CHAPTER 7

Marshall-Olkin Gumbel bivariate distributions and their applications in stress-strength reliability analysis and autoregressive minification processes

7.1 Introduction

The theory of reliability deals with the failure law of systems. Whenever the system is put in operation, sometime or other it is bound to fail either due to external causes or due to an internal structural defect. In reliability, the stress strength model describes the life of a component having a random strength (Y) and is subjected to a random stress (X). The component survives as long as Y is greater than X. The corresponding chance of

Some of the results in this chapter are published in Jose and Rani (2010).
occurrence, \( R = P(X < Y) \), is known as the reliability of the component.


### 7.1.1 Autoregressive minification processes

In a series of papers Tavares (1980) introduced two stationary markov processes with similar structural form which he had found useful in hydrological applications. In one of these, the observations \( \{X_n : n = 0, 1, 2, \ldots \} \) are generated by the equation

\[
X_n = K \min(X_{n-1}, \varepsilon_n), \quad n \geq 1, \tag{7.1.1}
\]

where \( \{\varepsilon_n, n \geq 1\} \) where \( k > 1 \) is a constant, and \( \{\varepsilon_n\} \) is an innovation process of independently and identically distributed random variables chosen to ensure that \( \{X_n\} \) is a stationary markov process with marginal distribution function \( F_X(x) \). Because of the structure of (7.1.1) the process \( \{X_n\} \) is called a minification process.

A stationary bivariate minification process with bivariate Marshall and Olkin exponential distribution is given by

\[
\begin{align*}
X_n &= k \min(X_{n-1}, Y_{n-1}, \eta_{n1}) \\
Y_n &= k \min(X_{n-1}, Y_{n-1}, \eta_{n2}) \tag{7.1.2}
\end{align*}
\]
where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$, $K > \frac{1}{\lambda_{12}}$, $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Here $\{(\eta_{n1}, \eta_{n2}), n \geq 1\}$ is a sequence of i.i.d random vectors such that $(X_n, Y_n)$ and $(\eta_{n1}, \eta_{n2})$ are independent for $m < n$.

In this chapter we are discussing two different structures of the minification processes. Consider a bivariate autoregressive minification process $(X_n, Y_n)$ having the structure $\{(X_n, Y_n)\}$ given by

$$(X_n, Y_n) = \begin{cases} 
(\varepsilon_n, \eta_n) & \text{w.p. } p, \\
\min(X_{n-1}, \varepsilon_n), \min(Y_{n-1}, \eta_n), & \text{w.p. } 1 - p 
\end{cases}$$

where $0 \leq p \leq 1$. Then $(X_n, Y_n)$ has MOGEM(m,p) stationary marginal distribution if and only if $(\varepsilon_n, \eta_n)$ has Morgenstern Gumbel exponential distribution with parameter $m$. This can be extended to k cases also.

Consider a bivariate autoregressive minification process $(X_n, Y_n)$ having the structure $\{(X_n, Y_n)\}$ given by

$$(X_n, Y_n) = \begin{cases} 
(\varepsilon_n, \eta_n) & \text{w.p. } p, \\
(\min(X_{n-1}, Y_{n-1}), (\varepsilon_n, \eta_n)), & \text{w.p. } 1 - p 
\end{cases}$$

Then $(X_n, Y_n)$ has MOGEM (m,p) stationary marginal distribution if and only if $(\varepsilon_n, \eta_n)$ has Morgenstern Gumbel exponential distribution with parameter $m$.

In this chapter a minification process with Marshall-Olkin Gumbel bivariate exponential distribution is developed.

7.1.2 Marshall-Olkin family of distributions

New parameters can be introduced to expand families of distributions for added flexibility or to construct covariate models. Introduction of scale parameter usually leads to the accelerated life model, and taking powers of the survival function introduces a parameter that leads to the proportional hazard model. Marshall and Olkin (1997) introduced a method of obtaining an extended family of distributions including one more parameter. For a random
variable with a distribution function $F(x)$ and survival function $\overline{F}(x)$, we can obtain a new family of distribution functions called univariate Marshall-Olkin family having cumulative distribution function $G(x)$ given by

$$G(x) = \frac{F(x)}{\alpha + (1 - \alpha)F(x)}; -\infty < x < \infty; 0 < \alpha < \infty.$$ 

