicity, the values are set to unity. That is, $\beta_1 = \beta_2 = 1$.

The energy density and pressure of the spinor field are obtained as [for a detailed derivation, see Chapter 2]

$$\rho_f = a_1 \left( m \langle \bar{\psi}\psi \rangle + V \right) - a_2 \left( m \langle \bar{\psi}\psi \rangle + V \right)^2 + a_2 \left( m \langle \bar{\psi}\psi \rangle + 2nV \right)^2$$ \quad (4.27)

and

$$p_f = a_1 \left( 2n - 1 \right) V + a_2 \left[ (2n - 1) V \right]^2.$$ \quad (4.28)
The EoS parameter can be written as:

\[ w_f^2 = \frac{a_1 (2n - 1) V + a_2 [(2n - 1) V]^2}{a_1 (m \langle \bar{\psi}\psi \rangle + V) - a_2 (m \langle \bar{\psi}\psi \rangle + V)^2 + a_2 (m \langle \bar{\psi}\psi \rangle + 2nV)^2}. \quad (4.29) \]

We have taken \( \lambda = 0.1, \beta_1 = \beta_2 = 1, n = 0.25, m = 0.01, a_1 = a_2 = 1, \) for numerically evaluating Eq. (4.29). Here also, we find that the EoS parameter satisfies the NEC; but its numerical value is not close to \(-1\). For the model with the

![Figure 4.5: EoS parameter \( w_f^2 \) vs. time, for a potential which is a combination of scalar and pseudo-scalar invariants](image)

perturbatively re-normalizable potential, we see that the EoS parameter initially oscillates and asymptotically becomes constant. For the model where the self-interaction potential is a combination of the scalar and pseudo-scalar invariants, the EoS parameter increases monotonically from an initial value greater than \(-1\) and oscillates about a value, that is between 0 and \(-1\), which indicates that the spinor condensate behaves like Quintessence for these two cases.

### 4.5 Perturbations

To study the stability of a dark energy model, we have to learn to what degree the system is stable both in classical and quantum level. Usually, systems with \( w < -1 \) show some nasty instabilities [Cai et al. (2008)]. In this section, we present the perturbation analysis of the model discussed in section 4.3.1; this has an EoS, \( w = -1 \) beyond a very early period. In a proper treatment of the problem, we
4.5: Perturbations

would perturb both metric and spinor and solve the linearized Einstein equations. Since the nature of the spinor makes this path cumbersome, we shall adopt the method of analysis followed by Armendariz-Picon and Greene (2003), where the spinor alone is perturbed in a given fixed space-time background.

Density perturbations \( \delta \rho \) are characterized by the variable:

\[
\zeta \equiv \frac{\delta \rho}{\rho + p}
\quad (4.30)
\]

where \( \delta \rho \) for our model, is given by

\[
\delta \rho = a_1 \left( m (\bar{\psi} \delta \psi + \psi \delta \bar{\psi}) + \delta V \right) - a_2 \left( m (\bar{\psi} \delta \psi + \psi \delta \bar{\psi}) + \delta V \right)^2 + a_2 \left( m (\bar{\psi} \delta \psi + \psi \delta \bar{\psi}) + n \delta V \right)^2 \quad (4.31)
\]

The power spectrum \( P(k) \), which is a measure of the fluctuations of the variable \( \zeta \) on the co-moving scale, is defined by the relation [Mukhanov et al. (1992)]

\[
\langle \zeta(t, \vec{x}) \zeta(t, \vec{x} + \vec{r}) \rangle = \int \frac{dk}{k} \sin kr k r P(k) . \quad (4.32)
\]

where \( \langle \rangle \) denotes the expectation value in an appropriately chosen vacuum state.

