Chapter 2

Various topologies on $\mathbb{Z}^2$†

2.1. Introduction

An important problem of digital topology is to provide the digital plane $\mathbb{Z}^2$ with a convenient structure for the study of geometric and topological properties of digital images. A basic criterion for such a convenience is the validity of an analogy of the Jordan curve theorem. It was in 1990 that a topology on $\mathbb{Z}^2$ convenient for the study of digital images was introduced by Khalimsky. A drawback of the Khalimsky topology is that the Jordan curves with respect to it can never turn at an acute angle. To overcome this deficiency, another topology on $\mathbb{Z}^2$ was introduced by Slapal. In this chapter we compare different topologies on $\mathbb{Z}^2$ by comparing their merits and demerits.

First of all we introduce some basic concepts needed for defining various topologies on $\mathbb{Z}^2$.

**Notation 2.1.1.** [45] Let $z = (x, y) \in \mathbb{Z}^2$. Put

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Then we put

\[ A_4(z) = H_2(z) \cup V_2(z) \]
\[ A_8(z) = H_2(z) \cup L_4(z) \cup R_4(z) \]
\[ = V_2(z) \cup D_4(z) \cup U_4(z) \]

and \( A'_4(z) = A_8(z) - A_4(z) \).

\( A_4(z) \) and \( A_8(z) \) are said to be 4-adjacent and 8-adjacent to \( z \) respectively.

\[ H_2(z), V_2(z), D_4(z), V_4(z), L_4(z), R_4(z) \]

and \( A'_4(z) \) are called horizontally 2-adjacent, vertically 2-adjacent, down 4-adjacent, up 4-adjacent, left 4-adjacent, right 4-adjacent and diagonally 4-adjacent to \( z \) respectively.
Now we introduce the basic topology on $\mathbb{Z}^2$:

## 2.2. Marcus Topology

### Definition 2.2.1.

Let $u$ be the Alexandroff $T_{1/2}$-topology on $\mathbb{Z}^2$ as follows.

For any $z = (x, y) \in \mathbb{Z}^2$

$$u(z) = \begin{cases} 
\{z\} \cup A_4(z) & \text{if } x + y \text{ is even} \\
\{z\} & \text{otherwise.}
\end{cases}$$

$(\mathbb{Z}^2, u)$ is called the Marcus topological space.

### Remark 2.2.2.

1. $u$ is a $T_{1/2}$ topology because the points $(x, y) \in \mathbb{Z}^2$ with $x + y$ even are open and all other points of $\mathbb{Z}^2$ are closed.

2. $(\mathbb{Z}^2, u)$ is a connected space in which a two point subset $\{z_1, z_2\} \subset \mathbb{Z}^2$ is connected if and only if $z_1$ and $z_2$ are 4-adjacent.

3. The Marcus topology which corresponds to 4-adjacency is very simple and therefore unsatisfactory for the purpose of digital image processing. There exist no topology on $\mathbb{Z}^2$ whose connectedness graph corresponds to 8-adjacency.
The Khalimsky topology whose connectedness corresponds to a combination of 4-adjacency and the 8-adjacency provides a convenient structure on \( Z^2 \). The Khalimsky topology is the Alexandroff topology \( v \) on \( Z^2 \) given as follows:

![Figure 1: A portion of the connectedness graph of the Marcus topology. The closed points are blackened.](image)

### 2.3. Khalimsky topology

**Definition 2.3.1.** For any \( z = (x, y) \in Z^2 \)

\[
v(z) = \begin{cases} 
\{z\} \cup A_8(z) & \text{if } x, y \text{ are even,} \\
\{z\} \cup H_2(z) & \text{if } x \text{ is even, and } y \text{ is odd} \\
\{z\} \cup V_2(z) & \text{if } x \text{ is odd and } y \text{ is even} \\
\{z\} & \text{otherwise}
\end{cases}
\]
The topological space \((Z^2, v)\) is called the Khalimsky topological space.