Then the corresponding survival function is

$$\overline{G}(x) = \frac{\alpha \overline{F}(x)}{1 - (1 - \alpha)\overline{F}(x)}; -\infty < x < \infty; 0 < \alpha < \infty.$$ 

This new family involves an additional parameter $\alpha$. In bivariate case if $(X, Y)$ be a random vector with joint survival function $\overline{F}(x, y)$, then

$$G(x, y) = \frac{F(x, y)}{\alpha + (1 - \alpha)F(x, y)}; -\infty < x < \infty; -\infty < y < \infty; 0 < \alpha < \infty.$$ 

constitute Marshall-Olkin bivariate family of distributions. The new parameter $\alpha$ results in added flexibility of distributions and influence the reliability properties.

In this chapter Morgenstern Gumbel bivariate distribution and its properties are discussed in section 7.2. Two Marshall Olkin Gumbel exponential minification processes are developed in section 7.3. Expressions for stress - strength reliability $R = P(X < Y)$ of a component is derived, where stress $(X)$ and strength $(Y)$ are independently distributed and stress $(X)$ consists of a two component system $X_1$ and $X_2$ are considered in section 7.4. The reliability of the system when the components $X_1$ and $X_2$ have Morgenstern Gumbel bivariate and Marshall-Olkin Gumbel bivariate exponential distributions are derived and evaluated for various parameter values in section 7.5. Bivariate failure rate is discussed in section 7.6. Conclusions are given in section 7.7.
7.2 Morgenstern Gumbel bivariate distribution and its properties

Consider the bivariate exponential distribution of Gumbel (1960) with joint cumulative distribution function

\[ F(x_1, x_2) = 1 - e^{-x_1} - e^{-x_2} + e^{-(x_1^m + x_2^m)\frac{1}{m}}, \quad x_1, x_2 > 0, m \geq 1 \] (7.2.1)

The survival function of the Morgenstern Gumbel bivariate distribution can be obtained as

\[ \overline{F}(x_1, x_2) = e^{-(x_1^m + x_2^m)\frac{1}{m}}. \]

Consider the survival function \( \overline{F} \). The new family of survival functions is constructed using Marshall-Olkin method by taking

\[ G(x; \alpha) = \frac{\alpha \overline{F}(x)}{1 - (1 - \alpha) \overline{F}(x)}; 0 < \alpha < \infty. \]

When \( \alpha = 1 \), we get

\[ \overline{G} = \overline{F}. \]

Marshall-Olkin Gumbel bivariate exponential distribution can be written as

\[ \overline{G}(x_1, x_2) = \frac{\alpha \overline{F}(x_1, x_2)}{1 - (1 - \alpha) \overline{F}(x_1, x_2)}; \alpha > 0, x_1 > 0, x_2 > 0 \]

\[ = \frac{\alpha e^{-(x_1^m + x_2^m)\frac{1}{m}}}{1 - (1 - \alpha) e^{-(x_1^m + x_2^m)\frac{1}{m}}}. \]

When \( \alpha = 1 \), it easily follows that \( \overline{G} = \overline{F} \).

From these we get

\[ \overline{G}_{X_1}(x_1) = \frac{\alpha e^{-x_1}}{1 - (1 - \alpha) e^{-x_1}}; \alpha > 0, x_1 > 0 \]

and

\[ \overline{G}_{X_2}(x_2) = \frac{\alpha e^{-x_2}}{1 - (1 - \alpha) e^{-x_2}}; \alpha > 0, x_2 > 0. \]
These are univariate Marshall-Olkin exponential distributions.

**Theorem 7.2.1.** Let \( N \) be a geometric random variable with \( p(N = n) = pq^{n-1} \),
\[ n = 1, 2..., 0 < p < 1, q = 1 - p \]

Consider a sequence \( \{(X_i, Y_i), i \geq 1\} \) of i.i.d random variables with common survival function \( F(x, y) \), \( N \) and \( (X_i, Y_i) \) are independent for all \( i \geq 1 \). Let \( U_N = \min_{1 \leq i \leq N} X_i \) and \( V_N = \min_{1 \leq i \leq N} Y_i \). Then the random vector \((U_N, V_N)\) is distributed as Marshall-Olkin Gumbel exponential (MOGE \((m, p)\)) if and only if \((X_i, Y_i)\) has the Gumbel bivariate distribution with parameter \( m \).

