CHAPTER 3

Connectedness and continuity in digital spaces with the

Khalimsky topology†

3.1. Introduction

Digital geometry is considered in $\mathbb{Z}^n$ whereas continuous geometry can be used in $\mathbb{R}^n$. To represent continuous geometrical objects in the computer we are limited to some sort of approximations. There are points in the Euclidean plane that can be described exactly on a computer. By introducing notions as connectedness and continuity on discrete sets we can represent discrete objects with the same accuracy as Euclid had in his geometry.

In this chapter we discuss connectedness and continuity in topological digital spaces and compactness in smallest neighborhood spaces. Intermediate value theorem and fixed point theorem for the Khalimsky line are also discussed.

Herman [14] gives a general definition of a digital space.

†Some results contained in this chapter were published in the International Journal of
Open Problems in Mathematics & Applications
**Definition 3.1.1.** [14] A digital space is a pair $(V, \pi)$ where $V$ is a non-empty set and $\pi$ is a binary symmetric relation on $V$ such that for any two elements $x$ and $y$ of $V$ there is a finite sequence $(x^0, \ldots, x^n)$ of elements in $V$ such that $x = x^0$ and $y = x^n$ and $(x^j, x^{j+1}) \in \pi$ for $j = 0, \ldots, n - 1$.

**Remark 3.1.2.** All topological spaces are not digital and all digital spaces are not topological.

**Example.**

(1) Topological spaces satisfying the $T_1$-axiom are not digital spaces

(2) Khalimsky line is a topological digital space

(3) $\mathbb{Z}^2$ with 8-adjacency is not topological but a digital space.

**3.2. Connectedness**

In this section we define a smallest neighborhood space and some important results related to connectedness are also discussed.

**Definition 3.2.1.** A topological space is said to be a smallest neighborhood space, or Alexandroff space, if arbitrary intersection of open sets is open.

This implies that the intersection of all neighborhood of a point $x$ is still a neighborhood of $x$.

The following theorem guarantees that a connected smallest neighborhood space is a digital space.
Theorem 3.2.2. Let $X$ be a connected smallest neighborhood space. Then for any pair of points $x$ and $y$ of $X$ there is a finite sequence $\{x^0, x^1, \ldots, x^n\}$ such that $x = x^0$ and $y = x^n$ and $\{x^i, x^{i+1}\}$ is connected for $i = 0, 1, \ldots, n - 1$.

**Proof.** Let $x$ be a point in $X$, and $Y$ denote the set of points which can be connected to $x$ by such a finite sequence. Clearly $x \in Y$. Suppose that $y \in Y$. Then $N(y) \subset Y$ and $\{\bar y\} \subset Y$. Thus $Y$ is open, closed and non-empty. Since $X$ is connected, this implies $Y = X$. □

Lemma 3.2.3. Let $X$ be a smallest neighborhood space. Suppose $y \in N(x)$. Then there is a path in $X$ that starts in $x$ and ends in $y$.

**Proof.** Let $\phi : I = [0, 1] \to X$,

$$
\phi(t) = \begin{cases} 
  x & \text{if } t = 0 \\
  y & \text{if } t > 0.
\end{cases}
$$

Suppose that $V$ is an open set in $X$. There are three cases:

1. $x \in V$, then $y \in N(x) \subset V$ so $\phi^{-1}(V) = I$
2. $y \in V$, $x \notin V$, then $\phi^{-1}(V) = (0, 1]$
3. $y \notin V$, then $\phi^{-1}(V) = \phi$

□
**Definition 3.2.4.** A topological space \( X \) is said to be Khalimsky arc connected if it satisfies the following conditions.

1. \( X \) satisfies \( T_0 \)-axiom

2. for all \( x, y \in X \), \( I = [a, b]_\mathbb{Z} \) and \( \phi : I \to X \) such that \( \phi(a) = x \), \( \phi(b) = y \) and \( \phi \) is a homeomorphism of \( I \) into \( \phi(I) \).

From the above definition we have the following result.

