CHAPTER I
INTRODUCTION
Chapter 1

Introduction

Linear and nonlinear partial differential equations (PDEs) of reaction-diffusion type arise in many applications which include physical sciences, developmental biology, ecology, physiology, finance, to name a few. Many problems in biology and chemistry modeled by partial differential equations (PDEs) have been extensively studied in the literature and their numerical solution can be accurately computed. The necessary coefficients, reaction excitations, initial and boundary data are provided in a deterministic way.

Reaction-Diffusion systems are usually coupled systems (multiple number) of parabolic partial differential equations. In applications to population biology, the reaction term models growth, and the diffusion term accounts for migration. Some of them are models for pattern formation in morphogenesis, for predator-prey and other ecological systems, for conduction in nerves, for epidemics, for carbon monoxide poisoning, and for oscillating chemical reactions.

Reaction-diffusion equations are used to simulate a variety of different phenomena, ranging from physics and engineering to mathematical biology. Haar
Wavelet-based schemes are used to solve partial differential equations characterized by widely varying length scales, and therefore hardly accessible by other numerical methods. The standard way to solve partial differential equations is to express the solution as a linear combination of so-called basis functions. These basis functions can for instance be plane waves, Gaussians or finite elements. Discretization of differential equations in this way makes it amenable to a numerical solution.

Wavelet transform and wavelet bases were originally conceived as a powerful tool for signal and image processing. More recently, wavelet analysis has been applied to the numerical solution of partial differential equations arising in various areas of engineering and physics. In particular, using wavelets based methods one can realize an effective multiscale analysis of functions and operators. The use of Haar wavelets in the engineering produces exciting results. The characteristics of wavelet transforms make them particularly useful for the approximation of functions with steep gradients or sharp spikes. Certainly the orthogonal and orthonormal properties of wavelet basis allow us to simplify the calculation of integrals. Haar wavelet in estimating depth profile of soil temperature has been obtained and the solutions are compared with the exact solutions. An accurate and efficient Haar transform or Haar wavelet method for some of the well-known Reaction-Diffusion (RD) problems has been developed. The equations include the Nowell-whitehead equation, Cahn-Allen equation, FitzHugh-Nagumo equation, Fisher’s equation, nonlinear-coupled reaction-diffusion equations, convection-diffusion equations, Burger’s equation, generalized Burger’s equation, one-dimensional reaction-diffusion equation and the Burgers-Fisher equation. The proposed scheme can be used to a wider class of nonlinear
equations. The power of this manageable method is confirmed. Also a comparative study of Haar wavelet method and a restrictive Taylor’s series method for solving Convection-Diffusion equation is established.

1.1 Partial Differential Equations (PDEs)

A partial differential equation (PDE) has the general form

\[ f(x, y, ..., u, u_x, u_y, ..., u_{xx}, u_{yy}, ...) = 0 \]  

involving several independent variables \( x, y, ..., \) an unknown function \( u \) of these variables, and the partial derivatives \( u_x, u_y, ..., u_{xx}, u_{yy}, ... \) of this function. Equation (1.1) is considered in a suitable domain \( D \) of the \( n \)-dimensional space \( \mathbb{R}^n \) in the independent variables \( x, y, ..., \). We seek one or more function of the form \( u = u(x, y) \) which satisfy (1.1). From these many possible solutions, we attempt to select a particular one by introducing suitable additional conditions, called initial and / or boundary conditions.

A partial differential equation is said to be linear if it is linear in the unknown function and all its derivatives with coefficients depending only on the independent variables; it is said to be quasilinear if it is linear in the highest-order derivative of the unknown function. The general quasi-linear second-order partial differential equation in one dependent variable \( u \), and two independent variables \( x, y \) may be written as

\[ Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \]  

where the coefficients are functions of \( x \) and \( y \); and \( A, B \) and \( C \) do not vanish.
simultaneously. We shall assume that the function \( u \) and the coefficients are twice continuously differentiable in an open set in \( \mathbb{R}^2 \). A PDE is said to be hyperbolic, parabolic, or elliptic at a point \((x_0, y_0)\) accordingly as

\[
B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0)
\] (1.3)

is positive, zero or negative respectively. If this is true at all points in some domain, then the equation is said to be hyperbolic, parabolic, or elliptic in that domain.

### 1.2 Reaction-Diffusion Equations (RDEs)

Reaction-Diffusion systems are usually coupled systems (multiple number) of parabolic partial differential equations which include pattern formation in morphogenesis, for predator-prey and other ecological systems, for conduction in nerves, for epidemics, for carbon monoxide poisoning, oscillating chemical reactions, pulse splitting and shedding, reactions and competitions in excitable systems and stability issues. RDEs in their simplest form are written as

\[
u_t = \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u)
\] (1.4)

where \( u = u(x, t) \) is the vector of dependent variables, \( f(u) \) is a non-linear vector-valued function of \( u \) (the reaction term), and \( D \) is the diffusion coefficient. The reaction term arise from any interaction between the components of \( u \). For example, \( u \) may be a vector of predator-prey interactions, competition or symbiosis. The diffusion terms may represent molecular diffusion or some “random” movement of individuals in a population. The reaction-diffusion system may be
extended to reaction-diffusion-convection type given by

\[ u_t = \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u) + C \frac{\partial u}{\partial x}, \quad (1.5) \]

where \( C \) is the convection coefficient.

It is known that for reaction-diffusion systems (1.4) [involving diffusion terms \( \frac{\partial^2 u}{\partial x^2} \)], the numerical treatment of the reaction terms is influential on the numerical results. In past years the study of reaction-diffusion equations has received a lot of attention due to its widespread areas of applications and the richness of their solution sets.

### 1.2.1 Derivation of Reaction-Diffusion Equations

Diffusion mechanism models the movement of many individuals in an environment or media. The individuals can be very small such as basic particles in physics, bacteria, molecules, or cells, or very large objects such as animals, plants or certain kind of events like epidemics, or rumors. The particles reside in a region, which we call \( \Omega \) is open set of \( \mathbb{R}^n \) (the \( n^{th} \) dimensional space with Cartesian coordinate system) with \( n \geq 1 \).

The main mathematical variable we consider here is the density function of the particles: \( P(t, x) \), where \( t \) is the time, and \( x \in \Omega \) is the location. The dimension of the population density usually is the number of particles or organisms per unit area (if \( n=2 \)) or unit volume (if \( n=3 \)). For example, the human population density is expressed in number of people per square kilometer.

Technically, we define the population density function \( P(t, x) \) as follows: let \( x \) be a point in the habitat \( \Omega \), and let \( \{O_n\} \) be a sequence of spatial regions surrounding \( x \); here \( O_n \) is chosen in a way that the spatial measurement \( |O_n| \) of \( O_n \) (length,
area, volume, or mathematically, the Lesbegue measure) tends to zero as \( n \to \infty \), and \( O_n \supset O_{n+1} \); then

\[
P(t, x) = \lim_{n \to \infty} \frac{N}{|O_n|}
\]  

(1.6)

if the limit exists. Here \( N \) denotes Number of organisms in \( O_n \) at time \( t \). The movement of \( P(t, x) \) is called the flux of the population density, which is a vector. By the principle of Fick’s law,

\[
J(t, x) = -d(x)\nabla_x P(t, x),
\]  

(1.7)

where \( J \) is the flux of \( P \), \( d(x) \) is called diffusion coefficient at \( x \), and \( \nabla_x \) is the gradient operator \( \nabla_x f(x) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}) \).