**Proof.** Let \( S(x, y) \) be the survival function of \((U_N, V_N)\). By definition
\[
S(x, y) = P(U_N > x, V_N > y) = \sum_{i=1}^{n} (F(x, y))^n pq^{n-1} = \frac{pF(x, y)}{1 - (1 - p)F(x, y)} = \frac{pe^{-(x^m + y^m)^{1/m}}}{1 - (1 - p)e^{-(x^m + y^m)^{1/m}}} \rightarrow \text{MOBGE}(m, p).
\]

Conversely let \((U_N, V_N)\) has \( \text{MOGE}(m, p) \) distribution. Then solving the equation we get,
\[
\frac{pF(x, y)}{1 - (1 - p)F(x, y)} = \frac{pe^{-(x^m + y^m)^{1/m}}}{1 - (1 - p)e^{-(x^m + y^m)^{1/m}}}
\]

Then we get,
\[
F(x, y) = e^{-(x^m + y^m)^{1/m}}
\]

which is the survival function of the Morgenstern Gumbel exponential distribution with parameter \( m \).
7.3 Marshall-Olkin Gumbel exponential minification processes

Model 1

Theorem 7.3.1. Consider a bivariate autoregressive minification process \((X_n, Y_n)\) having the structure \(\{(X_n, Y_n)\}\) given by

\[
(X_n, Y_n) = \begin{cases} 
(\varepsilon_n, \eta_n) & \text{w.p. } p, \\
\min(X_{n-1}, Y_{n-1}), (\varepsilon_n, \eta_n)) & \text{w.p. } 1-p
\end{cases}
\]

Then \((X_n, Y_n)\) has MOGEM \((m, p)\) stationary marginal distribution if and only if \((\varepsilon_n, \eta_n)\) has Morgenstern Gumbel exponential distribution with parameter \(m\).

Proof. The survival function of \((X_n, Y_n)\) is

\[
F_{X,Y}(x_1, x_2) = pF_{\varepsilon,\eta}(x_1, x_2) + (1-p)F_{X_{n-1}, Y_{n-1}}(x_1, x_2)F_{\varepsilon,\eta}(x_1, x_2).
\]

Under stationarity we have,

\[
F_{X,Y}(x_1, x_2) = \frac{pF_{\varepsilon,\eta}(x_1, x_2)}{1 - (1-p)F_{\varepsilon,\eta}(x_1, x_2)} \tag{7.3.1}
\]

We take

\[
F_{\varepsilon,\eta}(x_1, x_2) = e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}.
\]

Then equation (7.3.1) becomes

\[
F_{X,Y}(x_1, x_2) = \frac{pe^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{1 - (1-p)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}},
\]

which follows MOBGE\((m, p)\) from equation (7.3.1)

\[
F_{\varepsilon,\eta}(x_1, x_2) = \frac{F_{X,Y}(x_1, x_2)}{p + (1-p)F_{X,Y}(x_1, x_2)}
\]
Hence,
\[
\frac{pe^{-(x_1^m + x_2^m)\frac{1}{m}}}{1 - (1-p)e^{-(x_1^m + x_2^m)\frac{1}{m}}} = e^{-(x_1^m + x_2^m)\frac{1}{m}},
\]
which is the survival function of the Morgenstern Gumbel bivariate distribution. Therefore \((X_n, Y_n)\) follows Marshall-Olkin Gumbel bivariate distribution.

This result can be extended to \(k\) variate cases having the structure
\[
(X_1 n, \ldots, X_k n) = \begin{cases} 
(\varepsilon_1 n, \ldots, \varepsilon_k n) & \text{w.p.} \ p, \\
\min((X_{1n-1}, \ldots, X_{kn-1}), (\varepsilon_1 n, \ldots, \varepsilon_k n)) & \text{w.p.} \ 1 - p
\end{cases}
\]

\[
F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = pF_{\varepsilon_1 n,\ldots,\varepsilon_k n}(x_1,\ldots,x_n) + (1 - p)F_{X_{1n-1},\ldots,X_{kn-1}}(x_1,\ldots,x_n)F_{\varepsilon_1 n,\ldots,\varepsilon_k n}(x_1,\ldots,x_n).
\]