**Theorem 3.2.5.** A \( T_0 \) topological digital space is Khalimsky arc connected.

**Proof.** Let \( X \) be a topological digital space and let \( a, b \in X \). We get a finite, connected sequence of points \( (x_0, \ldots, x_n) \) such that \( x_0 = a \) and \( x_n = b \). Since the sequence is finite, we get a subsequence \( Y = \{y_0, \ldots, y_m\} \), that is minimal with respect to connectedness, and such that \( y_0 = a \) and \( y_m = b \). Hence \( Y \) has no connected proper subset containing \( a \) and \( b \).

Suppose that \( i > j \) and \( \{y_i, y_j\} \) is connected. Then \( i = j + 1 \) for if not, the sequence \( \{y_0, \ldots, y_j, y_i, \ldots, y_m\} \) would be a connected proper subset of the minimal sequence, which is a contradiction.

So there are two possibilities for \( N_Y(y_0) \); either \( N_Y(y_0) = \{y_0\} \) or \( N_Y(y_0) = \{y_0, y_1\} \). Re-index the sequence \( \{y_i\} \) so that \( y_s \) is its first element and \( y_{m+s} \) its last. Let \( s = 1 \) in the first case and \( s = 0 \) in the second case. We will show that the function \( \varphi : I = [s, s + m]_\mathbb{Z} \to Y \), is a homeomorphism.
\[ Y \text{ is } T_0 \text{ since } X \text{ is } T_0. \varphi \text{ is surjective and by minimality, injective also.} \]

To show that \( \varphi \) is a homeomorphism, it is sufficient to show that

\[ N_Y(y_i) = \varphi(N_I(i)) \text{ for every } i \in I. \quad (1) \]

For \( i = s \) this holds by the choice of \( s \). We use finite induction. Suppose that \( s < k \leq s + m \), and that (1) holds for \( i = k - 1 \). We consider two cases.

**Case 1.** \( k \) is odd. Then \( N_I(k) = \{k\} \). We must show that \( y_{k-1} \) and \( y_{k+1} \) are not in \( N_Y(y_k) \). But since equation (1) holds for \( i = k - 1 \), and \( k \in N_I(k - 1) \), we know that \( y_k \in N_Y(y_{k-1}) \). So \( y_{k-1} \notin N_Y(y_k) \), \( (Y \text{ is } T_0) \) and also that \( y_{k+1} \notin N_Y(y_k) \) since \( N_Y(y_{k-1}) \cap N_Y(y_k) \) is a neighborhood of \( y_k \) which does not include \( y_{k+1} \).

**Case 2.** \( k \) is even. We must show that \( y_{k-1} \in N_Y(y_k) \) and \( y_{k+1} \in N_Y(y_k) \) provided \( k < s + m \).

Now \( N_Y(y_{k-1}) = \{y_{k-1}\} \) by assumption, so clearly \( y_{k-1} \) belongs to \( N_Y(y_k) \). If \( k < s + m \) and \( y_k \in N_Y(y_{k+1}) \) then \( N_Y(y_k) \cap N_Y(y_{k+1}) \) would be a neighborhood of \( y_k \) not containing \( y_{k-1} \). This is a contradiction and hence \( y_{k+1} \in N_Y(y_k) \). \( \square \)
3.3. Continuous functions

This section contains a few results related to continuity of functions from $Z$ to $Z$ and $Z^n$ to $Z$.

We recall the definition of continuous function in a smallest neighborhood space.

**Definition 3.3.1.** A function $f : X \to Y$ from one smallest neighborhood space into another is continuous at a point $x$ if and only if the direct image of $N_X(x)$ is contained in $N_Y(f(x))$, or the inverse image of $N_Y(f(x))$ contains $N_X(x)$.

$$f(N_X(x)) \subseteq N_Y(f(x)),$$

equivalently $N_X(x) \subseteq f^{-1}(N_Y(f(x)))$.

$N_X(x), N_Y(f(x))$ denote the smallest neighborhood of $x \in X$ and $y \in Y$ respectively.

**Note 3.3.2.** A binary relation $\sim$ is defined in $Z$ such that $a \sim b$, if $a - b$ is even. That is, if $a$ and $b$ have the same parity.

**Theorem 3.3.3.** A function $f : Z \to Z$ is continuous if and only if

1. $f$ is Lip-1
2. For all every $x$, $f(x) \not\sim x$ implies $f(x \pm 1) = f(x)$.