On the other hand, the number of particles at any point may change because of other reasons like birth, death, hunting, or chemical reactions. We assume that the rate of change of the density function due to these reasons is \( f(t, x, P) \), which we usually call the reaction rate. Now we derive a differential equation using the balanced law. We choose any region \( O \), then the total population in \( O \) is \( \int P(t, x)dx \), and the rate of change of the total population is

\[
\frac{d}{dt} \int_O P(t, x)dx.
\]  

(1.8)

The net growth of the population inside the region \( O \) is

\[
\int_O f(t, x, P(t, x))dx,
\]  

(1.9)

and the total out flux is

\[
\int_{\partial O} J(t, x)n(x)dS,
\]  

(1.10)
where $\partial O$ is the boundary of $O$, and $n(x)$ is the outer normal direction at $x$. Then the balance law implies

$$\frac{d}{dt} \int_{O} P(t, x) dx = -\int_{\partial O} J(t, x). n(x) dS + \int_{O} f(t, x, P(t, x)) dx \quad (1.11)$$

From the divergence theorem in multi-variable calculus, we have

$$\int_{\partial O} J(t, x). n(x) dS = \int_{O} div(J(t, x)) dx. \quad (1.12)$$

Combining the above equations, and interchanging the order of differentiation and integration, we obtain

$$\int_{O} \frac{\partial P(t, x)}{\partial t} dx = \int_{O} [div(d(x) \nabla_x P(t, x)) + f(t, x, P(t, x))] dx. \quad (1.13)$$

Since the choice of the origin $O$ is arbitrary, then the differential equation

$$\frac{\partial P(t, x)}{\partial t} = [div(d(x) \nabla_x P(t, x)) + f(t, x, P(t, x))] \quad (1.14)$$

holds for any $(t, x)$. The above equation is called a reaction-diffusion equation. Here $div(d(x) \nabla_x P(t, x))$ is the diffusion term, which describes the movement of the individuals, and $f(t, x, P(t, x))$ is the reaction term, which describes the birth-death or reaction occurring inside the habitat or reactor. The diffusion coefficient $d(x)$ is not a constant in general since the environment is usually heterogeneous. But when the region is approximately homogeneous, we can assume that $d(x) = d$ and the above equation can be simplified to

$$\frac{\partial P}{\partial t} = d \Delta P + f(t, x, P), \quad (1.15)$$
where \( \Delta P = \text{div}\nabla P = \sum \frac{\partial^2 P}{\partial x_i^2} \) is the Laplacian operator. When no reaction occurs, this equation is diffusion equation:

\[
\frac{\partial P}{\partial t} = d\Delta P, \tag{1.16}
\]

In classical mathematical physics, the equation \( T_t = \Delta T \) is called heat equation, where \( T \) is the temperature function. So sometimes (1.16) is also called a nonlinear heat equation. The conduction of heat can be considered a form of heat.

### 1.2.2 Importance of Reaction-Diffusion problems

**(i) Civil Engineering**

The aggregate alkali reaction in fluid leaching processes is of special interest in analysis of concrete dams in civil engineering and the numerical investigations carried out here are directed towards a better understanding of the model. The particular interest are effects of surface imperfection on the subsequent coupled moisture transport and the reactive formation of gel in the granular porous medium. The numerical experiments also consider the influence of periodic and nonperiodic inhomogeneities in material diffusivity properties. Concrete is a complex material containing an aggregate, a cement matrix, residual active silica, alkali within the cement, and water which acts both as an active ingredient and as a medium for the reaction. The reaction essentially converts active silica into a gel which causes swelling that significantly weakens the concrete. Significant modeling and experimental difficulties arise in that there are at least two competing reactions in concrete (concrete hydration) essentially occurring in an aqueous fractured medium of variable temperature, whose properties change markedly as
the reaction progresses; these properties affect contact between reactive compo-
nents and largely determine the extent and time evolution of the reaction.

(ii) Chemical Engineering

Reaction-Diffusion equations arise in many chemical and biological settings. Solutions to these equations exhibit a wide variety of structures, including pat-
tern formation and traveling waves. In ground water aquifers, reaction-diffusion
equations govern kinetic absorption and the growth and the transport of bio-film
forming microbes, and the equations may contain advective terms.

In hydrology, equations of this characteristics model of transport and the fate of
absorbing contaminants and microbe-nutrient systems in ground water.

(iii) Mechanical Engineering

A simplified kinematical description of a rigidly rotating spiral induced in
a generic two-component reaction-diffusion medium is elaborated by application
of a free-boundary approach. It is shown that all characteristics of a rigidly ro-
tating spiral (including its rotation period) are determined by the value of the
slow component near the spiral front. Fundamental properties of the reaction-
diffusion model of a laminar flame have stimulated a good deal of interest among
mathematical physicists. The potential energy generated by an external force as
a result of a deformation is propagated among mass points by the principle of
reaction-diffusion. The novelty of the methodology is that the reaction-diffusion
techniques are established to describe the potential energy of deformation and
to extrapolate internal forces of a deformed object. Reaction-Diffusion model is
developed for the natural propagation of the energy generated by the external
force. A new method is evolved to derive the internal forces. An improved
reaction-diffusion model is developed to propagate the energy in a natural manner. A material flux based method is also presented to derive the internal forces from the potential energy distribution. Reaction-Diffusion concept was first presented to describe the growth and form in embryology. It describes the non-linear spatiotemporal structures propagating through a medium and has been widely used to describe many natural structures, forms, patterns and behaviours, especially in biology. The reaction-diffusion system exhibits the electrical behaviour of real biological tissues such as cardiac muscle and brain tissues, and has been used to represent the structure of tissues and the membrane dynamics.

(iv) Electrical Engineering

Knowledge of the characteristics of electric and magnetic fields produced by lightning discharges is needed for studying the effects of the potentially deleterious coupling of lightning fields to various circuits and systems. Sensitive electronic circuits are particularly vulnerable to such effects. The computation of lightning, electric and magnetic fields requires the use of a model that specifies current as a function of time at all points along the radiating lightning channel. The computed fields can be used as an input to electromagnetic coupling models, the latter, in turn, being used for the calculation of lightning induced voltages and currents in various circuits and systems. It is now generally accepted that a typical lightning stroke begins with the propagation of a negatively charged channel, called a stepped leader, from cloud to the ground. But before this downward leader reaches the ground, an upward leader begins to proceed from the ground and meets the downward-moving leader at the junction point. Once a stepped leader has established a connection to earth, the so-called return stoke moves swiftly up the ionized channel prepared by the stepped leader like a traveling wave on a
high-voltage transmission line and a heavy current occurs. However, the physical models derived from the experimental data or from the information determined directly from experimental data have often been obtained more on the basis of intuition than on the basis of detailed quantitative analysis.

(v) Biological Engineering

Transports of molecular oxygen from the blood plasma to the living tissue of the skeletal muscle or brain across the capillary walls are nowadays a very important topic. Answers to several questions such as (i) what factors affect the supply of oxygen tissue cell respirations? (ii) what happens when we inhale oxygen at low concentration (iii) what is the influence of axial and radial diffusion of oxygen in blood, oxygen diffusivity in tissue etc, can be given by modeling the system through RDEs. Recent research indicates that the classical diffusion equation is inadequate to model many real situations, where a particle plume spreads faster than the classical model predates and may exhibit significant asymmetry. This situation is called anomalous diffusion.

A framework for modeling gliomas growth and their mechanical impact on the surrounding brain tissue (the so-called, mass-effect) has an Eulerian continuum approach that results in a strongly coupled system of nonlinear Partial Differential Equations (PDEs): a reaction-diffusion model for the tumor growth and a piecewise linearly elastic material for the background tissue.

Fisher’s assumptions for a sexually reproducing species lead to a Huxley reaction-diffusion equation, with cubic logistic source term for the gene frequency of a mutant advantageous recessive gene. Fisher’s equation more accurately represents the spread of an advantaged mutant strain within an asexual species.


1.2.3 Reaction-Diffusion Modelling

Reaction-Diffusion models provide a good framework for studying questions about the ways that habitat geometry and the size or variations in vital parameters influence population dynamics. This also provide a way to translate local assumptions or data about the movement, mortality, and reproduction of individuals into global conclusions about the persistence or extinction of populations and the coexistence of interacting species. They can be derived mechanistically via rescaling from models of individual movement, which are based on random walks. Reaction-diffusion models are spatially explicit and typically incorporate quantities such as dispersal rates, local growth rates, and carrying capacities as parameters, which may vary with location or time.

The theoretical advances in nonlinear analysis and the theory of dynamical systems which have occurred in the last thirty years make it possible to give a reasonably complete analysis of many reaction-diffusion models. Those advances include developments in bifurcation theory, the formulation of reaction-diffusion models as dynamical systems (Henry, 1981), the creation of mathematical theories of persistence or permanence in dynamical systems, and the systematic incorporation of ideas based on monotonicity into the theory of dynamical systems.

The reaction-diffusion models that are the subject of this thesis are partial differential equations, which describe how population densities in space change over time. Since they describe the way that things change over time, it is natural to think of them as dynamical systems; however, as noted, the state space for a reaction-diffusion model will be a set of functions representing the possible spatial densities of a spatially distributed population. Thus, to formulate reaction-diffusion models as dynamical systems we need to define appropriate
state spaces of functions and determine how the models act on them. In general we will not be able to solve reaction-diffusion models explicitly, but that is also the case with many nonlinear systems of ordinary differential equations. What we can do in many cases is to determine when a model predicts persistence and when it predicts extinction, and perhaps describe some features of its dynamics, by using methods from the theory of dynamical systems. However, there are some new technical issues that arise in formulating reaction-diffusion models as dynamical systems. Many of those are related to the fact that the state spaces for reaction-diffusion models are infinite dimensional. Others have to do with problems such as verifying that the set of non-negative densities is invariant. (Since negative population densities don’t make sense, good models should predict that densities, which are initially nonnegative, remain so.)