Under stationarity \(F_{X}(x_1,\ldots,x_n) = \frac{pF_{\varepsilon n}(x_1,\ldots,x_n)}{1 - (1-p)F_{\varepsilon n}(x_1,\ldots,x_n)}\)

**Model 2**

**Theorem 7.3.2.** Consider a bivariate autoregressive minification process \((X_n, Y_n)\) having the structure \(\{(X_n, Y_n)\}\) given by
\[
(X_n, Y_n) = \begin{cases} 
(\varepsilon_n, \eta_n) & \text{w.p.} \ p, \\
\min(X_{n-1}, \varepsilon_n), \min(Y_{n-1}, \eta_n) & \text{w.p.} \ 1 - p
\end{cases}
\]

Then \((X_n, Y_n)\) has \(\text{MOGEM}(m, p)\) stationary marginal distribution if and only if \((\varepsilon_n, \eta_n)\) has Morgenstern Gumbel exponential distribution with parameter \(m\).

**Proof.** Let \(G(x, y)\) and \(F(x, y)\) be the survival functions of \((X_n, Y_n)\) and \((\varepsilon_n, \eta_n)\) respectively. From the definition of the process, we have
\[
G_n(x, y) = P(X_n > x, Y_n > y) = pF(x, y) + (1 - p)G_{n-1}(x, y)F(x, y)
\]
CHAPTER 7. MARSHALL-OLKIN GUMBEL BIVARIATE DISTRIBUTIONS AND THEIR APPLICATIONS IN STRESS-STRENGTH RELIABILITY ANALYSIS AND AUTOREGRESSIVE MINIFICATION PROCESSES

Under stationarity

\[ \overline{G}(x, y) = (p + (1 - p)\overline{G}(x, y))\overline{F}(x, y) \]

Replacing \( \overline{F} \) with the survival function of random vector with MOGEM \((m, p)\) distribution, we have

\[ \overline{F}(x, y) = e^{-(x^m + y^m)^m} \]

and solving we get,

\[ G_{X,Y}(x, y) = \frac{pe^{-(x^m + y^m)^m}}{1 - (1 - p)e^{-(x^m + y^m)^m}} \]

which is the survival function of MOGEM \((m, p)\) distribution. Conversely,

\[ \overline{F}(x, y) = \frac{\overline{G}(x, y)}{p + (1 - p)\overline{G}(x, y)} \]

On solving we get

\[ \overline{F}(x, y) = e^{-(x^m + y^m)^m} \]

which is the survival function of the Morgenstern Gumbel bivariate distribution. Therefore \((X_n, Y_n)\) follows Marshall Olkin gumbel bivariate distribution.

### 7.4 Determination of reliability under Marshall-Olkin Gumbel bivariate exponential (MOGBE) model

Let \(X_1\) and \(X_2\) be two non negative random variables jointly following MOGBE distribution with survival function

\[ \overline{F}(x_1, x_2) = \frac{\alpha e^{-(x_1^m + x_2^m)^m}}{1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^m}} \]

Also let \(Y\) (Strength) be a non-negative random variable following the exponential distribution with Survival function
\[ \overline{G}(y) = e^{-y} \]

and its pdf is

\[ g(y) = e^{-y}; \quad 0 < y < \infty \]

**Case(i): Stress components are in series**

We define, \( U = \min(X_1, X_2) \).

The survival function for \( U \) is given by

\[ \overline{F}_U(x) = \frac{\alpha e^{-(2x^m) \frac{1}{m}}}{1 - (1 - \alpha)e^{-(2x^m) \frac{1}{m}}} . \]

The cumulative distribution function is

\[ F_U(x) = \frac{1 - e^{-(2x^m) \frac{1}{m}}}{1 - (1 - \alpha)e^{-(2x^m) \frac{1}{m}}} . \]

Then reliability can be obtained as

\[ R = P(U < Y) = \int_0^\infty \left\{ \int_0^y f(u) du \right\} g(y) dy \]
\[ = \int_0^\infty \frac{1 - e^{-(2y^m) \frac{1}{m}}}{1 - (1 - \alpha)e^{-(2y^m) \frac{1}{m}}} e^{-y} dy \]
\[ = \int_0^\infty \frac{e^{-y} - e^{-(k+1)y}}{1 - (1 - \alpha)e^{-yk}} dy \]

where \( k = (2)^{\frac{1}{m}} \), graph is drawn for various values of \( \alpha \) and \( m \).