**Proof.** Suppose that the function $f : Z \to Z$ is continuous.
(1) If possible, $f$ is not Lip-1. Then $|f(n+1) - f(n)| \geq 2$. So $f\{n, n+1\}$ is not connected. But $\{n, n+1\}$ is connected. This is impossible since $f$ is continuous.

(2) Suppose that $x$ is even and $f(x)$ is odd. Then $U = f(\{x\})$ is open. Then $V = f^{-1}(U)$ is open and in particular that the smallest neighborhood $N(\{x\}) = \{x, x+1\}$ is contained in $V$ or in otherwords $f(x+1) = f(x)$.

A similar arguments holds when $f(x)$ is even and $x$ is odd.

Conversely, let $A = \{y-1, y, y+1\}$ where $y$ is even be any sub base element. We must show that $f^{-1}(A)$ is open. If $x \in f^{-1}(A)$ is odd then $\{x\}$ is a neighborhood of $x$. If $x$ is even, we have two cases. First if $f(x)$ is odd, by condition 2, $f(x+1) = f(x)$ so that $\{x-1, x, x+1\} \subset f^{-1}(A)$ is a neighborhood of $x$. Second, if $f(x)$ is even, then $f(x) = y$ and the Lip-1 condition implies $|f(x+1)-y| \leq 1$ so that again $\{x-1, x, x+1\} \subset f^{-1}(A)$ is a neighborhood of $x$. Thus $f$ is continuous. □

Above result holds in any dimension.

**Theorem 3.3.4.** A continuous function $f : \mathbb{Z}^n \to \mathbb{Z}$ is Lip-1 with respect to the $l^\infty$ metric.

**PROOF.** We use induction over the dimension. Suppose that the statement holds in $\mathbb{Z}^{n-1}$. Let $f : \mathbb{Z}^n \to \mathbb{Z}$ be continuous, $x' \in \mathbb{Z}^{n-1}$, $x_n \in \mathbb{Z}$...
and \( x = (x', x_n) \in \mathbb{Z}^n \). Assume that \( f(x) = 0 \). Consider the cases when \( x_n \) odd and \( x_n \) even. If \( x_n \) is odd then \( f(x + (0, \ldots, 0, 1)) = 0 \), and by the induction hypothesis \( f(x + (1, \ldots, 1, 1)) \leq 1 \) On the other hand, it is always true, by the induction hypothesis, that \( f(x + (1, \ldots, 1, 0)) \leq 1 \). If \( x_n \) is even and \( f(x + (1, \ldots, 1, 0)) = 1 \), then also \( f(x + (1, \ldots, 1)) = 1 \). This shows that \( f \) can increase atmost 1, if we take a step in every co-ordinate direction, and by a trivial modification of the argument, also if we step only in some directions. By a similar argument, we can get a lower bound, and hence \( f \) is Lip-I. \(
\)

For Khalimsky continuous functions, we have the following theorem.

**Theorem 3.3.5.** \( f : \mathbb{Z}^n \to \mathbb{Z} \) is continuous if and only if \( f \) is separately continuous.

**Proof.** If \( f : \mathbb{Z}^n \to \mathbb{Z} \) is continuous, then clearly \( f \) is separately continuous. Conversely suppose that \( f \) is separately continuous. We want to prove that the inverse image of a subbasis element, \( A = \{y-1, y, y+1\} \), where \( y \) is even is open. Suppose \( x \in f^{-1}(A) \). We show that \( N(x) \subset f^{-1}(A) \).

\[
N(x) = \begin{cases} 
    z \in \mathbb{Z}^n; & |x_i - z_i| \leq 1 \text{ if } x_i \text{ is even} \\
    z_i = x_i & \text{if } x_i \text{ is odd.}
\end{cases}
\]

Let \( z \in N(x) \) and \( I = \{i_0, \ldots, i_k\} \) be the indices for which \( |x_i - z_i| = 1 \).
Let \((x_0, \ldots, x_k)\) be the sequence of points in \(\mathbb{Z}^n\) such that \(x_0 = x, x_k = z\) and

\[x^{j+1} = x^j + (0, 0, \ldots, 0, \pm 1, 0, \ldots, 0)\]

for \(j = \{0, \ldots, k - 1\}\), so that \(x^{j+1}\) is one step closer to \(z\) than \(x_j\) in the \(i_{j}^{th}\) co-ordinate direction.