1.2.4 Traditional methods of solving RDE characterizing Physical and Engineering Phenomena

Analytical methods include method of separation of variables, Method of characteristics, Superposition principle, Adomain Decomposition method and transform methods such as Laplace, Fourier, Wavelet, Hankel, Cole-Hopf etc.

Matrix methods include differential transfer matrix method (DTMM), Relaxation method, Restrictive Taylor’s series method, Conjugate Gradient method, and Kryslow’s method.

Numerical methods include the finite difference method, the method of lines, the finite element method, the finite volume method, the spectral method, mesh free methods, domain decomposition methods, perturbation methods and multi-grid methods.
1.3 Mathematical Techniques to solve the Reaction-Diffusion (RD) Problems

(A) Method of finite differences (FDM)

In the absence of exact solution or analytical methods for the problem, numerical methods help us to provide approximate solutions. Modern computers pave the way for the development of efficient and more general numerical techniques, which may permit solutions for the most difficult problems of heat transfer. From the numerical methods available for solving nonlinear partial differential equations, finite differences are more frequently used and more universally applicable than any other method. Moreover, FDM provide numerical solutions on a simple and efficient manner.

In the methods of finite differences, the region of integration of the governing equations are divided into a system of rectangular meshes formed by two sets of lines, parallel to the coordinate axes. The numerical values of the dependent variables are obtained at the intersecting points, which are called mesh points or nodal points. The philosophy of the finite difference methods is to replace the partial derivatives appearing in the governing equations with algebraic difference quotients, yielding a system of algebraic equations, which can be solved for the flow-field variables at the specific discrete grid points in the flow. Accuracy can be improved by increasing the number of grid points.

Different approximations to derivatives lead to different finite difference methods. Explicit and implicit are the two different techniques in finite difference methods for solving any nonlinear partial differential equations. A formula, which expresses an unknown nodal value directly in terms of known nodal values is called an explicit formula.
This method is very simple to set up and program, but is conditionally stable. A method in which the calculation of the of an unknown value necessitates the solution of a set of simultaneous equations is called an implicit method. This procedure leads to a set of simultaneous equations in tridiagonal form. Implicit methods are more complicated to set up and program, but are unconditionally stable.

There is no guarantee that the solutions obtained by FDM approach will be accurate or even stable. So FDM must satisfy the basic requirements such as stability, compatibility and convergence. A detailed explanation is as follows:

(i) **Compatibility**: Finite difference equations are derived using the Taylor’s series expansion for two variables, neglecting the higher order terms in the series. These terms contribute a truncation error. It is required that the truncation error should tend to zero as the mesh sizes approaches to zero. Otherwise, the finite difference scheme is said to be incompatible or inconsistent with the partial differential equation. In this case, the finite difference solution is not likely to approach the desired solution.

(ii) **Stability**: If to carry out calculations to an infinite number of decimal places is possible and if the initial and boundary values are specified exactly, the numerical calculations will produce the exact solution of the difference equations. In practice, each calculation is carried out to a finite number of decimals and hence round off errors are introduced. The solution thus computed may not be the exact solution of the finite difference equation. Thus, a set of finite difference equation is said to be stable when the cumulative effect of all rounding error is negligible, otherwise it is said to be unstable.
(iii) **Convergence:** Let us assume that Compatibility is ensured. Next arises the question of whether the solution of difference equation converges to the solution of the partial differential equation, as the mesh sizes tend to zero. The finite difference solution is said to be convergent when the exact solution of the approximating difference equation tends to the exact solution of the partial differential equation as the mesh sizes tend to zero. More about these criteria are given in Smith [170], Mitchell [140], and Carnahan et al. [32].

One dimensional heat flow equation and solutions obtained through this approach have been compared with the solutions obtained through Haar wavelets.

**(B) Integral Transforms**

The classical methods of solution of initial and boundary value problems in physics and engineering sciences have their roots in Fourier’s pioneering work. An alternative approach through integral transform methods emerged primarily through Heaviside’s effort on operational techniques. In addition to bring up great theoretical interest to mathematicians, integral transform methods have been found to provide easy and effective ways of solving a variety of problems arising in engineering and physical science. A problem involving derivatives can be reduced to a simpler problem involving only multiplication by polynomials in the transform variable by taking an integral transform, solving the problem in the transform domain and then finding an inverse transform.

In this thesis Haar wavelet transform is very much highlighted due to its vast and recent applications in all fields as this transforms provide many features such as accuracy, simplicity, speed, flexibility, comfortability and less computation costs.
(C) Perturbation Methods

Perturbation methods comprises mathematical methods that are used to find an approximate solution to a problem which can not be solved exactly, by starting from the exact solution of exact solution of a related problem. Perturbation method is applicable if the problem at hand can be formulated by adding a “small” term to the mathematical description of the exactly solvable problem. When adding a secondary effect to a model, the model equations may acquire additional terms that are smaller in magnitude than those in the original system of equations. This may make the perturbed equations much more difficult to solve than the original system of equations.

Perturbation method leads to an expression for the desired solution in terms of a power series in some “small” parameter that quantifies the derivation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem. Formally, we have for the approximation to the full solution in a series of the small parameter (here called $\epsilon$), like the following:

$$ A = A_0 + \epsilon A_1 + \epsilon^2 A_2 + ... $$

In this example, $A_0$ would be the known solution to the exactly solvable initial problem and $A_1, A_2$ represents the “higher orders” which are found iteratively by some systematic procedure. For small $\epsilon$ these higher orders become successively more unimportant. They need for taking up perturbation techniques in future research needs to be highlighted.

(D) Adomain Decomposition Method (ADM)

An attempt is made to combine the advantages of the ADM and Haar
wavelets. The obtained ADM results have been validated against the Haar wavelet solutions. Good agreement with the exact solution has been observed. The Adomain decomposition method (ADM) is a creative and effective method for exactly solving functional equations of various kinds.

It is important to note that a large amount of research work has been devoted to the application of the ADM to a wide class of linear and nonlinear, ordinary or partial differential equations. The decomposition method provides the solution as an infinite series in which each term can be easily determined. The rapid convergence of the series obtained by this method is thoroughly discussed by Cherruault et al. [42]. Wazwaz [195] used the Adomain decomposition method for a reliable treatment of the Bratu-type equations and the Fisher’s equation. In this thesis ADM approach is given to one-dimensional reaction-diffusion problem and the Fisher’s reaction-diffusion problem. The solutions have been compared with the available theoretical solutions in the literature and the Haar wavelet solutions.

(E) Restrictive Taylor’s (RT) method

In my work, a new explicit method (Restrictive Taylor’s approximation method of the exponential matrix) for solving convection-diffusion equation with the initial and boundary conditions has been established. Numerical solutions have been compared with the Haar wavelet method. The power of the proposed method is confirmed.
1.4 Review of research approaches pertaining to Reaction-Diffusion Problems

In order to gain some insight into solving Reaction-Diffusion problems by modeling them through partial differential equations, the author has conducted a brief literature survey, which is presented in this section. In this review of Wavelet schemes for Reaction-Diffusion problems our attention is confined to the following:

(i) Fundamental aspects of Reaction-diffusion equations
(ii) Wavelet schemes for solving partial differential equations
(iii) Haar wavelet schemes for solving partial differential equations
(iv) Nonlinear partial differential equations and their applications
(v) Other wavelet schemes for solving reaction and diffusion problems

(i) Fundamental aspects of Reaction-diffusion equations


- V.S Manoranjan [134](1984) presented Bifurcation studies in reaction-diffusion equation II.

- G.F. Carey and Yun Shen [30](1994) showed least-squares finite element approximation of Fisher’s reaction-diffusion equations.


• N.S.Panchal and V.J.Daoo [153] (2001) established estimation of soil thermal characteristics from soil temperature measurement at Trombay.


• A.Mark et al. [136] (2005) established Strip-Tillage effect on seedbed soil temperature and other soil physical properties.

• D.Olmos [149](2007) carried out a Ph.D. Thesis on “Pseudospectral solutions of Reaction-Diffusion equations that model excitable media: Convergence of solutions and applications”.


• R.William Herb et al. [202](2008) presented ground surface temperature simulation for different land covers.

• M.C.Zhou (2008) [211] presented an application of traveling wave analysis in economic growth model.

(ii) Wavelet schemes for solving partial differential equations


• Chen and Hsiao [37](1997) established Haar wavelet method for solving lumped and distributed parameter systems.