**Case(ii): Stress components are in parallel**

In this case, we consider \( V = \max(X_1, X_2) \).
The cumulative distribution function for $V$ is given by

$$F_V(x) = \frac{\alpha e^{-kx}}{1 - (1 - \alpha)e^{-kx}} + \frac{1 - \alpha e^{-x}(1 + \alpha)}{1 - (1 - \alpha)e^{-x}}$$

Then,

$$R = P(V < Y) = \int_0^\infty \left\{ \int_0^y f(v)dv \right\} g(y)dy = \int_0^\infty \left\{ \frac{\alpha e^{-ky}}{1 - (1 - \alpha)e^{-ky}} + \frac{1 - \alpha e^{-y}(1 + \alpha)}{1 - (1 - \alpha)e^{-y}} \right\} e^{-y}dy$$

where $k = (2)^{\frac{1}{m}}$, graph is drawn for various values of $\alpha$ and $m$.

### 7.5 Determination of reliability under Morgenstern Gumbel bivariate distribution

The survival function of the Morgenstern Gumbel bivariate distribution is

$$\bar{F}(x_1, x_2) = e^{-(x_1^m + x_2^m)\frac{1}{m}}$$

Let $Y$ (strength) be a non-negative random variable following the exponential distribution with survival function

$$\bar{G}(y) = e^{-\lambda y}$$

and its pdf is

$$g(y) = \lambda e^{-\lambda y}; 0 < y < \infty, \lambda > 0$$

**Case(i): Stress components are in series**

Then $U = \min(X_1, X_2)$. 

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The survival function for $U$ is given by

$$F_U(x) = e^{-(2x^m)^{\frac{1}{m}}}$$

The reliability can be obtained as

$$R = P(U < Y) = \int_0^\infty \{ \int_0^y f(u)du \} g(y)dy = \int_0^\infty \{ 1 - e^{-(2y^m)^{\frac{1}{m}}} \} \lambda e^{-\lambda y} dy = \frac{k}{k + \lambda}$$

where $k = 2^{\frac{1}{m}}$, graph is drawn for various values of lambda and m.

**Case(ii): Stress components are in parallel**

Then $V = \max(X_1, X_2)$.

The corresponding cumulative distribution function is given by

$$F_V(x) = -2e^{-x} + 1 + e^{-kx}$$

where $k = 2^{\frac{1}{m}}$.

Then the reliability is

$$R = P(V < Y) = \int_0^\infty \{ \int_0^y f(v)dv \} g(y)dy = \int_0^\infty \{-2e^{-y} + 1 + e^{-ky}\} \lambda e^{-\lambda y} dy$$

where $k = 2^{\frac{1}{m}}$, graph is drawn for various values of $\lambda$ and $m$. 

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7.6 Bivariate failure rate

We assume that \((X_1, X_2)\) represent the lives of the components in a two component system. The bivariate failure rate at \((x_1, x_2)\) is defined by Basu(1971) is

\[
a(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)}; \text{ for } x_1 > 0, x_2 > 0
\]

Where the reliability is given by

\[
R(x_1, x_2) = \frac{\alpha e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}
\]

and

\[
f(x_1, x_2) = \frac{\partial^2 R}{\partial x_1 \partial x_2}
\]

\[
a(x_1, x_2) = \frac{1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{\alpha e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}} \left\{ \frac{(x_1^m + x_2^m)^{\frac{1}{m}}x_1^m x_2^m e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{x_1 x_2 (x_1^m + x_2^m)^2 (1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})} \right. \\
- \frac{(x_1^m + x_2^m)^{\frac{1}{m}} x_1^m x_2^m e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{x_1 x_2 (x_1^m + x_2^m)^2 (1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})} \\
+ \frac{(x_1^m + x_2^m)^{\frac{1}{m}} x_1^m x_2^m (1 - \alpha)(e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2}{(x_1 x_2)(x_1^m + x_2^m)^2 (1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2} \\
- \frac{(x_1^m + x_2^m)^{\frac{1}{m}} x_1^m x_2^m (1 - \alpha)^2 (e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^3}{x_1 x_2 (x_1^m + x_2^m)^2 (1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^3} \\
+ \frac{(x_1^m + x_2^m)^{\frac{1}{m}} x_1^m x_2^m (1 - \alpha)(e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2}{x_1 x_2 (x_1^m + x_2^m)^2 (1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2} \\
- \frac{(x_1^m + x_2^m)^{\frac{1}{m}} x_1^m x_2^m (1 - \alpha)m(e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2}{x_1 x_2 (x_1^m + x_2^m)^2 (1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2} \right\}
\]
The graph is drawn for various values of $\alpha$ and $m$.