Now if \(f(x)\) is odd, then by separate continuity and by Theorem 3.3.3 it follows that \(f(x^{j+1}) = f(x^j)\). In particular \(f(z) = f(x)\) and hence \(z \in f^{-1}(A)\). If \(f(x)\) is even, then \(f(x^{j+1}) = f(x^j) \pm 1\), for some \(j\). But then \(f(x^{j+1})\) is odd and must be constant on the remaining elements of the sequence. Therefore \(f(z) \in A\) and hence \(z \in f^{-1}(A)\). \(\square\)

We observe that the following functions are continuous.

**Theorem 3.3.6.** Let \(f : \mathbb{Z} \to \mathbb{Z}\), then

1. for all \(x \in \mathbb{Z}\), \(f(x) = a \in \mathbb{Z}\) is continuous,

2. for all \(x \in \mathbb{Z}\), \(f(x) = \pm x + C \in \mathbb{Z}\) where \(C\) is an even constant is continuous.

**Theorem 3.3.7.** Let \(f : \mathbb{Z} \to \mathbb{Z}\), then \(f(x) = x + 1\) is discontinuous.

**Proof.** In this case \(1 \to 2, 2 \to 3\) etc.

\{2\} is closed while \{1\} is open; \{3\} is open while \{2\} is closed. Hence the proof. \(\square\)
In a more general setting, this generalizes to the following.

**Theorem 3.3.8.** Let \( X \) be a topological space and \( f : X \to Z \) be a continuous mapping. Suppose that \( x_0 \in X \) if \( f(x_0) \) is odd, then \( f \) is constant on \( N(x_0) \) and \(|f(x) - f(x_0)| \leq 1 \) for all \( x \in C(x_0) \). If \( f(x_0) \) is even, then \( f \) is constant on \( C(x_0) \) and \(|f(x) - f(x_0)| \leq 1 \) for all \( x \in N(x_0) \) where \( N(x_0) \) denotes the intersection of all open sets containing \( x_0 \) and \( C(x_0) \) denote the intersections of all closed sets containing \( x_0 \).

**Proof.** Let \( y_0 = f(x_0) \) be odd, then \( \{y_0\} \) is an open set. Hence \( f^{-1}\{y_0\} \) is open and therefore \( N(x_0) \subset f^{-1}\{y_0\} \). Hence \( f(N(x_0)) = \{y_0\} \). Moreover the set \( A = \{y_0, y_0 \pm 1\} \) is closed. But then the set \( f^{-1}(A) \) is closed also and this implies that \( f(x) \in A \) for all \( x \in C(x_0) \). This complete the proof since the even case is dual. \( \square \)

As in Real Analysis there is an intermediate value theorem for the Khalimsky line, which is introduced below.

**Theorem 3.3.9.** Let two continuous functions \( f, g : I \to Z \) be given on a Khalimsky interval \( I = [a, b]_Z \). Assume that there are points \( s, t \in I \) with \( f(s) \geq g(s) \) and \( f(t) \leq g(t) \). Then there exists a point \( p \) intermediate between \( s \) and \( t \) such that \( f(p) = g(p) \).
PROOF. Without loss of generality assume that $s \leq t$. Define $M = \{x \in \mathbb{Z}, s \leq x \leq t$ and $f(x) \geq g(x)\}$.

Then $s \in M$, so $M$ is non-empty. Let $p = \max_{x \in M} x$. If $p = t$, then $f(t) = g(t)$ and hence the theorem.

If not, $p + 1 \leq t$, then we have $f(p) \geq g(p), f(p + 1) < g(p + 1)$.

We claim that $f(p) = g(p)$. Otherwise we should have

$$f(p) \geq g(p) + 1 \text{ and } f(p + 1) \leq g(p + 1) - 1.$$  

Because of the Lipschitz continuity, the only possibility then would be $f(p) = g(p) + 1$ and $f(p + 1) = g(p + 1) - 1$. But this situation is impossible.

