CHAPTER 6

Homotopy properties of digital simple closed curves and
digital fundamental groups†

6.1. Introduction

Digital homotopy theory has a goal to characterize properties of digital spaces analogous to that of classical homotopy theory. Knowledge of the digital fundamental group is important for Image Analysis, as the fundamental group of a digital image tells us something about the form of the image. Kong’s digital fundamental groups are defined only for dimensions 2 and 3 whereas Boxer’s digital fundamental groups are defined for digital images of all dimensions.

In this chapter we discuss some properties of digital simple closed curves, digital fundamental groups and homotopy equivalence.

Firstly we introduce the definition of a digital simple closed curve.

**Definition 6.1.1.** Let $X \subset Z^k$ have an adjacency relation $k$. Let $n$ be a positive integer. $X$ is a digital simple closed $n$-curve if there is an integer

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$m > 3$ and a $(2, k)$ continuous function $f : [0, m]_Z \to X$ such that

- $f$ is onto.
- $f(0) = f(m)$
- $0 \leq t_0 < t_1 \leq m$ and $t_1 - t_0 < m$ implies $f(t_0) \neq f(t_1)$
- for all $t \in [0, m]_Z$ the only $k$-neighbors of $f(t)$ in $f[0, m]_Z$ are $f((t - 1) \mod m)$ and $f((t + 1) \mod m)$

A digital simple closed 8-curve is digitally contractible. An example is given below.

**Example 6.1.2.** Let $X \subset Z^2$ be the set $X = \{(0, 0), (1, -1), (2, 0), (1, 1)\}$. Since $X$ used 8 adjacency, $X$ is a digital 8-curve.

Let $H : X \times [0, 2]_Z \to X$ be defined by

$$H(x, 0) = x \text{ for all } x \in X$$

$$H(0, 0, 1) = H(1, 1, 1) = (0, 0),$$

$$H(1, -1, 1) = H(2, 0, 1) = (1, -1)$$

$$H(x, 2) = (0, 0) \text{ for all } x \in X$$

is a pointed homotopy in $X$ between $\text{id}_X$ and the constant function with image $\{(0, 0)\}$. Thus $X$ is digitally contractible.
A classical theorem of Euclidean topology due to L. E. J. Brouwer, states that a $d$-dimensional sphere $s^d$ is not contractible. We present a digital analog for $d = 1$, and we present some related results also.

**Remark 6.1.3.** Let $X \subset Z^2$ be a digital simple closed 4-curve. Then $X$ is not digitally contractible.

**Proposition 6.1.4.** Let $S_a$ be a digital simple closed $K_a$ curve $a \in \{0, 1\}$. Let $f : S_0 \to S_1$ be a $(k_0, k_1)$ continuous function. If $|S_0| = |S_1|$, then the following are equivalent.

1. $f$ is one-to-one
2. $f$ is onto
3. $f$ is a $(k_0, k_1)$ isomorphism

**Proof.** Since $|S_0| = |S_1|$, the equivalence of (1) and (2) follows from the fact that $S_0$ is a finite set. (3) implies both (1) and (2) follows from the definition of isomorphism. So we can complete the proof by showing that (2) implies (3).

Let $S_a = \{x_{a,i}\}_{i=0}^{n-1}$, where the points of $S_a$ are circularly ordered, $a \in \{0, 1\}$. Let $x_{1,u} \in S_1$ and let $x_{0,v} = f^{-1}(x_{1,u})$. Then the $k_1$ neighbors of $x_{1,u}$ in $S_1$ are $x_{1,(u+1)} \mod n$ and $x_{1,(u-1)} \mod n$ and the $k_0$ neighbors of $x_{0,v}$ in $S_0$ are $x_{0,(v+1)} \mod n$ and $x_{0,(v-1)} \mod n$. Since $f$ is a continuous bijection $f(\{x_{0,(v-1)} \mod n, x_{0,(v+1)} \mod n\}) = \{x_{1,(u-1)} \mod n, x_{1,(u+1)} \mod n\}$. 

83
Thus
\[ f^{-1}(\{x_{1,(u-1)} \mod n, x_{1,(u+1)} \mod n\}) = \{x_{0,(v-1)} \mod n, x_{0,(v+1)} \mod n\} \]

Since \( u \) is an arbitrary index, \( f^{-1} \) is \((k_1, k_0)\) continuous, so \( f \) is a \((k_0, k_1)\) isomorphism. \( \square \)

**Theorem 6.1.5.** Let \( S \) be a simple closed \( k \) curve and let \( H : S \times [0, m] \to S \) be a \((k, k)\) homotopy between an isomorphism \( H_0 \) and \( H_m = f \), where \( f(S) \neq S \). Then \( |S| = 4 \).