**Remark 2.3.2.**

1. The Khalimsky topology \(v\) can be obtained as the product of two copies of the Alexandroff topology \(u\) on \(Z\) given by \(u(x) = \{x - 1, x, x + 1\}\) for any even \(x \in Z\) and \(u(x) = \{x\}\) for any odd \(x \in Z\).

2. Khalimsky topology is a \(T_0\) topology but not a \(T_{1/2}\) topology because the points with one co-ordinate even and the other odd are neither closed nor open, they are called mixed points. All the other points of \(Z^2\) are closed or open, they are called pure points.

3. A portion of the connectedness graph of the Khalimsky topology is shown in Figure 2.

![Figure 2. The closed points are ringed and the mixed ones boxed.](image)

4. In the Khalimsky topological space, any simple closed curve having atleast four points is a Jordan curve

**Remark** 4 immediately results in the following theorem.
Theorem 2.3.3. Any cycle $J$ in the square diagonal graph of type 2 is a Jordan curve in the Khalimsky topological space if and only if $J$ does not turn at any of its points at the acute angle $\frac{\pi}{4}$.

PROOF. Let $J$ be a cycle in the square-diagonal graph of type 2 which does not turn at any point $z \in J$ at the acute angle $\frac{\pi}{4}$. Then $J$ is a simple closed curve having at least four points. Thus $J$ is a Jordan curve.

Conversely let $J$ be a Jordan curve in the Khalimsky topological space. At a mixed point such a curve never turn and at a pure point, they turn at angles $\frac{\pi}{2}$ and $\frac{3\pi}{4}$ only. So it never turn at an acute angle $\frac{\pi}{4}$. □

Here we define another topology on $Z^2$ with respect to which each circuit in the square-diagonal graph is a Jordan curve.
2.4. Slapal’s topology

**Definition 2.4.1.** [44] Let \( w \) be the Alexandroff \( T_{1/2} \) topology on \( \mathbb{Z}^2 \) defined as follows.

For any point \( z = (x, y) \in \mathbb{Z}^2 \)

\[
 w(z) = \begin{cases} 
\{z\} \cup A_8(z) & \text{if } x = 4k, y = 4l, k, l \in \mathbb{Z} \\
\{z\} \cup A'_4(z) & \text{if } x = 2 + 4k, y = 2 + 4l, k, l \in \mathbb{Z} \\
\{z\} \cup D_4(z) & \text{if } x = 2 + 4k, y = 1 + 4l, k, l \in \mathbb{Z} \\
\{z\} \cup U_4(z) & \text{if } x = 2 + 4k, y = 3 + 4l, k, l \in \mathbb{Z} \\
\{z\} \cup R_4(z) & \text{if } x = 3 + 4k, y = 2 + 4k, k, l \in \mathbb{Z} \\
\{z\} \cup H_2(z) & \text{if } x = 2 + 4k, y = 4l, k, l \in \mathbb{Z} \\
\{z\} \cup V_2(z) & \text{if } x = 4k, y = 2 + 4l, k, l \in \mathbb{Z} \\
\{z\} & \text{otherwise}
\end{cases}
\]

The square-diagonal graph is a natural structure on \( \mathbb{Z}^2 \) and therefore be used for processing digital pictures provided there is a topology with respect to which its circuits are Jordan curves. It turns out that \( w \) is such a topology:

**Theorem 2.4.2.** Any cycle in the square-diagonal graph of type 4 is a Jordan curve in \( (\mathbb{Z}^2, w) \).
PROOF. Any cycle in the square-diagonal graph of type 4 is a simple closed curve in $(Z^2, w)$. Let $z = (x, y) \in Z^2$ be a point such that $x = 4k + p$ and $y = 4l + q$ for some $k, l, p, q \in Z$ with $pq = \pm 2$. Then we define the fundamental triangle $T(z)$ to be the nine-point subset of $Z^2$ given as follows.