• G.Strang and T.Nguyen [173](1997) presented Wavelets and Filter Banks.

• A.C.Gilbert [66] (1997) carried out a Ph.D thesis on “Multiresolution Homogenization schemes for differential equations and applications”.

• C. Cattani [34] (2001) showed evolotion equations by the Haar wavelet method.

• A.Cohen [45](2003) established numerical analysis of wavelet methods.


• M.G.Mayes [137] (2005) carried out a Ph.D thesis on “Wavelet signal processing techniques for efficient finite difference time-domain computation”.


• Min xu [139] (2006) carried out a Ph.D thesis on “Function approximation methods for optimal control problems”.

• M.R.Islam et al. [100] (2009) showed comparison of wavelet approximation order in different smoothness spaces.
(iii) Haar wavelet schemes for solving partial differential equations

- Chen and Hsiao [37] (1997) established Haar wavelet method for solving lumped and distributed parameter systems.


- Chen and Hsiao [37] (2001) presented a Wavelet approach to optimizing dynamic systems.


• N.M.Bujurke et al. [29] (2008) established computation of elgen values and solutions of regular Sturm-Liouville problems using Haar wavelets.

• U. Lepik [122](2009) had solved evolution equations by Haar wavelet method.

• J.L.Wu [204](2009) established a wavelet operational method for solving fractional partial differential equations numerically.


(iv) Nonlinear partial differential equations and their applications

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, optical fibers, biology, fluid dynamics and chemical kinetics. The concepts like solitons, peakons, kinks, breathers, cusps and compactons are now thoroughly investigated in the scientific literature. A variety of powerful methods, such as inverse scattering method, bilinear transformation, Backland transformation, a bilinear form, the tanh-sech method, extended tanh method, sine-cosine method, homogeneous balance method, Exp-function method, the tanh method, Adomian decomposition method, the tanh-coth method, Jacobi elliptic functions, and a Lax pair have been used independently by which soliton and multi-soliton solutions are obtained.

Wazwaz has published more than 100 papers in the area of nonlinear partial differential equations. He solved nonlinear PDE by using Adomain Decomposition Method (ADM) and the Variational Iteration Method (VIM). The solutions compared with the other classical methods.

• W. Malfliet and W. Hereman [132](1996)presented the tanh method I: Exact solutions of nonlinear evolution and wave equations.
• L. Debnath [48](1997) presented a book on “Nonlinear differential equations for scientists and engineers”.

• A.M. Wazwaz [191](2002) published a book on “Partial Differential Equations: Methods and applications”

• A.M. Wazwaz [193](2004) showed an analytical study of Fisher’s equation by using Adomian decomposition method.

• A.M. Wazwaz [192](2004) established the tanh method for travelling wave solutions of nonlinear equations.

• A.M. Wazwaz [197](2007) solved solitons and kink solutions for nonlinear parabolic equations.

• A.M. Wazwaz [198](2008) presented an analytical study on Burgers, Fisher, Huxley equations and combined forms of these equations.

(v) Other wavelet schemes for solving reaction and diffusion problems

(Except Haar wavelet scheme)

• M.S. El-Azab [52](2005) showed Rothe-Wavelet solution for nonlinear diffusion-reaction equations

• M.A. Pinsky [154](2005) showed integrability of the continuum wavelet scheme kernel.

A few other research papers have also been referred to study the gene propagation, depth profile of soil temperature, soil moisture and seasonal indices.

Haar Wavelet (HW) method, FDM approach, Restrictive Taylor’s (RT) method, Adomain Decomposition method (ADM), upwind finite difference method and
Integral transform methods were found to be convenient tools to obtain useful results.

During 1970s, Walsh functions and their cousins Haar wavelets received considerable attention in dealing with various problems of dynamical systems. Initially, using orthogonal functions to construct operational matrices for solving optimization problems of dynamical systems was established. The pioneering work in system analysis via Haar wavelets was initiated by Chen and Hsiao [37], who first derived a Haar operational matrix for integration. Since then, many operational matrices based on various orthogonal functions, such as Walsh, block-pulse, Laguerre, Legendre, Chebyshev, and Fourier have been developed. The main characteristic of this technique is to convert a differential equation into an algebraic equation, as a result of which, the solution procedures are greatly reduced or simplified. All orthogonal functions are supported on the whole interval $[a, b]$. This kind of global support makes them unsuitable for certain analysis, involving abrupt variations lasting for a very short duration. The operational matrix established for Haar wavelets eliminates all the drawbacks caused by the whole range support. Hsiao [93] proposed a simple and effective algorithm based on the STHWS for solving only linear stiff systems. The essential features of STHWS lies in representing the time-varying functions and their derivatives using only the first term of the Haar wavelet series and using the locality and orthonormality properties of Haar wavelets in transforming stiff systems into a system of algebraic equations. Haar wavelets have been applied extensively for signal processing in communications and physics research, and proved to be a useful mathematical tool.
1.5 Wavelet preliminaries

The theory behind wavelets has been developed during the last twenty to thirty years independently by mathematicians, scientists and engineers working in the areas of harmonic analysis theory (Calderon, 1964), filter bank theory (Esteban and Galand, 1977; Smith and Barnwell, 1986; Vetterli, 1984), and quantum mechanics (Aslaksen and Klauder, 1968). Morlet (1983) proposed the use of wavelets for analysis of seismic data and first coined the term “wavelets”. From 1987 to 1992, synthesis of these cross-disciplinary approaches evolved into wavelet analysis. Wavelet analysis has been used in a variety of applications, including image compression (DeVore, Jawerth and Lucier, 1992), signal denoising (Donoho and Johnstone, 1994), noise reduction (Esteban and Galand, 1977) speech and music processing (Kronland-Martinet, 1988), sound pattern analysis (Kronland-Martinet, Morlet and Grossmann, 1987) and sound synthesis (Miner, 1998).

Wavelets provide a tool for time-scale analysis of stationary (linear-time invariant) or nonstationary signals. They are finite in duration and therefore provide analysis of local signal features. Many systems are monitored and evaluated for their behavior using time signals. Additional information about the properties of a time signal can be obtained by representing the time signal by a series of coefficients, based on an analysis function. One example of a signal transformation is the transformation from the time domain to the frequency domain. The oldest and probably the best known method for this is the Fourier transform developed in 1807 by Joseph Fourier. An alternative method with some attractive properties is the wavelet transform, first mentioned by Alfred Haar in 1909. Since then a lot of research into wavelets and the wavelet transform is performed.
Though the Fourier transform is able to retrieve the global frequency content of a signal, its limitation is that the time information is lost. This is overcome by the short time Fourier transform (STFT) which calculates the Fourier transform of a windowed part of the signal and shifts the window over the signal. The short time Fourier transform gives the time-frequency content of a signal with a constant frequency and time resolution due to the fixed window length. This is often not the most desired resolution. For low frequencies often a good frequency resolution is required over a good time resolution. For high frequencies, the time resolution is more important. A multi-resolution analysis becomes possible by using wavelet analysis.

The wavelet analysis procedure is to adopt a wavelet prototype function, called an analyzing wavelet or mother wavelet. Temporal analysis is performed with a contracted, high frequency version of the prototype wavelet, while frequency analysis is performed with a dilated, low frequency version of the same wavelet. Because the original signal or function can be represented in terms of a wavelet expansion (using coefficients in a linear combination of the wavelet functions), data operations can be performed using just the corresponding wavelet coefficients. Other applied fields that are making use of wavelets include astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, magnetic resonance imaging, speech discrimination, optics, fractals, turbulence, earthquake-prediction, radar, human vision, and pure mathematics applications such as solving partial differential equations.
1.5.1 Wavelet Basis

Here we give a brief introduction to wavelets and the needed results are taken from Chui’s book [44]. We consider the space $L^2(\mathbb{R})$ of Lebesgue measurable functions $f$, defined on the real line $\mathbb{R}$, that by definition satisfy

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty.$$  

The wavelet basis is composed of functions $\Psi_{jk}(t)$ given by translation and dilation of a single function $\Psi(t)$, for instance,

$$\Psi_{jk}(t) = 2^j \psi(2^j t - k), j, k \in \mathbb{Z}, t \in (-\infty, \infty) \quad (1.17)$$

where $\mathbb{Z}$ denotes the set of all integers, i.e., $\mathbb{Z} = \ldots, -1, 0, 1, \ldots$ and $\Psi(t)$ is a fixed function in $L^2(\mathbb{R})$, the so-called mother wavelet. Therefore, for $j, k \in \mathbb{Z}$, $\Psi(2^j t - k)$ is obtained from $\Psi(t)$ by a dilation of $2^j$ and a translation by $\frac{k}{2^j}$.