Another approach to defining bivariate failure rate is provided by Johnson and Kotz (1975), where it is taken as the vector valued function,

$$
h(x_1, x_2) = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) H(x_1, x_2)
$$

$$
= \left( \frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2} \right)
= (h_1(x_1, x_2), h_2(x_1, x_2)),
$$

where $H = -\log R(x_1, x_2)$.

Now,

$$
h_1(x_1, x_2) = \frac{-\partial R}{R \partial x_1}
= \frac{(x_1^m + x_2^m)^{\frac{1}{m} - 1} x_1^{m-1}}{1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}},
$$

$$
h_2(x_1, x_2) = \frac{-\partial R}{R \partial x_2}
= \frac{(x_1^m + x_2^m)^{\frac{1}{m} - 1} x_2^{m-1}}{1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}},
$$

and we observe that $h_i(x_1, x_2) = \frac{\partial R}{\partial x_i}/R$.

According to Galambos and Kotz (1978) under suitable assumptions on the survival function $a(x_1, x_2)$ defined above can be expressed as

$$
a(x_1, x_2) = \frac{\partial H}{\partial x_1} + \frac{\partial H}{\partial x_2} + \frac{\partial^2 H}{\partial x_1 \partial x_2},
$$

where $H = H(x_1, x_2) = -\log R(x_1, x_2)$.

$$
\frac{\partial H}{\partial x_1} = \frac{-\partial R}{R \partial x_1}
= \frac{(x_1^m + x_2^m)^{\frac{1}{m} - 1} x_1^{m-1}}{1 - (1 - \alpha)e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}} \tag{7.6.1}
$$
and \[
\frac{\partial H}{\partial x_2} = -\frac{\partial R}{R \partial x_2} = \frac{(x_1^m + x_2^m)^{\frac{1}{m} - 1} x_2^{m-1}}{1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}} \tag{7.6.2}
\]

\[
\frac{\partial^2 H}{\partial x_1 \partial x_2} = \frac{\partial R}{R^2 \partial x_2 \partial x_1} - \frac{\partial^2 R}{R \partial x_1 \partial x_2} \tag{7.6.3}
\]

\[
\frac{\partial^2 H}{\partial x_1 \partial x_2} = \frac{(x_1^m + x_2^m)^{\frac{1}{m} - 1} (x_1 x_2)^{m-1}}{(1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2} \frac{1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{\alpha e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}} - \frac{\alpha (x_1^m + x_2^m)^{\frac{1}{m}} (x_1 x_2)^{m} e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{x_1 x_2 (x_1^m + x_2^m)^2 (1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})} + \frac{\alpha (x_1^m + x_2^m)^{\frac{1}{m}} (x_1 x_2)^{m} e^{-(x_1^m + x_2^m)^{\frac{1}{m}}}}{(x_1 x_2) (x_1^m + x_2^m)^2 (1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})} + \frac{3 \alpha (x_1 x_2)^m (1 - \alpha) ((x_1^m + x_2^m)^{\frac{1}{m}})^2 (e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2}{x_1 x_2 (x_1^m + x_2^m)^2 (1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2} + \frac{2 \alpha (x_1 x_2)^m (1 - \alpha)^2 (e^{(x_1^m + x_2^m)^{\frac{1}{m}}})^3 ((x_1^m + x_2^m)^{\frac{1}{m}})^2}{(1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^3 (x_1 x_2) (x_1^m + x_2^m)^2} - \frac{\alpha (1 - \alpha) (x_1 x_2)^m (e^{(-x_1^m + x_2^m)^{\frac{1}{m}}})^2 (x_1^m + x_2^m)^{\frac{1}{m}}}{(1 - (1 - \alpha) e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2 (x_1 x_2) (x_1^m + x_2^m)^2} + \frac{\alpha (1 - \alpha) (x_1 x_2)^m (x_1^m + x_2^m)^{\frac{1}{m}} (e^{-(x_1^m + x_2^m)^{\frac{1}{m}}})^2}{(1 - (1 - \alpha) e^{(x_1^m + x_2^m)^{\frac{1}{m}}})^2 (x_1 x_2) (x_1^m + x_2^m)^2}
\]

