Chapter 5

Generalized Bosonic Oscillators and their Coherent States

5.1 A Deformed Jaynes-Cummings Model

In Chapter 1 a non-linear realization of the deformed and para oscillators in terms of
the bosonic operator was used. That construction made it possible to get the thermal
vacuum for these systems through a unitary transformation.

A pair of bosonic operators [69][37-39]

\[
b = \sqrt{\frac{N+1}{f(N+1)}} a \quad b^\dagger = a^\dagger \sqrt{\frac{N+1}{f(N+1)}}
\]  

(5.1)
can be constructed from the generalized bosonic operator

\[
[a, N] = a \quad [a^\dagger, N] = -a^\dagger
\]

\[
[a, a^\dagger] = g(N)
\]  

(5.2)

where \(g(N) = f(N+1) - f(N)\) and

\[
a|n> = \sqrt{f(N)}|n-1> \quad a^\dagger|n> = \sqrt{f(N+1)}|n+1>
\]  

(5.3)

The operators \(b\) and \(a\) are related by a similarity transformation

\[
b = S a S^\dagger ,
\]
as the relation (1) can be written in this form with

\[
S = \sqrt{\frac{f(N)!}{N!}} , \quad S^{-1} = \sqrt{\frac{N!}{f(N)!}}
\]
The form of $f(N)$ for the $q$-deformed and the para oscillator have already been given. And $f(N!)|n > = f(n) \cdots f(1)|n >$.

Such a realization is used to study the deformed version of the Jaynes-Cummings model [43]. The Jaynes-Cummings (JC) model [44-45] describes the interaction of a two level atom with a single cavity mode. The Hamiltonian of the model is

$$H = \omega_c N + \omega_a S_z + g(S^+ b + S^- b^\dagger)$$

where $\omega_c$ is the energy of the mode and $\omega_a$ the energy gap of the two level atom. $g$ is a coupling constant, $S_z$, $S^+$ and $S^-$ are the operators of the atom. $S_z$ gives the atomic inversion and $S^+$ and $S^-$ are the raising and lowering operators. They obey the algebra

$$[S^+, S^-] = 2S_z$$
$$[S^\pm, S_z] = \pm S^\pm$$

(5.4)

The model is exactly solvable and the solution shows interesting quantum mechanical effects like the collapse and revival of atomic inversion, the inhibition of decay of excited states in a cavity and Rabi oscillations [46-48]. Another model for the interaction of a two level atom with cavity mode is due to Buck and Sukumar (BS) [49-51]. It is also exactly solvable and its solutions can be given in closed form and are more tractable analytically. The model has the Hamiltonian

$$H = \omega_c N + \omega_a S_z + g [S^+ (1 + N)^{1/2} b + S^- b^\dagger (1 + N)^{1/2}]$$

(5.5)

It is an effective model for a non-linearly coupled system.

It can now be shown [43] that the deformed version of this model,

$$H = \omega_c N + \omega_a S_z + g [S^+ (1 + N)^{1/2} a + S^- a^\dagger (1 + N)^{1/2}]$$

(5.6)

interpolates between the two models given above as the deformation parameter goes from 0 to 1. Here, $N$ is the number operator for the deformed oscillator. It has the form of
an infinite series [68] in terms of \(a, a^\dagger\). In terms of the corresponding bosons it is simply 
\(b^\dagger b\). Earlier studies of deformed JC model simply replaced \(N\) by \(a^\dagger a\) [52-54].

The interpolation is shown below. For the deformation [37-42]

\[ aa^\dagger - qa^\dagger a = 1 \]

that is considered \(f(N) = \frac{1 - q^N}{1 - q}\), when \(q = 0\), \(f(N) = 1\) and \(b = (N + 1)^{1/2}a\). Then (5.6) is just the JC model in terms of \(b\) and \(b^\dagger\) when \(q = 1\), \(f(N) = N\) and \(b = a\). Hence (5.6) in this case is the BS model in terms of \(b\) and \(b^\dagger\). For intermediate values of \(q\) it is a deformed model.

The expressions for the evolution of the atomic inversion \(\langle S_z(t) \rangle\) for this deformed model have been obtained for various initial states of the atom and the radiation mode [43]. When the radiation is in the usual coherent state, the evolution of \(\langle S_z(t) \rangle\) has the same features as in JC or BS models. The evolution has also been studied when the radiation is in the \(q\)-coherent state, the eigenstate of operator \(a\). The collapse and revival features are blurred in this case.

The eigenstates of the generalized annihilation operators can be constructed easily when a canonical conjugate \(A^\dagger\) is available, such that \([a, A^\dagger] = 1\) [37-39][69].