If $p$ and $f(p)$ are of different parity, then $f(p + 1) = f(p)$, which require a jump of two units in $g$. If on the other hand, $p$ and $f(p)$ are of same parity, then $p$ and $g(p)$ are of different parity, so that $g(p + 1) = g(p)$ requiring a jump of two units in $f$. This contradiction gives the only possibility

$$f(p) = g(p). \quad \square$$

3.4. Fixed point theorem

In this section we discuss fixed point property of certain subsets of the Khalimsky plane.
**Definition 3.4.1.** A topological space $X$ has the fixed point property if every continuous mapping $f : X \rightarrow X$ possesses a fixed point; that is, there exists a point $p$ such that $f(p) = p$.

**Note 3.4.2.** In a finite set $X$ with $N$ points there are $N^N$ self mappings $X \rightarrow X$. Out of these $(N - 1)^N$ do not have fixed points; thus there are $N^N - (N - 1)^N$ mappings which have a fixed point. By introducing a topology on $X$, the number of continuous mappings $C$ can be evaluated.

<table>
<thead>
<tr>
<th></th>
<th>Continuous</th>
<th>Discontinuous</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed point</td>
<td>$C$</td>
<td>$N^N - (N - 1)^N - C$</td>
<td>$N^N - (N - 1)^N$</td>
</tr>
<tr>
<td>No fixed point</td>
<td>0</td>
<td>$(N - 1)^N$</td>
<td>$(N - 1)^N$</td>
</tr>
<tr>
<td>Sum</td>
<td>$C$</td>
<td>$N^N - C$</td>
<td>$N^N$</td>
</tr>
</tbody>
</table>

**Example 3.4.3.** For a Khalimsky interval $\{a, a + 1\}$ consisting of two points, there are four mappings; the two constant mappings, the identity, and the one interchanging $a$ and $a + 1$. The first three have a fixed point; the fourth does not. But it is discontinuous.

<table>
<thead>
<tr>
<th></th>
<th>Continuous</th>
<th>Discontinuous</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed point</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>No fixed point</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Sum</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

**Example 3.4.4.** For the Khalimsky square $\{0, 1\}^2 \subset Z^2$, there are $N^N = 4^4 = 256$ self mappings of which $(N - 1)^N = 3^4 = 81$ do not have fixed
points. The remaining $256 - 81 = 175$ have a fixed point. Of the 16 mappings $\{0, 1\}^2 \to \{0, 1\}$, 6 are continuous. There are therefore $6^2 = 36$ continuous mappings $\{0, 1\}^2 \to \{0, 1\}^2$ and they all have fixed points.

<table>
<thead>
<tr>
<th></th>
<th>Continuous</th>
<th>Discontinuous</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed point</td>
<td>36</td>
<td>139</td>
<td>175</td>
</tr>
<tr>
<td>No fixed point</td>
<td>0</td>
<td>81</td>
<td>81</td>
</tr>
<tr>
<td>Sum</td>
<td>36</td>
<td>220</td>
<td>256</td>
</tr>
</tbody>
</table>

Of the 6 continuous mappings $\{0, 1\}^2 \to \{0, 1\}$, five maps $\{0, 0\}$ to 0; the remaining one is the constant 1. Therefore, of the 36 continuous mappings $\{0, 1\}^2 \to \{0, 1\}^2$, $\{0, 0\}$ is a fixed point except when one of the component is the constant 1. Thus they all have a fixed point.

We shall now prove that a continuous mapping of an interval into itself has a fixed point.

**Remark 3.4.5.** By $C(Z, Z)$ we denote the set of all continuous functions from $Z$ to $Z$, $C\#(Z, Z)$ be the subset of $C(Z, Z)$ such that

$$C\#(Z, Z) = \{ f \in C(Z, Z); \exists s \in Z, f(s) \geq s \text{ and } \exists t \in Z, f(t) \leq t \}$$

then $f \in C(Z, Z)$ has a fixed point if and only if $f \in C\#(Z, Z)$.

**Theorem 3.4.6.** Every bounded Khalimsky interval has the fixed point property.
PROOF. Let \( f : I \to I \) be a continuous mapping, where \( I = [a, b] \) is a bounded interval. Extend \( f \) to a mapping \( g : Z \to Z \) by defining \( g(x) = f(a) \) for \( x < a \) and \( g(x) = f(b) \) for \( x > b \). Then \( g \) is continuous and \( g \) belongs to \( C_\#(Z, Z) \). Thus it has a fixed point \( p \in Z \), but \( p \in \text{im } g \subset I \), \( p \) is a fixed point of \( f \) also. \( \Box \)

3.5. Locally finite and locally countable spaces

**Definition 3.5.1.** A smallest neighborhood space is called locally finite if every point in it has a finite adjacency neighborhood. If every point has a countable adjacency neighborhood, it is called locally countable.