\[
a(x_1, x_2) = (7.6.1) + (7.6.2) + (7.6.3).
\]
Table 7.1: Reliability under Marshall Olkin Gumbel bivariate exponential (MOGBE) model where stress components are in series.

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<thead>
<tr>
<th>m</th>
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<th>$\alpha = .05$</th>
<th>$\alpha = .1$</th>
<th>$\alpha = .2$</th>
<th>$\alpha = .5$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 2$</th>
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<td>0.93500</td>
<td>0.89813</td>
<td>0.84649</td>
<td>0.75354</td>
<td>0.66666</td>
<td>0.57079</td>
</tr>
<tr>
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<td>0.97251</td>
<td>0.91356</td>
<td>0.86005</td>
<td>0.80094</td>
<td>0.68754</td>
<td>0.58578</td>
<td>0.47833</td>
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<tr>
<td>3</td>
<td>0.96964</td>
<td>0.90523</td>
<td>0.85379</td>
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<td>0.66368</td>
<td>0.55750</td>
<td>0.44728</td>
</tr>
<tr>
<td>4</td>
<td>0.96812</td>
<td>0.90033</td>
<td>0.84736</td>
<td>0.77501</td>
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<td>0.54321</td>
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<td>0.89812</td>
<td>0.84341</td>
<td>0.76958</td>
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<td>0.53640</td>
<td>0.42263</td>
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<td>0.89251</td>
<td>0.83527</td>
<td>0.75847</td>
<td>0.62897</td>
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<td>30</td>
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<td>0.82968</td>
<td>0.75089</td>
<td>0.61881</td>
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<td>0.61677</td>
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<td>0.38988</td>
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<td>100</td>
<td>0.96332</td>
<td>0.88725</td>
<td>0.82769</td>
<td>0.74820</td>
<td>0.61524</td>
<td>0.50173</td>
<td>0.38808</td>
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</table>

<table>
<thead>
<tr>
<th>m</th>
<th>$\alpha = 5$</th>
<th>$\alpha = 10$</th>
<th>$\alpha = 30$</th>
<th>$\alpha = 50$</th>
<th>$\alpha = 100$</th>
<th>$\alpha = 300$</th>
<th>$\alpha = 500$</th>
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<tbody>
<tr>
<td>1</td>
<td>0.44196</td>
<td>0.35149</td>
<td>0.23199</td>
<td>0.18788</td>
<td>0.13919</td>
<td>0.84448</td>
<td>0.66448</td>
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<tr>
<td>2</td>
<td>0.34256</td>
<td>0.25389</td>
<td>0.14707</td>
<td>0.11131</td>
<td>0.07475</td>
<td>0.38308</td>
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<td>3</td>
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<td>0.22479</td>
<td>0.12420</td>
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<td>0.05946</td>
<td>0.28679</td>
<td>0.20215</td>
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<td>4</td>
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<td>0.21103</td>
<td>0.11382</td>
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<td>0.05287</td>
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<td>0.17121</td>
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<td>0.28714</td>
<td>0.20304</td>
<td>0.10793</td>
<td>0.07815</td>
<td>0.04920</td>
<td>0.19262</td>
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<td>10</td>
<td>0.26974</td>
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<td>0.06913</td>
<td>0.04264</td>
<td>0.01890</td>
<td>0.01274</td>
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<td>30</td>
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<td>0.17790</td>
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<td>0.03742</td>
<td>0.01608</td>
<td>0.01068</td>
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</table>

Table 7.2: Reliability under Marshall Olkin Gumbel bivariate exponential (MOGBE) model where stress components are in parallel.