The operator \(A^\dagger\) conjugate to \(a\) is

\[ A^\dagger = \frac{a^\dagger(N + 1)}{f(N + 1)} \]

It satisfies the canonical commutation relation on the Fock space. We also have \([A, a^\dagger] = 1\). In terms of \(A^\dagger\), the expression for the number operator is \(A^\dagger a\). The eigenstate of \(a\) is \(\exp \alpha A^\dagger |0\rangle\) and that of \(A\) is \(\exp \alpha a^\dagger |0\rangle\).
5.2 A Unified Approach to Multiphoton Coherent States

Such a construction of a canonical conjugate can be also carried out for the powers of the bosonic annihilation operator [61]. These operators denoted by $F$ have the general form $F = \phi(N)b^m$ and are called multiphoton annihilation operators in quantum optics literature. $\phi(x)$ is some function of $x$ which does not have zeros for positive integral values of $x$ including zero. $F$ could also be a multimode annihilation operator.

As in the case of deformed and para oscillators it is possible to construct $G^\dagger$ satisfying

$$[F, G^\dagger] = 1.$$  \hspace{1cm} (5.7)

Note that $G^\dagger \neq F^\dagger$ in general. The operator $G^\dagger$ can be used to construct the eigenstates of $F$ starting from the states annihilated by $F$. Since $F$ contains $b^m$, there are $(m-1)$ states that are annihilated by it. Denoting them by $|\nu_i\rangle, i = 1, \ldots, m-1,$

$$F|\nu_i\rangle = 0$$

from which it follows that

$$\exp(fG^\dagger)F \exp(-fG^\dagger)|f_i\rangle = (F - f)|f_i\rangle = 0$$

$$F|f_i\rangle = f|f_i\rangle$$  \hspace{1cm} (5.8)

where $|f_i\rangle = \exp(fG^\dagger)|\nu_i\rangle$. The use of canonical conjugate allows the construction of the eigenstates of $F$ in a manner analogous to the construction of the eigenstate of $b$, the coherent state. Since $G^\dagger \neq F^\dagger$, we have another distinct relation


From the explicit construction of $G^\dagger$ given below it can be seen that $|\nu_i\rangle$ are the vacuua of $G$ also.
Hence, the commutation relation (5.7) is two faced and it generates two sets of eigenstates.

When the operator $G^\dagger$ is constructed below we find it has a different form in each sector. A sector is the set of number states $|mn + i\rangle$, $n = 0, 1, \ldots$ generated from the vacuum $|v_i\rangle$ by the application of $F^\dagger$.

The operator

$$N_c = \frac{1}{m} [a^\dagger a + \text{a constant}]$$

which satisfies

$$[F, N_c] = F$$

is used in the construction of $G^\dagger$. The constant, $c$, is any real number. Now, let $G^\dagger_c$ be an operator consisting of $a^\dagger m$ and some function of $N$ such that

$$FG^\dagger_c = N_c.$$

Then, from $[F, N_c] = [F, FG^\dagger_c] = F$ it follows that $[F, G^\dagger_c] = 1 + X$. The operator $X$ satisfies $FX = 0$. Such an operator can be constructed from the states annihilated by $F$ as

$$X = \sum_{i=0}^{m-1} \mu_i |i\rangle <i|.$$

Only diagonal operators are chosen because $[F, G^\dagger_c]$ is diagonal for the general form of $F$ and $G^\dagger_c$. Using the orthogonality of the states $|i\rangle$, we get the coefficients $\mu_i$.

$$[F, G^\dagger_c]|i\rangle = FG^\dagger_c|i\rangle = (1 + \mu_i)|i\rangle.$$

But since $FG^\dagger_c = N_c$,

$$<i|N_c|i> = \frac{1}{m} (i + m - c) = (1 + \mu_i)$$

$$\mu_i = \frac{1}{m} (i - c).$$

(5.11)
We have

$$[F, G^\dagger_i] = 1 + \sum_{i=0}^{m-1} \mu_i |i><i|$$

and the action of the commutator on all number states is a unit operator except for $|i\rangle$. The freedom to choose $c$ is now made use of to ensure that $[F, G^\dagger_i] = 1$ hold on all states of a sector. If we set $c = i$, then the operator obtained from

$$FG^\dagger_i = N_i$$

is the canonical conjugate of $F$ in the whole of the sector generated from $|i\rangle$. Explicitly,

$$G^\dagger_i = \frac{1}{m} F^\dagger \frac{1}{FF^\dagger} (b^\dagger b + m - i)$$

when $F = b^2$, for example.

$$G^\dagger_o = \frac{1}{2} b^{12} \frac{1}{1 + N}$$

in the even number sector generated from $|0\rangle$ and

$$G^\dagger_1 = \frac{1}{2} b^{12} \frac{1}{N + 2}$$

in the odd number sector generated from $|1\rangle$.

The eigenstates of $a^2$ generated from the vacua $|0\rangle$ and $|1\rangle$ are $\exp fG^\dagger_o |0\rangle$ and $\exp fG^\dagger_1 |1\rangle$. They are known as the cat states in literature. They can be expressed as the linear combinations of coherent states which are macroscopic states.