**Proof.** Let \( S = \{x_i\}_{i=0}^{n-1} \) where the points of \( S \) are circularly ordered.

There exists \( w \in [1, m] \) such that
\[ w = \min\{t \in [0, m] | H_t(S) \neq S\} \]

Without loss of generality, assume that \( x_1 \not\in H_\omega(S) \). Then the induced function \( H_{\omega-1} \) is a bijection, so there exists \( x_u \in S \) such that \( H(x_u, \omega - 1) = x_1 \). By Proposition 6.1.4
\[ H_{\omega-1}(\{x_{(u-1)} \mod n, x_{(u+1)} \mod n\}) = \{x_0, x_2\} \]

and the continuity property of homotopy implies \( H(x_u, \omega) \in \{x_0, x_2\} \). Without loss of generality
\[ H(x_{(u-1)} \mod n, \omega - 1) = x_0 \quad (1) \]
and \( H(x_u, \omega) = x_2 \) \( (2) \)

Suppose \( n > 4 \). Equation (2) implies \( H(x_{(u-1) \mod n}, \omega) \in \{x_1, x_2, x_3\} \) but this is impossible, because

- \( H(x_{(u-1) \mod n}, \omega) \neq x_1 \) by the choice of \( x_1 \)
- \( H(x_{(u-1) \mod n}, \omega) \notin \{x_2, x_3\} \) from equation (1), because \( n > 4 \)

implies neither \( x_2 \) nor \( x_3 \) is \( k \)-adjacent to \( x_0 \).

The contradiction arose from the assumption that \( n > 4 \). Therefore, we must have \( n \leq 4 \). Since a digital simple closed curve is assumed to have at least 4 points, we must have \( n = 4 \). \( \square \)

**Theorem 6.1.6.** Let \( (S, k) \) be a simple closed \( k \)-curve such that \( |S| > 4 \).

Then \( S \) is not \( k \)-contractible.

**Proof.** By Theorem 6.1.5 it follows that if \( |S| > 4 \), then there cannot be a \( (k, k) \) homotopy between \( 1_S \) and a constant map in \( S \). \( \square \)

### 6.2. Digital fundamental group

In this section, we discuss digital fundamental group derived from a classical notion of algebraic topology.

**Definition 6.2.1.** For a pointed image \((X, x_0)\), a \( k \)-loop based at \( x_0 \) is a \((2, k)\) continuous function \( f : [0, m)_Z \rightarrow X \) with \( f(0) = x_0 = f(m) \). The
number \( m \) depends on the loop. Let

\[
F_1^k(X, x_0) = \{ f | f \text{ is a } k\text{-loop based at } x_0 \}
\]

For members \( f : [0, m_1]_Z \to X, g : [0, m_2]_Z \to X \) of \( F_1^k(X, x_0) \), we get a map \( f * g : [0, m_1 + m_2]_Z \to X \) defined by

\[
f * g(t) = \begin{cases} 
  f(t) & \text{if } 0 \leq t \leq m_1 \\
  g(t - m_1) & \text{if } m_1 \leq t \leq m_1 + m_2
\end{cases}
\]

The \( k \)-homotopy class of a pointed loop \( f \) is denoted by \([f]\). We have \( g \in [f] \) if and only if there is a homotopy, holding the end point fixed, between trivial extension \( F, G \) of \( f, g \) respectively, where a trivial extension \( F \) of \( f \) is a map that follows the same path as \( f \) with pauses for rest. The * operation preserves homotopy classes in the sense that if \( f_1, f_2, g_1, g_2 \in F_1^k(X, p) \), \( f_1 \in [f_2] \) and \( g_1 \in [g_2] \) then \( f_1 * g_1 \in [f_2 * g_2] \)

i.e., \([f_1 * g_1] = [f_2 * g_2]\).

Then we can see that

\[
\pi_1^k(X, x_0) = \{ [f] | f \in F_1^k(X, x_0) \}
\]
is a group with the operation. \([f] \cdot [g] = [f * g]\) known as the \(k\)-fundamental group of \((X, x_0)\).