If the function $\Psi$ has unit length, then all of the functions $\Psi_{jk}$ also have unit length, that is,

$$\|\Psi_{jk}\|_2 = \|\Psi\|_2 = 1, j, k \in \mathbb{Z}, \quad (1.18)$$

where $\|\Psi\|_2 = (\int_{-\infty}^{\infty} \psi(t)^2 dt)^{\frac{1}{2}}$.

**Definition 2.1.1**

A function $\psi \in L^2(\mathbb{R})$ is called an orthogonal wavelet, if the family $\psi_{jk}$ is an orthonormal basis of $L^2(\mathbb{R})$, that is,

$$\langle \psi_{jk}, \psi_{lm} \rangle = \int_{-\infty}^{\infty} \psi_{jk}(t) \psi_{lm}(t) dt = \delta_{jl} \delta_{km}, j, k, l, m \in \mathbb{Z} \quad (1.19)$$

where $\delta_{jk}$ is the Kronecker symbol, and every $f \in L^2(\mathbb{R})$ can be written as
\[ f(t) = \sum_{j,k=-\infty}^{\infty} c_{jk} \psi_{jk}(t), \quad (1.20) \]

where the convergence of the series in (1.20) is in \( L^2(\mathbb{R}) \), namely

\[
\lim_{M_1,N_1,M_2,N_2 \to \infty} \left\| f - \sum_{j=-M_2}^{N_2} \sum_{k=-M_1}^{N_1} c_{jk} \psi_{jk}(t) \right\|_2 = 0 \quad (1.21)
\]

For each \( j \in \mathbb{Z} \), let \( W_j \) denote the closure of the linear span of the basis \( \{\psi_{jk} : k \in \mathbb{Z}\} \),

\[ W_j = \text{span} \{\psi_{jk} : k \in \mathbb{Z}\}. \]

Then it is clear that subspaces \( W_j \) of \( L^2(\mathbb{R}) \) are mutually orthogonal. We use the notation

\[ W_j \perp W_l, \quad j \neq l. \]

Consequently, \( L^2(\mathbb{R}) \) can be decomposed as an orthogonal sum of the subspaces \( W_j \)

\[ L^2(\mathbb{R}) = \oplus_{j \in \mathbb{Z}} W_j = \ldots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots, \]

In the sense that any function \( f \in L^2(\mathbb{R}) \) has a unique decomposition ( [44], page 15 )

\[ f(t) = \sum_{j,k=-\infty}^{\infty} c_{jk} \psi_{jk}(t). \quad (1.22) \]

### 1.5.2 Multi-Resolution Analysis (MRA)

A function, \( \phi(x) \in L^2(\mathbb{R}) \), is called a scaling function that generates a multi-resolution analysis (MRA) in the subspaces, \( \ldots, V_{n-1}, V_n, V_{n+1}, \ldots \), if the following conditions are satisfied.
i) $V_j \subset V_{j+1}, \forall j$;

ii) $f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}$;

iii) $f(x) \in V_n \Leftrightarrow f(x + 2^{-n}) \in V_n$;

iv) $\lim_{n \to \infty} V_n = \bigcup_n V_n$ is dense in $L^2(\mathbb{R})$;

v) $\lim_{n \to \infty} \bigcap_n V_n = \{O\}$;

vi) The set $\{\phi(x - k)\}_{k \in \mathbb{Z}}$ forms a Riesz or unconditional basis for $V_0$, i.e., there exist constants $A$ and $B$, with $0 < A \leq B < \infty$, such that,

$$A \sum_{k \in \mathbb{Z}} |c_k|^2 \leq \left\| \sum_{k \in \mathbb{Z}} c_k \phi(x-k) \right\|_2^2 \leq B \sum_{k \in \mathbb{Z}} |c_k|^2$$

for any sequence $\{c_k\} \in l_2$, the subspace of all square summable sequences ($A=B=1$ for an orthonormal basis).

A scaling function, $\phi(x)$, and a set of related coefficients, $p(k)_{k \in \mathbb{Z}}$, are constructed such that they satisfy the so-called two-scale relation or refinement equation,

$$\phi(x) = \sum_k p(k)\phi(2x - k) \quad (1.23)$$

and some additional conditions. We say that the scaling function $\phi(x)$ has compact support if and only if finitely many coefficients $p(k)$ are non-zero.

Translations of the scaling function, $\{\phi(x - k)\}$, from a Riesz or unconditional basis of a subspace $V_0 \subset L^2(\mathbb{R})$. Furthermore, through translation of $\phi$ by a factor of $2^n$ and dilation by a factor of $k.2^{-n}$, a Riesz basis,

$$\{\phi_{n,k}(x)\}_{k \in \mathbb{Z}},$$

is obtained for the subspace $V_n \subset L^2(\mathbb{R})$, where

$$\phi_{n,k}(x) = 2^n \phi(2^nx - k) \quad (1.24)$$

corresponding to resolution level $n$.

Thus, the scaling function, $\phi(x)$, generates a set of basis for a sequence of nested subspaces of $L^2(\mathbb{R})$, and tends to $L^2(\mathbb{R})$, as the resolution level $n$, goes to infinity.
1.5.3 Accuracy and Approximation

While a wavelet scheme is applied to any dynamic phenomenon described by a differential equation, a function \( f(t) \) is projected onto a scaling space \( V_j \) using wavelets. Here the index \( j \) stands for the time scale \( \Delta t = 2^{-j} \) in the calculations. The scaling functions are given by \( 2^j \phi(2^j t) \) and they translate by \( k \Delta t \). Thus it is enough if one basis for \( V_j \) is calculated. \( f_j(t) \) is the projection of \( f(t) \) on \( V_j \) is that subspace. Hence \( f_j(t) \) can be expressed as a combination of basis functions. That is, \( \forall j, \)

\[
f_j(t) = \sum_{-\infty}^{\infty} a_{jk} 2^j \phi(2^j t - k)
\]

(1.25)

Multiresolution combines the splitting functions (in several scales) at level zero through \( j - 1 \) with the coarse average at level zero. Hence in terms of subspaces,

\[ V_j = V_0 \oplus W_0 \oplus ... \oplus W_{j-1}. \]

Except for \( V_0 \), the basis functions are now wavelets and the approximation of \( f \) at level \( j \) is a projection of \( f \) onto \( V_j \), and is given by,

\[
f_j(t) = \sum_k c_{0k} \phi(t - k) + d_{0k} \psi(t - k) + d_{1k} \psi(2t - k) + ... \]

(1.26)

In practice, the level \( j \) is determined by balancing accuracy with cost. As there are twice as many basis functions and twice as many coefficients, the cost and bit rate are approximately doubled between one level and the next one. The accuracy not only depends on the coefficients \( c_{0k} \) but also on \( d_{0k} \), the connection coefficients. Hence a smooth function would be able to approximate it. So properties of \( f(t) \) will contribute towards the error. Hence error estimate involved in \( p \)-th derivative of \( f(t) \) can be given as
\[ \| f(t) - f_j(t) \| \approx C (\Delta t)^p \| f^p(t) \|. \]

Here a constant C and p depend on a choice of wavelets. Also the step from \( \Delta t = 2^{-j} \) to \( \Delta t = 2^{-(j+1)} \) divide the error by \( 2^p \).

Thus the choice of p must make the asymptotic error estimate, and accurate one. When there are regions where \( f^p(t) \) is small and the region is subjected to sudden change the global error estimate can be made as a local one. The increase of j in the region of sudden change will homogenize the error where we have an adaptive mesh.

Multiscale method, based on the approximations \( (f_j)_{j \geq 0} \) give raise multiscale decomposition. Through the expansion of \( f(t) \) into the some of the cosets of the approximation and addition details. In practice, these approximations and decompositions can be defined and implemented in various ways.

### 1.5.4 The superiority of Wavelet transform

1. **The basis set can be improved in a systematic way:** If one wants the solution of the differential equation with higher accuracy one can just add more wavelets in the expansion of the solution. This will not lead to any numerical instabilities.

2. **Different resolutions can be used in different regions of space:** If the solution of the differential equation is varying particularly rapidly in a particular region of space one can increase the resolution in this region by adding more high-resolution wavelets centered around this region.

3. **There are few topological constraints for increased resolution regions:**
   The regions of increased resolution can be chosen arbitrarily, the only requirement being that a region of higher resolution be contained in a region of the next lower
The matrix elements of the differential operators are more easy to calculate.