$$T(z) = \begin{cases} 
(r, s) \in Z^2; \ y - 1 \leq s \leq y + 1 - |r - x| & \text{if } x = 4k + 2 \\
\text{and } y = 4l + 1 \text{ for some } k, l \in Z \\
(r, s) \in Z^2; \ y - 1 + |r - x| \leq s \leq y + 1 & \text{if } x = 4k + 2 \text{ and } y = 4l - 1 \text{ for some } k, l \in Z \\
(r, s) \in Z^2; \ x - 1 \leq r \leq x + 1 - |s - y| & \text{if } x = 4k + 1 \\
\text{and } y = 4l + 2 \text{ for some } k, l \in Z \\
(r, s) \in Z^2; \ x - 1 + |s - y| \leq r \leq x + 1 & \text{if } x = 4k - 1 \\
\text{and } y = 4l + 2 \text{ for some } k, l \in Z.
\end{cases}$$

Figure 3. A portion of the connectedness graph of $w$. The closed points are ringed.
Graphically, the fundamental triangle $T(z)$ consists of the point $z$ and the eight points lying on the triangle surrounding $z$. The four types of fundamental triangles are represented in figure 4.

![Diagram of fundamental triangles](image)

**Figure 4**

The four types of fundamental triangle are represented by $T(z_1)$, $T(z_2)$, $T(z_3)$ and $T(z_4)$.

1. Every fundamental triangle is connected (so that the union of two fundamental triangles having a common side is connected).
2. If we subtract from a fundamental triangle some of its sides, then the resulting set is still connected.
3. If $S_1, S_2$ are fundamental triangles having a common side $D$, then the set $(S_1 \cup S_2) - M$ is connected whenever $M$ is the union of some sides of $S_1$ or $S_2$ different from $D$.
4. Every connected subset of $(\mathbb{Z}^2, w)$ having atmost two points is a subset of a fundamental triangle.
For every cycle $C$ in the square-diagonal graph of type 4, there are sequences $S_F, S_I$ of fundamental triangles, $S_F$ finite and $S_I$ infinite such that whenever $S \in \{S_F, S_I\}$ the following two conditions are satisfied.

(a) Each member of $S$, excluding the first one, has a common side with at least one of its predecessors.

(b) $C$ is the union of those sides of fundamental triangles from $S$ that are not shared by two different fundamental triangles from $S$.

To prove (5) let $S = \{S_1, S_2, \ldots\}$ be a finite or infinite sequences of fundamental triangles defined as follows. Let $T_1$ be an arbitrary fundamental triangle. For any $k \in \mathbb{Z}, k \geq 1$, if $S_1, S_2, \ldots, S_k$ are defined, let $S_{k+1}$ be any fundamental triangle having a side which is disjoint from $C$ and common with at least one of the triangles $S_1, S_2, \ldots, S_k$. If for certain $k \geq 1$, there is no such a fundamental triangle $S_{k+1}$, then $S_k$ is considered to be the last member of $S$, i.e., we have $S = (S_1, S_2, \ldots, S_k)$. Otherwise if $S_{k+1}$ is defined whenever $k \geq 1$, we have $S = (S_i)_{i=1}^\infty$. Further, let $S'_1 = S(z)$ be a fundamental triangle such that $z \notin S$ whenever $S$ is a member of $S$. Having defined $S'_1$, let $S' = (S'_1, S'_2, \ldots)$ be a sequence of fundamental triangles defined analogously to $S$. Then one of the sequences $S, S'$ is finite and the other is infinite. $S$ is finite or infinite if and only if its first member
equals such a fundamental triangle $S(z)$ for which $z = (k, l) \in \mathbb{Z}^2$ has the property that the cardinality of the set, $\{(x, l) \in \mathbb{Z}^2, x > k\} \cap C$ is odd or even respectively. The same is true for $S'$. If we put $\{S_F, S_I\} = \{S, S'\}$ where $S_F$ is finite and $S_I$ is infinite, then the conditions (a) and (b) are satisfied.