**Lemma 6.2.2.** The operation ‘\(\cdot\)’ is associative on \(\pi^k_1(X, p)\).

**Proof.** Let \(f_i : [0, m_i] \rightarrow X\) be a digital path \(i \in \{1, 2, 3\}\) such that \(f_1(m_1), f_2(0)\) and \(f_2(m_2) = f_3(0)\). Then each of \((f_1 \circ f_2) \circ f_3\) and \(f_1 \circ (f_2 \circ f_3)\) is identically equal to the path defined \(F : [0, m_1 + m_2 + m_3] \rightarrow X\) defined by

\[
F(t) = \begin{cases} 
  f_1(t) & \text{if } 0 \leq t \leq m_1 \\
  f_2(t - m_1) & \text{if } m_1 \leq t \leq m_1 + m_2 \\
  f_3(t - m_1 - m_2) & \text{if } m_1 + m_2 \leq t \leq m_1 + m_2 + m_3 
\end{cases}
\]

\[\Box\]

**Lemma 6.2.3.** Let \((X, p)\) be a pointed digital image. Let \(\bar{p} : [0, m] \rightarrow X\) be a constant function with image \(\{p\}\). Then \([\bar{p}]_X\) is an identity element for \(\pi^k_1(X, p)\).

**Proof.** Let \([f]_X \in \pi^k_1(X, p)\) have representative \(f : [0, m'] \rightarrow X\). Both \(f \cdot \bar{p}\) and \(\bar{p} \cdot f\) are trivial extensions of \(f\), hence members of \([f]_X\). (This is because each of \(f \cdot \bar{p}\) and \(\bar{p} \cdot f\) follows the path of \(f\) with an extra pause.}

87
for rest during the portion of the loop represented by \( \bar{p} \). Hence \([\bar{p}]_X\) is an identity element for \( \pi^k_1(X, p) \).

Lemma 6.2.4. If \( f : [0, m]_Z \to X \) represents an element of \( \pi^k_1(X, p) \), then the function \( g : [0, m]_Z \to X \) defined by \( g(t) = f(m - t) \) for \( t \in [0, m]_Z \) is an element of \([f]^{-1}_X\) in \( \pi^k_1(X, p) \).

PROOF. Since \( g(t) = f(m - t) \), \( g \) is digitally continuous and has the same base point \( p \) as \( f \). By Lemma 6.2.3 it is sufficient to show that each of \( f \cdot g \) and \( g \cdot f \) is in the same loop class as the constant function \( \bar{p} : [0, 2m]_Z \to (X, p) \) whose image is \( \{p\} \).

\( (f \cdot g) : [0, 2m]_Z \to X \) is a loop defined by

\[
(f \cdot g)(t) = \begin{cases} 
  f(t) & \text{if } 0 \leq t \leq m \\
  f(2m - t) & \text{if } m \leq t \leq 2m 
\end{cases}
\]

Let \( H : [0, 2m]_Z \times [0, m]_Z \to X \) be defined by

\[
H(t_1, t_2) = \begin{cases} 
  (f \cdot g)(t_1) & \text{if } 0 \leq t_1 \leq m - t_2 \text{ or } m + t_2 \leq t_1 \leq 2m \\
  (f \cdot g)(m - t_2) & \text{otherwise.}
\end{cases}
\]

For all \( t_2 \in [0, m]_Z \), we have \( H(0, t_2) = p \) and \( H(2m, t_2) = p \). Thus \( H \) is a pointed digital homotopy between \( H_0 \), which is identical to \( f \cdot g \) and
\( H_m, \) which is identical to \( \bar{p}. \) Thus \( f \cdot g \in [\bar{p}]_X. \) Similarly we can prove that \( g \cdot f \in [\bar{p}]_X. \) Hence the proof. \( \square \)

**Theorem 6.2.5.** \( \pi_f^k(X, p) \) is a group under the ‘·’ product operation, the fundamental group of \( (X, p). \)

**Proof.** This follows from Lemmas 6.2.2, 6.2.3 and 6.2.4. \( \square \)

From the next result it follows that in a connected digital image \( X, \) the choice of basepoint is immaterial in determining the digital fundamental group.

**Theorem 6.2.6.** Let \( X \) be a digital image with adjacency relation \( k \) and let \( p \) and \( q \) members of the same \( k \)-component of \( X. \) Then with respect to \( k, \) \( \pi_1(X, p) \) and \( \pi_1(X, q) \) are isomorphic groups.

**Proof.** Let \( f : [0, m]_Z \to X \) be a digital \( k \)-path such that \( f(0) = p, f(m) = q. \) Let \( g : [0, m]_Z \to X \) be the \( k \)-path from \( q \) to \( p \) defined by

\[
g(t) = f(m - t) \text{ for all } t \in [0, m]_Z.
\]

We know that \( f \cdot g \in [\bar{p}]_{(X,p)} \) and \( g \cdot f \in [\bar{q}]_{(X,q)} \) (by Lemma 6.2.4).

Let \( F : \pi_1(X, p) \to \pi_1(X, q) \) be defined by

\[
F([h]_{(X,p)}) = [g \cdot h \cdot f]_{(X,q)}.
\]
We can show that $F$ is a group isomorphism.