The numerical effort scales linearly with respect to system size: As three-dimensional problems of realistic size require a very large number of basis functions. It gains more importance, that the numerical effort scales only linearly (and not quadratically or cubically) with respect to the number of basis functions. The iterative matrix techniques require (i) Matrix vector multiplications which are necessary for all iterative methods can be done with linear scaling (ii) The number of matrix vector multiplications is independent of the problem size. The first requirement is fulfilled since the matrix representing the differential operator is sparse. The second requirement is related to the availability of a good preconditioning scheme, which can be easily found by analyzing the Fourier properties of wavelets.

(iii) To solve a PDE numerically, we first need to find finite-dimensional approximation space for the solutions, and then discretize the PDE to a system of algebraic equations in this space so that the numerical solutions can be obtained.

1.6 Haar Wavelets

Haar basis functions are based on pulses in space. The Haar scaling function, $\phi$, as well as the Haar mother wavelet, $\psi_0$, are presented in Figure 1.2. The scaling function is simply a pulse function over a given domain. The wavelet function is based on the scaling function, and consists of two pulses, each of half the domain of the scaling function and of the opposite magnitude. The inner product of either function with itself is 1, while the inner product of the other two functions
The Haar wavelets of higher resolution levels are based on the mother wavelet. For each level of resolution the number of wavelets is doubled while the domain of each is halved. The magnitude of each function is modified so that the inner product of each wavelet function with itself is one. The inner product of any wavelet coefficient with any other wavelet coefficient, at any resolution level, or with the scaling function, is 0.

Figure 1.2 presents the wavelet coefficients for wavelet resolution levels 1 and 2. We assume that the maximum used wavelet resolution is $r_{\text{max}}$.

The reconstruction of the wavelets yields some interesting properties. When the coefficients of the expansion are summed to determine field values, the function appears as a pulse train. The pulses have the domain of half of the highest resolution wavelet. Furthermore, these pulses overlap the constant valued sections of the highest resolution wavelets. A linear combination of the wavelet/scaling functions has as many degrees of freedom as the number of coefficients used.

In 1910 Alfred Haar introduced a function, which presents a rectangular pulse pair. After that various generalizations were proposed (a state-of-the-art about Haar transforms can be found in [76]). In the 1980s it turned out that the Haar function was in fact the Daubechies wavelet of order 1. It is the simplest orthonormal wavelet with compact support. An essential shortcoming of the Haar wavelets is that they are not continuous. The derivatives do not exist in the points of discontinuity; therefore it is not possible to apply the Haar wavelets directly to solving differential equations. There are at least two possibilities of ending this impasse. First, the piecewise constant Haar functions can be regularized with interpolation splines; this technique has been applied by Cattani...
<table>
<thead>
<tr>
<th>S.No.</th>
<th>Haar functions</th>
<th>Integrals of Haar functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>h₀</td>
<td>t</td>
</tr>
<tr>
<td>2.</td>
<td>h₁</td>
<td>t</td>
</tr>
<tr>
<td>3.</td>
<td>h₂</td>
<td>t</td>
</tr>
<tr>
<td>4.</td>
<td>h₃</td>
<td>t</td>
</tr>
<tr>
<td>5.</td>
<td>h₄</td>
<td>t</td>
</tr>
<tr>
<td>6.</td>
<td>h₅</td>
<td>t</td>
</tr>
<tr>
<td>7.</td>
<td>h₆</td>
<td>t</td>
</tr>
<tr>
<td>8.</td>
<td>h₇</td>
<td>t</td>
</tr>
</tbody>
</table>

Figure 1.1: First eight Haar scaling functions and their integrals.

This greatly complicates the solution process and the main advantage of the Haar wavelets - their simplicity - gets lost. Another possibility was proposed by Chen and Hsiao [37]. They recommended to expand into the Haar series not the function itself, but its highest derivative appearing in the differential equation;
the other derivatives (and the function) are obtained through integrations. All these ingredients are then incorporated into the whole system, discretized by the Galerkin or collocation method. Chen and Hsiao [37] demonstrated the possibilities of their method by solving linear systems of ordinary differential equations.
(ODEs) and partial differential equations (PDEs). In [38] an optimal control problem with the quadratic performance index is discussed. In [95] Hsiao and Wang applied this method to solving singular bilinear and nonlinear systems. Nonlinear stiff systems were examined in [94]. In [92] Hsiao demonstrated that the Haar wavelet approach is effective also for solving variational problems. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.

Recently, Haar wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system. The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [37], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [97], who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wu [96] proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix. Kalpana and Raja Balachandar [108] presented Haar wavelet based method of analysis for observer design in the generalized state space or singular system of transistor circuits.
1.6.1 The Haar System

In this, we discuss the wavelet approximation of a given function $f \in L^2(\mathbb{R})$ in the Haar wavelet system. The Haar wavelet family for $t \in [0,1]$ is defined as follows.

$$
\begin{align*}
    h_i(t) = \begin{cases} 
        1 & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\
        -1 & \text{for } t \in \left(\frac{k+0.5}{m}, \frac{k+1}{m}\right] \\
        0 & \text{elsewhere}
    \end{cases}
\end{align*}
$$

Haar scaling function is given by

$$
\begin{align*}
    \phi(t) = \begin{cases} 
        1 & \text{for } 0 \leq t < 1, \\
        0 & \text{otherwise}
    \end{cases}
\end{align*}
$$

Let $f$ be a function in $L^2(\mathbb{R})$ and $I_{jk} = [k2^{-j}, (k+1)2^{-j}]$. We can define piecewise constant approximation $f_j$ of $f$ at scale $2^{-j}$ by

For all $x \in I_{j,k}$, $k \in \mathbb{Z}$

$$
    f_j(x) = 2^j \int_{I_{j,k}} f(t) dt, 
$$

ie., $f$ is approximated by its mean value on each interval $I_{j,k}$, $k \in \mathbb{Z}$.

Remark: 1.6.1

The choice of the mean value makes $f_j$ the $L^2$-orthogonal projection of $f$ on
the space

\[ V_j = \{ f \in L^2 \}. \]

Here \( f \) is constant on \( I_{j,k}, k \in \mathbb{Z} \).

Indeed, an orthogonal basis for \( V_j \) is given by the family

\[ \phi_{j,k} := 2^j \phi \left( 2^j - k \right), k \in \mathbb{Z}, \quad (1.30) \]

where \( \phi := \chi_{[0,1]} \), and clearly \( f_j \) can be written as

\[ f_j = \sum_{k \in \mathbb{Z}} \langle f, \phi_{j,k} \rangle \phi_{j,k}, \quad (1.31) \]

with the usual notation \( \langle f, g \rangle = \int f(t)g(t)dt \).

We will thus denote \( f_j \) by \( p_j f \) where \( P_j \) is the orthogonal projector onto \( V_j \). We shall also use the notation

\[ c_{j,k} = c_{j,k}(f) := \langle f, \phi_{j,k} \rangle = \int_{I_{j,k}} f(t)\phi_{j,k}dt, \quad (1.32) \]

for the normalized mean values which are the coordinates of \( P_j f \) in the basis \( (\phi_{j,k})_{k \in \mathbb{Z}} \).

**Remark: 1.6.2**

The above approximation process is local in the sense that the value of \( P_j f \) on \( I_{j,k} \) is only influenced by the value of \( f \) on the same interval. In particular, we can still use (1.29) to define \( P_j f \) when \( f \) is only locally integrable, or when \( f \) is only defined on a bounded interval such as \([0,1]\).