**Note 3.5.2.** Let \( f : X \to Z \) be a continuous function defined on the smallest neighborhood space \( X \). Define the equivalence relation \( x \sim y \) if and only if \( N(x) = N(y) \) and the quotient space is \( \tilde{X} = X/\sim \). The space \( \tilde{X} \) is a \( T_0 \) space.

**Theorem 3.5.3.** Let \( X \) be a locally finite space. If \( X \) is connected, then \( X \) is countable.

PROOF. If \( X \) is not \( T_0 \), consider the quotient space \( \tilde{X} \), where points with identical neighborhoods have been identified. \( \tilde{X} \) is \( T_0 \) and \( X \) is countable if \( \tilde{X} \) is countable. It is therefore sufficient to prove the result for \( T_0 \) spaces.
Let $x$ be any point in $X$. The ball

$$B_n(x) = \{ y \in X; \rho_X(x, y) \leq n \}$$

where $\rho_X$ is the arc metric on $X$ is finite for every $n \in \mathbb{N}$. Since $X = \bigcup_{n=0}^{\infty} B_n(x)$, the result is true, since the countable union of finite sets is countable.

In a similar way, we can characterize countable smallest neighborhood spaces, which is given below.

**Theorem 3.5.4.** A smallest neighborhood space $X$ is countable, if and only if, it is locally countable and has countably many connectivity components.

**Proof.** A countable space is always locally countable and has countably many components. The countable axiom of choice implies that countable unions of countable sets are countable. So if $X$ is countable and locally countable by Theorem 3.5.3, we can say that $X$ is countable. If $X$ has countably many connectivity components and each component is countable, then $X$ is countable.

The following theorem states that the sets of open points is dense in a locally finite space.
Theorem 3.5.5. Let $X$ be a smallest neighborhood space and let $S \subset X$ be the set of open points in $X$ and $T$ be the set of closed points. If $X$ is $T_0$ and locally finite then $X = C(S) = N(T)$ where $C(S)$ is the closure of $X$ and $N(T)$ is the neighborhood of $T$.

**Proof.** Let $X = C(S)$. Take $y_0 \in X$ and $Y_0 = N(y_0)$. If $Y_0$ is a singleton set, then $y_0 \in S$ otherwise for $k \geq 0$, choose $y_{k+1} \in Y_k - \{ y_k \}$ and let $Y_{k+1} = N(y_{k+1})$. Then $Y_{K+1} \subset Y_K$ and since $X$ is $T_0$, $y_k \not\in Y_{K+1}$. Repeat the above process until $Y_k$ is a singleton set, atmost $\text{Card}(Y_0) - 1$ steps are needed. $y_k$ is an open set and that $y_0 \in C(y_k)$, since $y_0$ is arbitrary, $X = C(S)$.

3.6. Join operator

To combine two topological spaces $X$ and $Y$ we usually take the disjoint union. The pieces $X$ and $Y$ are completely independent in this construction. Melin [9] introduce another way of combining two spaces by using the join operation.

**Definition 3.6.1.** [9] Let $X$ and $Y$ be two topological spaces. The join of $X$ and $Y$ denoted by $X \vee Y$ is a topological space over the disjoint set union of $X$ and $Y$ where a subset $A \subset X \vee Y$ is open if either
(1) \( A \cap X \) is open in \( X \) and \( A \cap Y = \phi \) or

(2) \( A \cap X = X \) and \( A \cap Y \) is open in \( Y \).

A set \( B \subset X \lor Y \) is closed if and only if

(1) \( B \cap X \) is closed in \( X \) and \( B \cap Y = Y \) or

(2) \( B \cap X = \phi \) and \( B \cap Y \) is closed in \( Y \).