For every $[k]_X \in \pi_1(X, q)$ represented by the $q$-based loop $k : [0, m_0]_Z \rightarrow X$, consider the $p$-based loop $h = f \cdot k \cdot g : [0, 2m + m_0]_Z \rightarrow X$. We claim

$$F([h]_{(X,p)}) = [k]_{(X,q)}.$$ 

To prove this, we proceed as follows.

For $v \in [0, m]_Z$, define functions $f_v, g_v : [0, m]_Z \rightarrow X$ by

$$f_v(s) = f(\max\{s, v\})$$

$$g_v(s) = g(\min\{s, m - v\}).$$

We have identically

$$f_0 = f, \quad f_m = \bar{q} = g_m, \quad g_0 = g.$$

Also for all $v$,

$$g_v(0) = f(m) = f_v(m) = q \quad \text{and} \quad g_v(m) = f(v) = f_v(0).$$

So the path $g_v \cdot f_v$ is defined.
Consider the function $H : [0, 4m+m_0]_Z \times [0, m]_Z \rightarrow X$ defined by

$$H(s, v) = (g_v \cdot f_v \cdot k \cdot g_v \cdot f_v)(s)$$

$$H(s, 0) = (g \cdot f \cdot k \cdot g \cdot f)(s) = (g \cdot h \cdot f)(s).$$

$H(s, m)$ represents $[k]$ in that

$$H(s, m) = (\bar{q} \cdot \bar{q} \cdot k \cdot \bar{q} \cdot \bar{q})(s)$$

so that $H(s, m)$ is a trivial extension of $k$, and for all $v \in [0, m]_Z$

$$H(0, v) = q = H(4m + m_0, v).$$

Thus $F$ is onto.

Similarly we can show that if $[k]_X \in \pi_1(X, p)$ is such that $F([k]) = [\bar{q}]_X$ then $[\bar{p}] = [k]$. This is done by showing that the trivial extension $\bar{p} \cdot \bar{p} \cdot k \cdot \bar{p} \cdot \bar{p}$ of $k$ is pointed homotopic to

$$f \cdot g \cdot k \cdot f \cdot g \in [f \cdot F([k]) \cdot g] = [f \cdot \bar{q} \cdot g],$$

then observing that $f \cdot \bar{q} \cdot g$ is a trivial extension of $f \cdot g$ which in turn is pointed homotopic to $\bar{p}$. Thus $F$ is one-to-one. Finally, to show that $F$ is a homomorphism, we note that $F([k_0] \cdot [k_1]) = [g \cdot k_0 \cdot k_1 \cdot f]$ has $g \cdot k_0 \cdot \bar{p} \cdot k_1 \cdot f$
as a trivial extension. The latter as above is pointed homotopic to

\[ g \cdot k_0 \cdot f \cdot g \cdot k_1 \cdot f \in [g \cdot k_0 \cdot f] \cdot [g \cdot k_1 \cdot f] = F([k_0]) \cdot F([k_1]) \]

Thus \( F \) is a group isomorphism. \( \square \)

### 6.3. Limitation of homotopy equivalence

In Euclidean topology, all simple closed curves are homeomorphic. But an analogous statement is false for digital simple closed curves, as a pair of digital simple closed curves need not have the same cardinality. This observation implies that a pair of digital simple closed curves need not have the same digital homotopy type.

**Definition 6.3.1.** A map \( f : X \to Y \) is called a homotopy equivalence if there is a map \( g : Y \to X \) such that \( f \circ g \simeq I \) and \( g \circ f \simeq I \). The spaces \( X \) and \( Y \) are said to be homotopy equivalent or to have the same homotopy type.

**Theorem 6.3.2.** Let \( X \subset Z^{a_0} \) and \( Y \subset Z^{a_1} \) be respectively, \( k \) and \( \lambda \) simple closed curves that are not contractible such that \( |X| \neq |Y| \). Then \( X \) and \( Y \) do not have the same \((k, \lambda)\) homotopy type.

**Proof.** Without loss of generality assume that \( |X| < |Y| \). Let \( f : X \to Y \) be a \((k, \lambda)\) continuous function. Since \( |X| < |Y| \), it follows that \( f(X) \) is a proper subset of \( Y \). Therefore \( f \) is a \( \lambda \)-contractible map in \( Y \).
It follows that if $g : Y \to X$ is any $(\lambda, k)$-continuous map, then $g \cdot f$ is a $k$-contractible map in $X$. Since $1_X$ is assumed not to be a $k$-contractible map, $1_X$ and $g \cdot f$ are not $k$-homotopic in $X$. Since $f$ is arbitrary, it follows that $X$ and $Y$ not $(k, \lambda)$ homotopic. □