The source of the density perturbations \( \delta \rho \) are the fluctuation \( \delta \psi \) of the spinor field around its homogeneous background value \( \psi \). The fluctuations \( \delta \psi \) is treated as a quantum field in an expanding universe whereas \( \psi \) is considered as a classical field [Parker (1971)]. That is,

\[
\delta \psi = \frac{1}{(2\pi)^{3/2}} \int d^3 k \sum_{\sigma} \left[ u(t, \vec{k}, \sigma) a(k, \sigma) e^{i \vec{k} \vec{x}} + v(t, \vec{k}, \sigma) b^\dagger(k, \sigma) e^{-i \vec{k} \vec{x}} \right] . \quad (4.33)
\]

The index \( \sigma \) runs over the two spin states of the spinor, and the operators \( a \) (\( a^\dagger \)) and \( b \) (\( b^\dagger \)) are particle and antiparticle annihilation (creation) operators, \( \{a(\vec{k}, \sigma), a^\dagger(\vec{k}', \sigma')\} = \{b(\vec{k}, \sigma), b^\dagger(\vec{k}', \sigma')\} = \delta^{(3)}(\vec{k} - \vec{k}') \delta_{\sigma \sigma'} \).

The time evolution of \( v \) is dictated by the equation of motion of \( \delta \psi \). The field \( \delta \psi \) itself satisfies the linearized Dirac equation:

\[
i \Gamma^0 D_0 \delta \psi + \Gamma^i D_i \delta \psi - m \delta \psi - V' \delta \psi = 0.
\]

For simplicity, \( \delta \psi \) is rescaled as \( \delta \psi_R = a^{3/2} \delta \psi \) instead of \( \delta \psi \). In particular, \( v_R \) satisfies

\[
i \gamma^0 \frac{dv_R}{d\tau} + \gamma^i k_i v_R - am v_R - a V' v_R = 0, \quad (4.34)
\]

where \( \tau \) denotes conformal time, \( d\tau = dt/a \). The second order differential equation from Eq.(4.34) yields

\[
v''_R + \left[ k^2 + a^2 m + a^2 V'^2 + i(a'm + (aV')') \right] v_R = 0. \quad (4.35)
\]
We see that \( v_R \propto e^{i k \tau} \) as \( \tau \to -\infty \).

The solutions to Eq. (4.35) in momentum space are the Hankel functions of the second kind \([Abramowitz and Stegun (1965)]\). Up to the factor \( a^{-3/2} \), the perturbations of the spinor field oscillate inside the Hubble radius. However, the above equation does not mean that the model is able to avoid any instabilities, since we have not done a detailed analysis of the perturbation equation.

## 4.6 Summary

In this chapter, we investigate whether a spinor condensate field with a suitable interaction potential can act as an alternative for the Cosmological Constant. We propose a model using spinor condensate field with an interaction potential of the form of a power-law and study the evolution of its EoS parameter with time. We find that for small values of the potential parameters \( V_0 \) and \( n \), the EoS parameter \( w_f \) of the dark energy field decreases monotonically from 0 to \(-1\), which is consistent with observations. With a proper choice of the potential parameters, we notice that the spinor condensate field is able to achieve the phantom divide line crossing without the introduction of any ghost fields. We have tracked the behavior of the density parameters (\( \Omega_m \) and \( \Omega_f \)) for the spinor condensate model with power-law interaction potential, starting from their presently accepted values obtained from WMAP-7 data. We find that the proposed model can reproduce the expected redshift behaviors of the density parameters of each constituent.

We have also analyzed the evolution of the EoS parameter (\( w_f \)) of the spinor condensate model with different interaction potentials to study the contrast with the above case. We have considered a perturbatively re-normalizable potential as well as a potential, which is a combination of scalar and pseudo-scalar invariants. In the above two cases, it is found that the EoS parameter initially oscillates and asymptotically becomes constant. It is to be noted, from the results of chapter 3 and also the present one that our model satisfies the NEC for a variety of potentials. Finally, we presented the perturbation analysis of this model.

To sum up: (a) the spinor model with self-interaction potential of the form of a power-law can be considered as an alternative to the Cosmological Constant (b) spinor condensate model with a perturbatively re-normalizable potential or a potential which is a combination of scalar and pseudo-scalar invariant behave like quintessence (c) the redshift behavior of density parameters for the power-law potential model matches well with that for the Λ-CDM model (d) the analysis of perturbations shows that the spinor field perturbations oscillate inside the Hubble radius.