Given a cycle $C$ in the square-diagonal graph of type 4, let $S_F$ and $S_I$ denote the union of all members of $S_F$ and $S_I$ respectively. Then $S_F \cup S_I = \mathbb{Z}^2$ and $S_F \cap S_I = C$. Let $S_F^*$ and $S_I^*$ be the sequences obtained from $S_F$ and $S_I$ by subtracting $C$ from each member of $S_F$ and $S_I$ respectively. Let $S_F^*$ and $S_I^*$ denote the union of all members of $S_F^*$ and $S_I^*$ respectively. $S_F^*$ and $S_I^*$ are connected by (1), (2) and (3) and it is clear that $S_F^* = S_F - C$ and $S_I^* = S_I - C$. So $S_F^*$ and $S_I^*$ are the two components of $\mathbb{Z}^2 - C$ by (4). $S_F - C$ is called the inside component and $S_I - C$ is the outside component. Hence the cycle $C$ is a Jordan curve in $(\mathbb{Z}^2, w)$. □

2.5. Quotient topologies of $w$

**Remark 2.5.1.** Given a topological space $(X, p)$, a set $Y$ and a surjection $e : X \rightarrow Y$, a topology $q$ on $Y$ is said to be the quotient topology of $p$ generated by $e$ if $q$ is the finest topology on $Y$ for which $e : (X, p) \rightarrow (Y, q)$ is continuous.
For Alexandroff topological spaces \((X, p)\) and \((Y, q)\), a map \(e : (X, p) \to (Y, q)\) is continuous if and only if \(e(p\{x\}) \subseteq q\{e(x)\}\) for every \(x \in X\).

We need the following lemma.

**Lemma 2.5.2.** Let \((X, p)\), \((Y, q)\) be Alexandroff topological spaces and let \(e : X \to Y\) be a surjection. Then the following condition is sufficient for \(q\) to be the quotient topology of \(p\) generated by \(e\).

For any pair of points \(x, y \in Y\), \(x \in q(y)\) if and only if there are \(a \in e^{-1}(x)\) and \(b \in e^{-1}(y)\) such that \(a \in p(b)\).

We require the following surjection for the forthcoming theorem.

**Notation 2.5.3.** Let \(f : Z^2 \to Z^2\) be a surjection given as follows.

For every \((x, y) \in Z^2\)

\[
f(x, y) = \begin{cases} 
(2k, 2l) & \text{if } (x, y) = (4k, 4l), \\
(2k, 2l + 1) & \text{if } (x, y) \in A_4(4k, 4l + 2), \\
(2k + 1, 2l) & \text{if } (x, y) \in A_4(4k + 2, 4l), \\
(2k + 1, 2l + 1) & \text{if } (x, y) \in A'_4(4k + 2, 4l + 2),
\end{cases}
\]

where \(k, l \in Z\).

**Theorem 2.5.4.** The Khalimsky topology \(t\) coincides with the quotient topology of \(w\) generated by \(f\).
PROOF. We can show that for any points $z_1, z_2 \in \mathbb{Z}^2$, $z_1 \in t(z_2)$ if and only if there are points $a \in f^{-1}(z_1)$ and $b \in f^{-1}(z_2)$ such that $a \in w(b)$.

This is true if $z_1 = z_2$. Therefore suppose that $z_1 \neq z_2$.

Let $z_1 \in t(z_2)$. Then $z_2$ is not a closed point in $(\mathbb{Z}^2, t)$, hence $z_2 = (x, y)$ where $x$ or $y$ is even. Thus we have the following three possibilities.

**Case 1.** $z_2 = (2k, 2l)$, for some $k, l \in \mathbb{Z}$ and $z_1 \in A_8(z_2) - \{z_2\}$. Then $f^{-1}(z_2) = (4k, 4l)$ and we get one of the following eight cases.