3.6.1. Properties. The join operator has the following properties for all smallest neighborhood spaces \( X, Y \) and \( Z \):

(1) \( X = \phi \lor X = X \lor \phi \)

(2) \( (X \lor Y) \lor Z = X \lor (Y \lor Z) \)

(3) \( (X \lor Y)' = Y' \lor X' \)

Proposition 3.6.2. Let \( X \) and \( Y \) be smallest neighborhood spaces,

(1) \( X \lor Y \) is \( T_0 \) if and only if \( X \) and \( Y \) are \( T_0 \)

(2) \( X \lor Y \) is compact if and only if \( Y \) is compact

(3) If \( X \neq \phi \) and \( Y \neq \phi \) then \( X \lor Y \) is connected.

Proof. (1) Let \( X \) be open in \( X \lor Y \), if \( x \in X \) and \( y \in Y \) then \( X \) is an open set containing \( x \) but not \( y \). Hence \( X \lor Y \) fail to be \( T_0 \) only for a pair of points in \( X \) or a pair of points in \( Y \). Hence if \( X \) and \( Y \) are \( T_0 \), then \( X \lor Y \) is \( T_0 \). Conversely if \( X \lor Y \) is \( T_0 \) then \( X \) and \( Y \) are \( T_0 \).
(2) Assume that $Y$ is not compact and $\{A_i\}_{i \in I}$ be an open cover of $Y$ without a finite subcover. Let $B_i = A_i \cup X$ for each $i \in I$, then $\{B_i\}_{i \in I}$ is an open cover of $X \cup Y$ without a finite subcover. This is impossible. Hence if $X \cup Y$ is compact, then $Y$ is compact.

Conversely suppose that $Y$ is compact and $\{B_i\}_{i \in I}$ be an open cover of $X \cup Y$. It induces an open cover of $Y$ with elements in $B_i \cap Y$. But this cover has a finite subcover $\{B_i \cap Y\}_{i=1}^n$ since $Y$ is compact. It follows that $\{B_i\}_{i=1}^n$ is a finite subcover of $X \cup Y$ since any $B_i$ where $B_i \cap Y \neq \phi$ covers $X$.

3 Assume that $x \in X$ and $y \in Y$. Since $x \in N(y)$, $x$ and $y$ are adjoint. If $a, b \in X$, then $\{a, y, b\}$ is a connected set. Similarly, $\{c, x, d\}$ is a connected set if $c, d \in Y$.

All the above properties except (3) are true not only small neighborhood spaces but also for general topological spaces. □

3.7. Extended Khalimsky line

Let $[-\infty, \infty]_Z$ be the set obtained by adjoining two new elements $+\infty$ and $-\infty$ to the set $Z$. We can extend the ordering of $Z$ by putting $-\infty < m < +\infty$ for all $m \in Z$. Then we can endow $[-\infty, \infty]_Z$ with a suitable topology so that it is a compactification of $Z$. Extended Khalimsky line is not a smallest neighborhood space.
**Definition 3.7.1.** [16] A compactification of a smallest neighborhood space $X$ is a compact topological space $Y$ such that $X$ is homeomorphic to a subset of $Y$ and $X$ is dense in $Y$.

**Remark 3.7.2.** Let $\{\{2m+1\}, \{2m, 2m \pm 1\}, [2m+1, +\infty]_Z, [-\infty, 2m+1]_Z, n \in Z\}$ be a basis for $[-\infty, +\infty]_Z$. Then $Z$ as a subspace of $[-\infty, +\infty]_Z$ is the ordinary Khalimsky line and the closure of $Z$ in this space is $[-\infty, +\infty]_Z$. Since $[-\infty, +\infty]_Z$ has both a largest and smallest element it constitutes a complete lattice.

It is possible to find a connected smallest neighborhood space which is a compactification of $Z$.

**Theorem 3.7.3.** In any compaction of $Z$, there exist a connected smallest neighbourhood space.

**Theorem 3.7.4.** There exist no compactification of $Z$ which is also a smallest neighbourhood space such that no point in $Z$ adjacent to a point at infinity.

**Proof.** Let $K = Z \cup X$ be a compactification of $Z$ where $X$ is not empty and $K \cap Z \cong Z$. Since every point in $Z$ is separated from every point in $X$ by the smallest neighbourhood property, any set open in $Z$ is open in $K$ and any set closed in $Z$ is closed in $K$. Hence since $K$ is a
smallest neighbourhood space and \( Z \) can be covered both by closed and open sets, \( Z \) constitutes a connectivity component which is not compact, i.e., \( K \) is not compact.

\( \square \)