The second set of coherent states are the eigenstates of $G_o$ and $G_1 = \exp ga^{12} |0\rangle$ and $\exp ga^{12} |1\rangle$. The former when normalized is the well known squeezed state [60] and the latter have also been studied recently [65]. The operator $G^\dagger_o$ of which the squeezed state is an eigenstate was also constructed earlier.

The same procedure holds for multimode operators. The operator $N_c$ is of the form

$$\{a^\dagger a + b^\dagger b + c\},$$

where $a, b, ..$ are the modes. $a$ here stands for one of the bosonic modes.
not the generalized oscillator of the previous section. For \( F = ab \), for example, there are three vacua of \( F \). They are \(|0, p\rangle, |p, 0\rangle\) and \(|0, 0\rangle\) where \( p \) is a positive integer. The canonical conjugates are

\[
G_1^\dagger = a^\dagger b^\dagger \frac{1}{1 + a^\dagger a}, \quad G_2^\dagger = a^\dagger b^\dagger \frac{1}{1 + b^\dagger b}, \quad G_3^\dagger = \frac{1}{2}(G_1^\dagger + G_2^\dagger).
\]

The eigenstates of \( ab \), \( \exp(f G_1^\dagger)|0, 0\rangle \), \( \exp(f G_2^\dagger)|0, p\rangle \), and \( \exp(f G_3^\dagger)|00\rangle \) are the pair coherent states \([55-57]\). These states are also the eigenstates of the operator \( a^\dagger a - b^\dagger b \), as can be easily seen. The eigenstates of \( G_1, G_2, \) and \( G_3 \) are the second set of special states associated with the operator \( ab \). The eigenstate of \( G_3 \), \( \exp(f a^\dagger b^\dagger)|0, 0\rangle \) is the well known Caves-Schumaker state \([57-59]\).

Pairs of canonical conjugate operators were earlier constructed by Brandt and Greenberg \([64]\). Both operators of the kind \( G^\dagger = F^\dagger \) and \( G^\dagger \neq F^\dagger \) were constructed. They have the form

\[ F = a^m \phi(N)^{(1-n)}, \quad G^\dagger = \phi(N)^n a^\dagger^n \]

where \( \phi(N) \) has the form \((\frac{N}{m})(N - m!) /N!\)^{1/2} with \([x]\) denoting the largest integer smaller than or equal to \( x \). They can be made symmetric by choosing \( n \) appropriately. But these constructions do not include pairs like

\[ F = \frac{1}{N+1} a^2, \quad G_0^\dagger = a^\dagger a \frac{(N+3)}{(N+2)(N+4)}. \]

The Brandt-Greenberg states were subsequently studied for non-classical properties like squeezing \([64]\). The canonical conjugates of \( F = a^m \) in the sector generated from \(|0\rangle\) were constructed by Buzek, Jex and Quang \([66]\).

This construction of the canonical conjugates can also be extended to the generalized operators \( a^m \) \([71]\). The operator \( A^\dagger \) constructed in section is used for this purpose.
Chapter 5. *Generalized Bosonic Oscillators and their Coherent States*

The pairs of operators \((a, A^\dagger)\) and \((A^\dagger, a)\) satisfy the same commutation relations as the bosonic operators. As shown earlier, a set of bosonic operators \((b, b^\dagger)\) can be constructed corresponding to every pair \((A, a^\dagger)\).

The action of any expression consisting of the operators \((b, b^\dagger, N)\) on any number state is the same when the set is replaced by \((a, A^\dagger, N)\) or by \((A^\dagger, a^\dagger, N)\). The canonical conjugates of \(F = \phi(N)a^m\) are obtained from the corresponding bosonic operator by replacing \((b, b^\dagger, N)\) by \((a, A^\dagger, N)\). For \(F = a^2\), for example,

\[
G_0^\dagger = \frac{1}{2} A^{12} \frac{1}{N + 1}
\]

\[
G_1^\dagger = \frac{1}{2} A^{12} \frac{1}{N + 2}
\]

The conjugate of \(G_0\) and \(G_1\) is \(a^{12}\).

When the bosonic operators are replaced by \((A, a^\dagger, N)\), we have \(F' = \phi(N)A^m\) and its conjugates \(G'_0, G'_1\) are

\[
\frac{1}{2} a^{12} \frac{1}{N + 1}
\]

and

\[
\frac{1}{2} a^{12} \frac{1}{N + 2}
\]

respectively. The eigenstates of \(F, G, F', G'\) can be constructed as before.

In conclusion, we have put to use the fact that the various generalized bosonic oscillators are related to the bosonic oscillator and a canonical conjugate and be constructed for any of them. This was applied to study a deformed version of the Jaynes-Cummings model.

Canonical conjugates have been constructed for operators of the form \(\phi(N)b^m\). A systematic procedure has been given for the construction of these operators in the various sectors. A variety of eigenstates which can be called generalized coherent states, as they
are the eigenstates of annihilation operators, are constructed easily. The relation between the closely related states occurring in pairs like the squeezed states and the cat states is explained.

These results have been easily extended to the generalized bosonic oscillators using the operator $A^\dagger$. 