1. $z_1 = (2k+1, 2l)$ hence $f^{-1}(z_1) = A_4(4k+2, 4l)$, $(4k+1, 4l) \in f^{-1}(z_1)$ and we have $(4k+1, 4l) \in w\{4k, 4l\}$

2. $z_1 = (2k-1, 2l)$ hence $f^{-1}(z_1) = A_4(4k-2, 4l)$, $(4k-1, 4l) \in f^{-1}(z_1)$ and we have $(4k-1, 4l) \in w\{4k, 4l\}$

3. $z_1 = (2k, 2l+1)$ hence $f^{-1}(z_1) = A_4(4k, 4l+2)$, $(4k, 4l+1) \in f^{-1}(z_1)$ and we have $(4k, 4l+1) \in w\{(4k, 4l)\}$

4. $z_1 = (2k, 2l-1)$ hence $f^{-1}(z_1) = A_4(4k, 4l-2)$, $(4k, 4l-1) \in f^{-1}(z_1)$ and we have $(4k, 4l-1) \in w\{(4k, 4l)\}$

5. $z_1 = (2k+1, 2l+1)$ hence $f^{-1}(z_1) = A'_4(4k+2, 4l+2)$, $(4k+1, 4l+1) \in f^{-1}(z_1)$ and we have $(4k+1, 4l+1) \in w\{(4k, 4l)\}$

6. $z_1 = (2k-1, 2l-1)$ hence $f^{-1}(z_1) = A'_4(4k-2, 4l-2)$, $(4k-1, 4l-1) \in f^{-1}(z_1)$ and we have $(4k-1, 4l-1) \in w\{(4k, 4l)\}$

7. $z_1 = (2k-1, 2l+1)$ hence $f^{-1}(z_1) = A'_4(4k-2, 4l+2)$, $(4k-1, 4l+1) \in f^{-1}(z_1)$ and we have $(4k-1, 4l+1) \in w\{(4k, 4l)\}$
8. \( z_1 = (2k-1, 2l-1) \) hence \( f^{-1}(z_1) = A'_4(4k-2, 4l-2), (4k-1, 4l+1) \in f^{-1}(z_1) \) and we have \((4k-1, 4l-1) \in w\{(4k, 4l)\}\).

**Case 2.** \( z_2 = (2k, 2l+1) \), for some \( k, l \in Z \) and \( z_1 \in H_2(z_2) - \{z_2\} \). Then

\[
f^{-1}(z_2) = A_4(4k, 4l + 2), \{(4k + 1, 4l + 2), (4k - 1, 4l + 2)\} \subset f^{-1}(z_2)
\]

and we get one of the following two cases

1. \( z_1 = (2k+1, 2l+1) \) hence \( f^{-1}(z_1) = A'_4(4k+2, 4l+2), (4k+1, 4l+1) \in f^{-1}(z_1) \) and we have \((4k + 1, 4l + 1) \in w\{4k + 1, 4l + 2\}\)

2. \( z_1 = (2k-1, 2l+1) \) hence \( f^{-1}(z_1) = A'_4(4k-2, 4l+2), (4k-1, 4l+3) \in f^{-1}(z_1) \) and we have \((4k - 1, 4l + 3) \in w\{(4k - 1, 4l + 2)\}\)

**Case 3.** \( z_2 = (2k + 1, 2l) \), for some \( k, l \in Z \) and \( z_1 \in V_2(z_2) - \{z_2\} \). Then

\[
f^{-1}(z_2) = A_4(4k + 2, 4l), \{(4k + 2, 4l + 2), (4k + 2, 4l - 1)\} \subseteq f^{-1}(z_2)
\]

and we get one of the following two cases

1. \( z_1 = (2k + 1, 2l + 1) \) hence \( f^{-1}(z_1) = A'_4(4k + 2, 4l + 2), (4k + 1, 4l + 1) \in f^{-1}(z_1) \) and we have \((4k +1, 4l + 1) \in w\{4k + 2, 4l + 2\}\)

2. \( z_1 = (2k + 1, 2l - 1) \) hence \( f^{-1}(z_1) = A'_4(4k + 2, 4l - 2), (4k + 1, 4l - 1) \in f^{-1}(z_1) \) and we have \((4k + 1, 4l - 1) \in w\{(4k + 2, 4l - 1)\}\) we have shown that whenever \( z_1 \in t\{z_2\} \) there are points \( a \in f^{-1}(z_1) \) and \( b \in f^{-1}(z_2) \) such that \( a \in w(b)\).
Conversely suppose that there are points \( a \in f^{-1}(z_1) \) and \( b \in f^{-1}(z_2) \) such that \( a \in w(b) \). Then \( f^{-1}(z_1) \) is not open in \((Z^2, w)\). Therefore we have the following three possibilities.