**Remark: 1.6.3**

Since \( V_j \subset V_{j+1} \), it is clear that \( P_{j+1} f \) contains ‘more information’ on \( f \) than the
coarser approximation $P_j f$. More precisely,

$$P_j f / I_{j,k} = [P_{j+1} f / j+1,2k + P_{j+1} f / j+1,2k+1] / 2 \quad (1.33)$$

We can also define the orthogonal projection $Q_j := P_{j+1} f - P_j f$ onto $W_j$, the orthogonal complement of $V_j$ into $V_{j+1}$. From (1.33), it is clear that $Q_j f$ ‘oscillates’ in the sense that

$$Q_j f / I_{j+1,2k} = -Q_j f / I_{j+1,2k+1} \quad (1.34)$$

The oscillation property (1.34) allows us to expand $Q_j f$ into

$$Q_j f = \sum_{k \in Z} d_{j,k} \psi_{j,k}, \quad (1.35)$$

where $\psi_{j,k} := 2^j \{ \psi / 2^j - k \}$ and

$$\psi(x) = \chi[0, 1/2] - \chi[1/2, 1]. \quad (1.36)$$

Since the $\chi_{j,k} \in Z$ are also an orthonormal system, they constitute an orthonormal basis for $W_j$ and we thus have

$$d_{j,k} = d_{j,k}(f) = \langle f, \psi_{j,k} \rangle. \quad (1.37)$$

We thus have re-expressed the ‘two-level’ decomposition of $P_{j+1}$ into the coarser approximation $P_j f$ and the additional fluctuations $Q_j f$, according to

$$\sum_k c_{j+1,k} \psi_{j+1,k} = \sum_k c_{j,k} \psi_{j,k} + \sum_k d_{j,k} \psi_{j,k}. \quad (1.38)$$
This decomposition can be iterated on an arbitrary number of levels. If \( j_0 < j_1 \), we can rewrite the orthogonal decomposition

\[
P_{j_1} f = P_{j_0} f + \sum_{j_0 \leq j_1} Q_j f,
\]

(1.39)

according to

\[
\sum_k c_{j_1,k} \phi_{j_1,k} = \sum_k c_{j_0,k} \phi_{j_0,k} + \sum_{j_0 \leq j < j_1} \sum_k d_{j,k} \psi_{j,k}
\]

(1.40)

The above equation gives a local description of each contribution and should be viewed as an orthonormal change of basis in \( V_{j_1} \): both \( \{ \phi_{j_1,k} \}_{k \in \mathbb{Z}} \) and \( \{ \phi_{j_0,k} \}_{k \in \mathbb{Z}} \cup \{ \psi_{j,k} \}_{j_0 \leq j < j_1, k \in \mathbb{Z}} \) are orthonormal bases for \( V_j \), and any function in \( V_j \) has thus a unique decomposition in each of these bases.

Note that different role played by the functions \( \phi \) and \( \psi \): the former is used to characterize the approximation of a function at different scales, while the latter is needed to represent the fluctuation between successive levels of approximation. In particular, we have \( \int \psi = 0 \), reflecting the oscillatory nature of these fluctuations. In the more general multiresolution context the function \( \phi \) is called 'scaling function' and \( \psi \) is called “mother wavelet”, in the sense that all the wavelets \( \psi_{j,k} \) are generated from translations and dilations of \( \psi \).

Clearly the union of the approximation spaces \( V_j \) is dense in \( L^2(\mathbb{R}) \), i.e.,

\[
\lim_{j \to \infty} \| f - P_j f \|_{L^2}^2 = 0,
\]

(1.41)

for all \( f \in L^2(\mathbb{R}) \). Combining (1.40) and (1.41), we obtain that the orthonormal family \( \phi_{j_0,k} \cup \psi_{j,k} \) is a complete orthonormal system of \( L^2(\mathbb{R}) \).
Any function \( f \in L^2(\mathbb{R}) \) can thus be decomposed into

\[
f = \sum_{k} c_{j0,k}(f)\phi_{j0,k} + \sum_{j \geq j_0} \sum_{k} d_{j,k}(f)\psi_{j,k},
\]

(1.42)

where the series converges in \( L^2 \).

### 1.6.2 Integration of Wavelets

In (1.27) integer \( m = 2^j \) \( (j = 0, 1, 2, \ldots J) \) indicates the level of the wavelet; \( k = 0, 1, 2, \ldots, m-1 \) is the translation parameter. Maximal level of resolution is \( J \). The index \( i \) is calculated according to the formula \( i = m + k + 1 \); in the case of minimal values \( m=1, k=0 \), we have \( i = 2 \), the maximal value of \( i \) is \( 2M = 2^{(J+1)} \). It is assumed that the value \( i=1 \) corresponds to the scaling function for which \( h_1 \equiv 1 \) in \([0, 1]\). Let us define the collocation points \( t_l = (l - 0.5)/2M, (1, 2, \ldots, 2M) \) and discretise the Haar function \( h_i(t) \): In this way we get the coefficient matrix \( H(i, l) = (h_i(t_l)) \), which has the dimension \( 2M \times 2M \). Each integer \( i \) has a unique decomposition into the two integers \( l \) and \( k \). Sample computations are shown in the following table.

| \( l \) | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | ... |
| \( k \) | 0 | 0 | 1 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 4 | ... |
| \( i = 2^l + k + 1 \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | ... |

Table 1.1: Index computation for Haar basis functions

The orthogonal set of Haar wavelets \( h_0(t) \) to \( h_7(t) \) is shown in Figure 1.1, which contains a family of single square wavelets. The first basis \( h_0(t) \) is called the scaling function, which is equal to 1 for the whole unit time interval. The second basis \( h_1(t) \) is the fundamental square wave. The others, \( h_2(t) \) to \( h_7(t) \) are
generated from \( h_1(t) \) via two operations: translation and dilation.

Haar wavelets have several useful properties such as,

1. the Haar set forms a local basis since each Haar function contains just one wavelet which nonzero over some subinterval and remains zero elsewhere in the interval \([0,1]\);

2. the Haar basis functions are orthogonal to one another;

3. the integration of Haar wavelets can be expandable into Haar series.

The operational matrix of integration \( P \), which is a \( 2M \) square matrix, is defined by the equation

\[
(PH)_{il} = \int_0^{l_i} h_i(t) \, dt \tag{1.43}
\]

\[
(QH)_{il} = \int_0^{l_i} dt \int_0^{l_i} h_i(t) \, dt \tag{1.44}
\]

The elements of the matrices \( H, P \) and \( Q \) can be evaluated according to (1.27), (1.43) and (1.44).

\[
H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\]
\[
P_2 = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}
\]
\[
H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}
\]
\[ P_4 = \frac{1}{16} \begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \]

\[ P_8 = \frac{1}{64} \begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 \end{bmatrix} \]

Chen and Hsiao [37] showed that the following matrix equation for calculating the matrix \( P \) of order \( m \) holds

\[
P_{(m)} = \frac{1}{2^m} \begin{bmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H_{(m/2)}^{-1} & O \end{bmatrix}
\]

where \( O \) is a null matrix of order \( \frac{m}{2} \times \frac{m}{2} \)

\[
H_{m \times m} \triangleq [h_m(t_0) \ h_m(t_1) - \cdots - h_m(t_{m-1})]
\]  (1.45)

and

\[
\frac{i}{m} \leq t < i + \frac{1}{m},
\]

\[
H_{m \times m}^{-1} = \frac{1}{m} H_{m \times m}^T \text{diag}(r)
\]
It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once; after that they will be applicable for solving whatever differential equations. Since $H$ and $H^{-1}$ contain many zeros, this phenomenon makes the Haar transform much faster than the Fourier Transform and it is even faster than the Walsh transform. This is one of the reasons for rapid convergence of the Haar wavelet series.

1.6.3 Function Approximation

Any function $y(x) \in L^2[0,1)$ can be decomposed as

$$y(x) = \sum c_n h_n(x)$$  \hspace{1cm} (1.46)

where the coefficients $c_n$ are determined by

$$c_n = 2^j \int_0^1 y(x) h_n(x) \, dx$$  \hspace{1cm} (1.47)

where $n = 2^j + k, j \geq 0, 0 \leq k < 2^j$. Especially $c_0 = \int_0^1 y(x) \, dx$.

The series expansion of $y(x)$ contains infinite number of terms. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite terms, that is

$$y(x) = \sum_{0}^{m-1} c_n h_n(x) = c_{(m)}^T h_{(m)}(x)$$  \hspace{1cm} (1.48)
where the coefficients $c_{(m)}^T$ and the Haar function vector $h_m(x)$ are defined as $c_{(m)}^T = [c_0, c_1, ..., c_{m-1}]$ and $h_m(x) = [h_0(x), h_1(x), ..., h_{m-1}(x)]^T$ where 'T' means transpose and $m = 2^j$

1.6.4 Features of Haar wavelet transform

We introduce a Haar wavelet schemes for solving a few reaction-diffusion problems, which will exhibit several advantageous features:

i) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods.

ii) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of significant wavelet coefficients.

iii) The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically. The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. It has been well demonstrated that in applying the nice properties of Haar wavelets, the differential equations can be solved conveniently and accurately by using Haar wavelet method systematically. According to this method the spatial operators are approximated by the Haar wavelet method and the time derivation operators by the finite difference method The main advantages of this method is its simplicity and small computation costs: it is due to the sparsity of the transform matrices and to the small number of significant wavelet coefficients. It is worth mentioning that Haar solution provides excellent results even for small values of $m$. For larger values of $m$ (that is, $m=16,m=32,m=64,m=128,m=256$) we can obtain the
results closer to the real values. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones.