<table>
<thead>
<tr>
<th>m</th>
<th>$\alpha = .001$</th>
<th>$\alpha = .05$</th>
<th>$\alpha = 1$</th>
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<th>$\alpha = 50$</th>
<th>$\alpha = 100$</th>
<th>$\alpha = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.99723</td>
<td>0.95165</td>
<td>0.83333</td>
<td>0.82166</td>
<td>0.87317</td>
<td>0.89769</td>
<td>0.94402</td>
</tr>
<tr>
<td>2</td>
<td>0.99841</td>
<td>0.97309</td>
<td>0.91421</td>
<td>0.91926</td>
<td>0.94974</td>
<td>0.96213</td>
<td>0.98273</td>
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<tr>
<td>10</td>
<td>0.99964</td>
<td>0.99414</td>
<td>0.98267</td>
<td>0.98547</td>
<td>0.99191</td>
<td>0.99424</td>
<td>0.99772</td>
</tr>
<tr>
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<td>0.99996</td>
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<td>0.99826</td>
<td>0.99854</td>
<td>0.99923</td>
<td>0.99945</td>
<td>0.99979</td>
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</tbody>
</table>
Table 7.3: Reliability under Morgenstern Gumbel bivariate model where stress components are in series.

<table>
<thead>
<tr>
<th>m</th>
<th>$\lambda = 0.5$</th>
<th>$\lambda = 1$</th>
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<th>$\lambda = 10$</th>
<th>$\lambda = 50$</th>
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<tbody>
<tr>
<td>1</td>
<td>0.80000</td>
<td>0.66666</td>
<td>0.28571</td>
<td>0.16666</td>
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<td>0.73879</td>
<td>0.58578</td>
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<td>0.12389</td>
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<td>5</td>
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<td>0.18681</td>
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<td>0.68188</td>
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<td>0.02098</td>
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<tr>
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<td>0.50866</td>
<td>0.17153</td>
<td>0.09381</td>
<td>0.02028</td>
</tr>
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<td>0.50346</td>
<td>0.16860</td>
<td>0.09206</td>
<td>0.01987</td>
</tr>
</tbody>
</table>

Table 7.4: Reliability under Morgenstern Gumbel bivariate model where stress components are in parallel.

<table>
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<th>m</th>
<th>$\lambda = .01$</th>
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<th>$\lambda = 1$</th>
<th>$\lambda = 10$</th>
</tr>
</thead>
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<td>1</td>
<td>0.98517</td>
<td>0.86580</td>
<td>0.33333</td>
<td>0.01515</td>
</tr>
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<td>0.98721</td>
<td>0.88422</td>
<td>0.41421</td>
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<td>5</td>
<td>0.98882</td>
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<td>0.46539</td>
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<td>0.08501</td>
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<td>0.98996</td>
<td>0.90795</td>
<td>0.49653</td>
<td>0.08975</td>
</tr>
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</table>
Figure 7.1: The graph of bivariate Gumbel distribution for $\alpha = 1.55, m = 0.76$

Figure 7.2: Reliability for stress under Morgenstern Gumbel bivariate model where stress components are in series (Fig 7.2a) and parallel (Fig 7.2b)

Fig 7.2a. Reliability as a function of parameter $m(\lambda = 0.5)$.

Fig 7.2b. Reliability as a function of parameter $m(\lambda = 1)$.
Figure 7.3: Reliability for stress under Marshall Olkin Gumbel bivariate model where stress components are in series (Fig 7.3a) and parallel (Fig 7.3b).

Figure 7.4: Bivariate failure rate for $\alpha = 6$ and $m=0.6$
7.7 Conclusions

In this chapter we have introduced a new family of bivariate distributions, namely, the Marshall-Olkin Morgenstern Gumbel bivariate distributions and their properties are studied. Two different models of minification processes with the above bivariate distribution as stationary marginal distribution are developed. The expressions for stress-strength reliability are obtained when stress components are in series as well as in parallel. The reliability values are also computed for different parameter values. Expression for the bivariate failure rate is also derived and graphical studies are conducted. Using these, bivariate system reliability can be estimated and also multi-component systems having optimum reliability properties can be designed. These facts can be utilized in system reliability studies for devising equipments in series and parallel, which will give optimum results. The models are useful in designing aeroplanes, rockets, space crafts etc.

References