**Case 1.** \( f^{-1}(z_1) = A_4(4k, 4l + 2) \) for some \( k, l \in Z \) hence \( z_1 = (2k, 2l + 1) \) and we get one of the following two cases

\[
(1) \quad z_2 = (2k, 2l + 2) \quad \text{because then} \quad f^{-1}(z_2) = \{(4k, 4l + 4)\},
\]

\[
a = (4k, 4l + 3) \in f^{-1}(z_1) \quad \text{and} \quad b = (4k, 4l + 4) \in f^{-1}(z_2)
\]

then we have \( z_1 \in t\{z_2\} \).

\[
(2) \quad z_2 = (2k, 2l) \quad \text{because then} \quad f^{-1}(z_2) = \{(4k, 4l)\}, \quad a = (4k, 4l + 1) \in f^{-1}(z_1) \quad \text{and} \quad b = (4k, 4l) \in f^{-1}(z_2).
\]

So \( z_1 \in t\{z_2\} \).

**Case 2.** \( f^{-1}(z_1) = A_4(4k+2, 4l) \) for some \( k, l \in Z \) hence \( z_1 = (2k+1, 2l) \) and we get one of the following two cases

\[
(1) \quad z_2 = (2k + 2, 2l) \quad \text{because then} \quad f^{-1}(z_2) = \{(4k + 4l, 4l)\},
\]

\[
a = (4k + 3, 4l) \in f^{-1}(z_1) \quad \text{and} \quad b = (4k + 4, 4l) \in f^{-1}(z_2) \quad \text{so}
\]

we have \( z_1 \in t\{z_2\} \).

\[
(2) \quad z_2 = (2k, 2l) \quad \text{because then} \quad f^{-1}(z_2) = \{(4k, 4l)\}, \quad a = (4k + 1, 4l) \in f^{-1}(z_1) \quad \text{and} \quad b = (4k, 4l) \in f^{-1}(z_2), \quad \text{then we have} \quad z_1 \in t\{z_2\}.
\]

**Case 3.** \( f^{-1}(z_1) = A'_4(4k + 2, 4l + 2) \) for some \( k, l \in Z \) hence \( z_1 = (2k + 1, 2l + 1) \) and we get one of the following four cases
(1) \( z_2 = (2k + 2, 2l + 2) \) because then \( f^{-1}(z_2) = \{(4k + 4, 4l + 4)\} \),
\[ a = (4k + 3, 4l + 3) \in f^{-1}(z_1) \text{ and } b = (4k + 4, 4l + 4) \in f^{-1}(z_2) \]
so we have \( z_1 \in t\{z_2\} \)

(2) \( z_2 = (2k, 2l + 2) \) because then \( f^{-1}(z_2) = \{(4k, 4l + 4)\} \),
\[ a = (4k + 1, 4l + 3) \in f^{-1}(z_1) \text{ and } b = (4k, 4l + 4) \in f^{-1}(z_2) \]
\[ z_1 \in t\{z_2\} \]

(3) \( z_2 = (2k, 2l) \) because then \( f^{-1}(z_2) = \{(4k, 4l)\} \),
\[ a = (4k + 3, 4l + 1) \in f^{-1}(z_1) \text{ and } b = (4k + 4, 4l) \in f^{-1}(z_2) \]
so we have \( z_1 \in t\{z_2\} \)

(4) \( z_2 = (2k, 2l) \) because then \( f^{-1}(z_2) = \{(4k, 4l)\} \), \( a = (4k+1, 4l+1) \in f^{-1}(z_1) \text{ and } b = (4k, 4l) \in f^{-1}(z_2) \), so we have \( z_1 \in t\{z_2\} \)

We have shown that \( a \in f^{-1}(z_1) \), \( b \in f^{-1}(z_2) \) and \( a \in w(b) \) imply \( z_1 \in t(z_2) \).