This thesis also confirmed the power of the Haar wavelet method in handling nonlinear equations in general. This method can be easily extended to find the solution of all other non-linear parabolic equations. Another benefit of our method is that the scheme presented here, with some modifications, seems to be easily extended to solve model equations including more mechanical, physical or biophysical effects, such as nonlinear convection, reaction, linear diffusion and dispersion. The complexity with respect to 3 dimensional spatial variable for solving other nonlinear parabolic problems can be solved easily.

1.7 Convergence and stability of Haar Wavelet Scheme

The first is the family of Haar functions \( \{h_2(x)\} \). This is a system of unconditional convergence by default, for according to the argument, the series

\[
\sum_{i=1}^{\infty} c_i h_2(x)
\]

has only a finite number of non-zero terms for almost every \( x \). Therefore it trivially converges unconditionally almost everywhere [85].

**Remark:** Since we have an orthonormal basis, the \( L^2 \)– convergence in (1.42) is unconditional, one can permutate the terms or change their signs without affecting the convergence of the series. If \( f \) is continuous, \( P_j f \) converges uniformly to \( f \) as \( j \) goes to \( +\infty \), so that we can define a summation process by letting \( j_1 \) go to \( +\infty \) in (1.41). The Haar wavelet method is always stable [139].
1.8 Computational complexity of Haar and other schemes

In order to establish the superiority of the Haar wavelet scheme the computational complexity of Haar wavelets have been compared with that of the other schemes as tabulated below.

Table 1.2: Comparison of algorithmic complexity of the proposed method with FFT and WT

<table>
<thead>
<tr>
<th>Series</th>
<th>Number of additions</th>
<th>Number of multiplications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Haar Transform (HT)</td>
<td>$2m - 2$</td>
<td>$m$</td>
</tr>
<tr>
<td>Walsh Transform (WT)</td>
<td>$m \log_2 m$</td>
<td>$m$</td>
</tr>
<tr>
<td>Fast Fourier Transform (FFT)</td>
<td>$m \log_2 m$</td>
<td>$m(\log_2 m + 1)$</td>
</tr>
</tbody>
</table>

The fast capability of Haar wavelet method should be impressive. Since $H$ and $H^{-1}$ contain many zeros, this phenomenon makes the Haar transform faster than the Fourier transform, and it is even faster than the Walsh transform. This is one of the reasons for rapid convergence of the Haar wavelet method.

In practical applications, a small number of terms increases the calculation speed and saves memory storage; a large number of terms improve resolution accuracy. Therefore, a trade-off between calculation speed, memory saving, and the resolution accuracy has been considered in the analysis.

1.9 Computational resources

All the computational work in this dissertation has been coded using Matlab7.0 and plots are generated using Matlab7.0 and sigma plot. All concerned Matlab programs to readily check/run for sample problems considered here as well as for problems not considered here. All the numerical experiments presented in
this thesis were computed in double precision with some MATLAB codes on a personal computer System with Processor Intel(R) Core(TM) 2 Duo CPU T5470 @ 1.60GHz(2CPUs) and 1 GB RAM.

1.10 Genesis of the thesis

The importance of Reaction-diffusion equations, due to its wide variety of applications has lead to the development of several mathematical methods, one among which is the theory of wavelets. This theory is mainly based on the two important wavelet systems namely

(i) The Haar system
(ii) The Daubechies system

While Haar’s simple-step wavelets exhibit jump discontinuities, Daubechies wavelets are for continuous phenomena. As a consequence of continuity of Daubechies wavelets, they approximate continuous functions more accurately than the Haar’s wavelets but at the cost of intricate algorithms based upon the sophisticated theoretical development [47].

Motivated by advantages of Haar’s simple-step methods and its less computational costs establishing mathematical models for various types of reaction-diffusion equations augers well and the utility in carrying out research studies, Chen et al. [37] suggested Haar wavelet system as the tracking approach in evaluating wavelet coefficients through the traditional calculus methods.

Haar wavelet approach is thus and is also expected to provide valuable information for further advances of tracking approach which is far from other recently developed approaches. In these lines a few research papers addressing the methodology of solving nonlinear partial differential equations have been referred
by the author [[38],[77],[78],[92],[93],[94],[95],[108],[122],[123]].

Also the research papers pertaining to other solution techniques have been referred by the author for the purpose of implementing them to carry out the comparative study.

Pioneering research carried out by Lepik [[120],[121]] and others has established Haar wavelets with reasonable simplifying assumptions. Their model designed for lumped and distributed parameter systems focused on convergent issues in addition to detailing the parameters involved in the physical phenomena, thus paving the way for the other researchers to work on designing models for other types of nonlinear phenomena appearing in various problems across the various engineering disciplines.

A few other research papers [[77],[78],[79],[80],[92],[94],[123]] are also been studied to analyse the convergence, stability and complexity of the proposed Haar scheme.

Various mathematical tools to solve coupled reaction-diffusion equations characterising the dynamic behaviour of reactions and diffusions have also been studied. Finite Difference Method (FDM), Adomain Decomposition Method (ADM), Integral transform methods, Restrictive Taylor’s (RT) method were found to be the convenient tools to obtain significant results. A brief literature survey is presented in section 1.4 and a brief description of the mathematical methods is presented in 1.3.
1.11 Aims and scope

From the literature review it is observed that investigators have studied Haar wavelet scheme from the perfective of solving linear/nonlinear PDE in order to obtain accurate solutions. However, problems with highly nonlinear PDE correspondence have not been addressed from application point of view. Hence the author has made an attempt to study the various reaction-diffusion problems that correspond to various engineering phenomena which in turn do not have any analytic solutions due to the complexity of governing equations.

Wavelet methods particularly Haar wavelet scheme paves the way to solve such problems. Since the stability and convergence have ensured previous authors. This scheme is employed to find more accurate solutions of the chosen problems.

To study the depth profile of soil temperature, an appropriate parabolic PDE is considered and the parameters involved therein have been studied together with the sensitivity analysis.

As the first case such problems are varied in nature from various engineering studies have been taken up for analysis using Haar wavelet schemes.

The choice of the parameters that increase the solution accuracy have been carefully maintained while obtaining the quality solutions. The solution obtained for all problems of the present study have been compared with available theoretical solutions through other methods found in the literature an by tracing appropriate profiles. The results are found to be in excellent agreement.
1.12 Organization of the thesis

The organization of this thesis follows that the Haar wavelet method for solving a few reaction-diffusion problems. The thesis considers the study of some Reaction-Diffusion problems of practical importance in various engineering and sciences. The RD equations are simplified by invoking suitable approximations that suit the physical nature of the problem.

Chapter 1: It begins with a brief introduction. It discusses the importance of reaction-diffusion problems. Derivation and traditional methods for solving reaction-diffusion equation equations are presented. Mathematical techniques to solve reaction-diffusion equation have been explained. Review of research approaches pertaining to reaction-diffusion equations have been showed. After that a brief introduction to wavelets as given by Chui [44]. The definition and properties of Haar wavelets have been discussed. The importance of Haar wavelets are its simplicity and capability of numerical computation, such as orthogonality and the operational matrix of integration [37]. Convergence and computational complexity of a proposed method have been discussed. At the end of the chapter, the computational resources and genesis of the thesis have been presented.

Chapter 2: It deals with Haar wavelet in estimating depth profile of soil temperature.

Chapter 3: It provides the Haar wavelet method for solving Fisher’s equation.

Chapter 4: It deals with Haar wavelet method for solving FitzHugh-Nagumo and Cahn-Allen equations.

Chapter 5: It is devoted to a comparative study of a Haar wavelet method and a Restrictive Taylor’s series method for solving convection-diffusion equations.

Chapter 6: It deals with Haar wavelet method for solving some nonlinear
parabolic equations.

Chapter 7: The next chapter contains a comparison of Haar wavelet and Adomain decomposition method for solving one-dimensional reaction-diffusion equations.

Chapter 8: It deals with Haar wavelet method for solving nonlinear coupled reaction-diffusion equations.

Chapter 9: It is devoted to Haar wavelet approach for solving one-dimensional Burgers’ equations.

Chapter 10: In chapter 10, solving lightning based traveling wave equations by the Haar wavelet method has been established.

Chapter 11: Chapter 11 provides future scope of the thesis which includes a list of few problems where

(1) Use of other wavelets such as Spline wavelets, Legendre wavelets, Morlet wavelets etc for solving other type of traveling wave equation may be taken up for study.

(2) In few studies concerning two-dimensional and three-dimensional turbulence, Haar wavelet schemes can be applied to discuss the solutions.

(3) Haar wavelets can be used to analyze the gene propagation using Fisher and Nagumo equations.

(4) Mathematical models can be developed for gene propagations to analyze the spread of an advantaged mutant strain within an asexual species.