By lemma 2.5.2, \( t \) is the quotient topology of \( w \) generated by \( f \). \qed

We need the following surjection for the coming theorem

**Notation 2.5.5.** Let \( g : \mathbb{Z}^2 \to \mathbb{Z}^2 \) be the surjection as follows. For any \((x, y) \in \mathbb{Z}^2 \)

\[
g(x, y) = \begin{cases} 
k + l, l - k & \text{if } (x, y) \in A_8(4k, 4l), k, l \in \mathbb{Z} \\
(k + l + 1, l - k) & \text{if } (x, y) = (4k + 2, 4l + 2), 
\end{cases}
\]

for some \( k, l \in \mathbb{Z} \) with \( k + l \) odd
or \((x, y) \in A_{12}(4k + 2, 4l + 2)\) for some \(k, l \in \mathbb{Z}\) with \(k + l\) even, where 
\[
A_{12}(k, l) = \{(x, y) \in \mathbb{Z}^2, x = k \text{ and } |y - l| \leq 3 \text{ or } y = l \text{ and } |x - k| \leq 3\}.
\]

Thus \(A_{12}\) consists of the point \((k, l)\) and the 12 nearest points to \((k, l)\) each of which has one co-ordinate common with \((k, l)\).

**Theorem 2.5.6.** The Marcus topology \(u\) coincides with the quotient topology of \(w\) generated by \(g\).

**Proof.** We will show that for any points \(z_1, z_2 \in \mathbb{Z}^2, z_1 \in u(z_2)\) if and only if there are points \(a \in g^{-1}(z_1)\) and \(b \in g^{-1}(z_2)\) such that \(a \in w(b)\). This is true if \(z_1 = z_2\). Therefore suppose \(z_1 \neq z_2\).

Let \(z_1 \in t(z_2)\). Then \(z_2\) is an open set in \((\mathbb{Z}^2, u)\) hence \(z_2 = (x, y)\) where one of the numbers \(x, y\) is even and the other is odd, and \(z_1 \in \mathcal{A}_4(z_2) - \{z_2\}\). It follows that \(z_2 = (k + l + 1, l - k)\) and \(g^{-1}(z_2) = \{(4k + 2, 4l + 2)\}\) for some \(k, l \in \mathbb{Z}, k + l\) odd, if \(x\) is even while \(g^{-1}(z_2) = A_{12}(4k + 2, 4l + 2)\) for some \(k, l \in \mathbb{Z}, k + l\) even, if \(x\) is odd. We have \((4k + 2, 4l + 2) \in g^{-1}(z_2)\) (in both the cases when \(x\) is even or odd) and one of the following four cases occurs.

1. \(z_1 = (k + l + 2, l - k)\), hence \(g^{-1}(z_1) = A_8(4k + 4, 4l + 4), (4k + 3, 4l + 3) \in g^{-1}(z_1)\) and we have \((4k + 3, 4l + 3) \in w\{4k + 2, 4l + 2\}\)

2. \(z_1 = (k + l, l - k)\), hence \(g^{-1}(z_1) = A_8(4k, 4l), (4k + 1, 4l + 1) \in g^{-1}(z_1)\) and we have \((4k + 1, 4l + 1) \in w\{4k + 2, 4l + 2\}\)
(3) \( z_1 = (k + l + 1, l - k + 1) \), hence \( g^{-1}(z_1) = A_8(4k, 4l + 4), (4k - 1, 4l + 3) \in g^{-1}(z_1) \) and we have \((4k - 1, 4l + 3) \in w\{(4k + 2, 4l + 2)\}\)

(4) \( z_1 = (k + l + 1, l - k - 1) \), hence \( g^{-1}(z_1) = A_8(4k + 4, 4l), (4k + 3, 4l + 1) \in g^{-1}(z_1) \) and we have \((4k + 3, 4l + 1) \in w\{(4k + 2, 4l + 2)\}\)

We have shown that whenever \( z_1 \in u(z_2) \) there are points \( a \in g^{-1}(z_1) \) and \( b \in g^{-1}(z_2) \) such that \( a \in w(b) \).

Conversely suppose that there are points \( a \in g^{-1}(z_1) \) and \( b \in g^{-1}(z_2) \) such that \( a \in w(b) \). Then \( g^{-1}(z_1) \) is not open in \((Z^2, w)\). Therefore \( g^{-1}(z_1) = A_8(4k, 4l) \) which means that \( z_1 = (k + l, l - k) \) for some \( k, l \in Z \).

Further we have one of the following four cases,

1. \( z_2 = (k + l + 1, l - k) = ((k - 1) + (l - 1) + 1, (l - 1) - (k - 1)) \)
   because then \( g^{-1}(z_2) = \{(4(k - 1) + 2, 4(l - 1) + 2)\} = \{4k - 2, 4l - 2\} \)
   or \( g^{-1}(z_2) = A_{12}(4k - 2, 4l - 2) \), \( a = (4k - 1, 4l - 1) \in g^{-1}(z_1) \) and \( b = (4k - 2, 4l - 2) \in g^{-1}(z_2) \). Then we have \( z_1 \in u\{z_2\} \)

3. \( z_2 = (k + l, l - k + 1) = (1 + (k - 1) + l, l - (k - 1)) \) because then \( g^{-1}(z_2) = \{(4(k - 1) + 2, 4l + 2)\} = \{4k - 2, 4l + 2\} \) or \( g^{-1}(z_2) = A_{12}(4k - 2, 4l) \),
\[ a = (4k - 1, 4l + 1) \in g^{-1}(z_1) \text{ and } b = (4k - 2, 4l + 2) \in g^{-1}(z_2). \] Then also \( z_1 \in u\{z_2\}. \)

4. \( z_2 = (k+l, l-k-1) = (k+(l-1)+1, (l-1)-k) \) because then \( g^{-1}(z_2) = \{4k+2, 4(l-1)+2\} = \{4k+2, 4l-2\} \) or \( g^{-1}(z_2) = A_{12}(4k+2, 4l-2), \)
\[ a = (4k+1, 4l-1) \in g^{-1}(z_1) \text{ and } b = (4k+2, 4l-2) \in g^{-1}(z_2). \] Then also \( z_1 \in u\{z_2\}. \)

We have shown that \( a \in g^{-1}(z_1), b \in g^{-1}(z_2) \) and \( a \in w(b) \) imply \( z_1 \in u\{z_2\}. \) By lemma 2.5.2, \( u \) is the quotient topology of \( w \) generated by \( g. \)

**Result 2.5.7.** 1. Marcus topological space can be obtained from the Khalimsky topological space. It is homeomorphic to the subspace of the Khalimsky topological space given by the pure points and it is got by putting
\[ \phi(x, y) = (x - y + 1, x + y + 1) \text{ for all } (x, y) \in \mathbb{Z}^2. \]

2. Both the Marcus topological space \((\mathbb{Z}^2, u)\) and Khalimsky topological space \((\mathbb{Z}^2, v)\) the following two conditions are satisfied for any pair of different points \((z_1, z_2) \in \mathbb{Z}^2. \)

\( a \) If \((z_1, z_2)\) is connected, then \( z_1 \) and \( z_2 \) are 8-adjacent.

\( b \) If \( z_1 \) and \( z_2 \) are 4-adjacent, then \( \{z_1, z_2\} \) is connected.

\( c \) The topology \( w \) satisfies only the condition \( (a) \), makes it less convenient for applications in digital topology.